COLLOQUIUM MATHEMATICUM

VOL. LVIII 1990 FASC. 2

ALGEBRAIC SL(2)-*ACTIONS: DEFORMATIONS OF INFINITE ISOTROPY SUBGROUPS*

BY

EWA DUMA (WARSZAWA)

Let SL(2) act algebraically on an algebraic variety X in such a way that for any point x of some curve $C \subset X$ the isotropy subgroup at x is conjugate to a fixed subgroup $H \subset \mathrm{SL}(2)$. If a point y belongs to $\bar{C}\backslash C$ and the isotropy subgroup at y is G_y, then we say that there exists a *deformation* of H to G_y (see Definition 1).

In [9] Richardson has given an example of a deformation of a cyclic group of order 3 to the group $\{e\}$ (e denotes the identity of SL(2)). In this paper we describe all possible deformations of any infinite subgroup $H \subset \mathrm{SL}(2)$. In Section 2 there are drawn diagrams which depict all our results.

0. Assumptions and notation. Throughout the paper $\boldsymbol{k}$ denotes an algebraically closed field of characteristic zero, and $\boldsymbol{k}^* = \boldsymbol{k}\backslash\{0\}$ denotes the multiplicative group of $\boldsymbol{k}$. All algebraic varieties and morphisms are supposed to be defined over $\boldsymbol{k}$.

Let an algebraic group G act on an algebraic variety X. Then X is called a *G-variety*. By X^G we denote the variety of fixed points, i.e.,

$$X^G = \{x \in X\colon\ \forall g \in G,\ gx = x\}.$$

If $x \in X$, then $G_x = \{g \in G\colon\ gx = x\}$ is the isotropy subgroup at a point x, G_x^0 is the component of the identity of G_x, and

$$Gx = \{y \in X\colon\ \exists g \in G,\ y = gx\}$$

is the orbit of x.

For any two subgroups H_1 and H_2 of G we write $H_1 \cong H_2$ if H_1 and H_2 are conjugate in G, and $H_1 \subseteq H_2$ if there exists a subgroup H_3 such that $H_3 \subset H_2$ and $H_1 \cong H_3$.

Let

$$\mathrm{SL}(2) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\colon\ ad-bc = 1,\ a, b, c, d \in \boldsymbol{k}\right\}$$

in its usual matrix representation. Let $\boldsymbol{k}(\varrho_i)$ be the vector space of homogeneous polynomials of degree i in two variables x and y, with SL(2)-action determined by the following equalities:

$$g(x) = ax+cy \quad \text{and} \quad g(y) = bx+dy$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2).$$

Let $k(\varrho_{i_1}\oplus\ldots\oplus\varrho_{i_s})$ denote the direct sum of the vector spaces $k(\varrho_{i_j})$, $j = 1, 2, \ldots, s$, with the direct sum of SL(2)-actions on summands. By $P(\varrho_{i_1}\oplus\ldots\oplus\varrho_{i_s})$ we denote the projectivization of the vector space $k(\varrho_{i_1}\oplus\ldots\ldots\oplus\varrho_{i_s})$ with the projectivization of the SL(2)-action.

1. Preliminaries.

DEFINITION 1. Let G be an algebraic group with subgroups H_1 and H_2. We say that there exists a *deformation* of H_1 to H_2 in G and we write $H_1 \to H_2$ if one of the following two equivalent conditions holds:

(a) There exist a G-variety X, a curve $C \subset X$ and a point $x_0 \in \bar{C}\setminus C$ such that, for any $x \in C$, $G_x \cong H_1$ and $G_{x_0} \cong H_2$.

(b) There exist an irreducible G-variety X, a non-empty open subset $U \subset X$ and a point $x_0 \in X$ such that, for any $x \in U$, $G_x \cong H_1$ and $G_{x_0} \cong H_2$.

Remark 1. In condition (a) we may assume that $\bar{C}$ is irreducible.

Remark 2. It is easy to show that in both conditions we may claim that $G_{x_0} = H_2$.

Proof of the equivalence of conditions (a) and (b).

(a)$\Rightarrow$(b). Take an irreducible curve C as in (a) and $Y = \overline{G(C)}$. Then Y is G-stable and constructible, so there exists a non-empty open subset $U \subset G(C)$ such that, for any $x \in U$, $G_x \cong H_1$. For $\overline{G(C)}$ and U, condition (b) is fulfilled.

(b)$\Rightarrow$(a). As a curve C one may take any curve contained in U such that $x_0 \in \bar{C}$.

The following result is well known:

THEOREM 1. *Let an algebraic group G act on an algebraic variety X. Then for any positive integer n the set $\{x \in X\colon \dim G_x \geqslant n\}$ is closed in X.*

For the proof see Part 0 in Section 3 of [7].

As an immediate consequence of Theorem 1 we get

COROLLARY 1. *If there exists a deformation of H_1 to H_2 in G, then* $\dim H_1 \leqslant \dim H_2$.

2. Main results. From now on through the whole paper we assume that $G = \mathrm{SL}(2)$. We describe all possible deformations of infinite subgroups of SL(2). Up to conjugation there are only five types of infinite subgroups in SL(2): SL(2) and

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} : a \in \boldsymbol{k}^*,\ b \in \boldsymbol{k} \right\},$$

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in \boldsymbol{k}^* \right\},$$

$$N(T) = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in \boldsymbol{k}^* \right\},$$

and

$$N_n = \left\{ \begin{pmatrix} \varepsilon & b \\ 0 & 1/\varepsilon \end{pmatrix} : \varepsilon^n = 1,\ b \in \boldsymbol{k} \right\}, \quad n = 1, 2, \ldots,$$

$N_1 = \boldsymbol{k}^+$ (see [8]).

Below we draw diagrams of all possible deformations of infinite subgroups of SL(2). By $H_1 \not\to H_2$ we mean that there is no deformation of H_1 to H_2 in SL(2).

(a) $$\mathrm{SL}(2) \begin{cases} \not\to H & \text{for any } H \subsetneq \mathrm{SL}(2), \\ \to \mathrm{SL}(2), \end{cases}$$

(b) $$B \begin{cases} \not\to H & \text{for any } H \neq B, \\ \to B, \end{cases}$$

(c) $$N_n \begin{cases} \not\to T,\ N(T),\ N_m \text{ with } n \nmid m, \\ \to \mathrm{SL}(2),\ B,\ N_{rn}, \end{cases}$$

(d) $$T \to H \quad \text{for any } H \subseteq \mathrm{SL}(2) \text{ such that } \dim H \geqslant 1,$$

(e) $$N(T) \begin{cases} \not\to T,\ N_{2k+1}\ (k = 0, 1, 2, \ldots), \\ \to N(T),\ \mathrm{SL}(2),\ B,\ N_{2k}\ (k = 1, 2, \ldots). \end{cases}$$

As an immediate consequence of Corollary 1 we infer that SL(2) cannot be deformed to any subgroup $H \subsetneq \mathrm{SL}(2)$. Thus (a) is proved.

THEOREM 2. *The Borel subgroup B can be deformed only to itself.*

Proof. Konarski has proved in [3] that, for any algebraic group acting on X, the set of all isomorphic projective orbits is closed in X. Hence the set

$$\{x \in X\colon G_x \cong B\} = \{x \in X\colon Gx \cong P^1\}$$

is closed in X. Therefore (b) is proved.

In the following three sections we will consider the cases: $H = N_n$, $H = T$ and $H = N(T)$.

3. Deformations of N_n. There exist deformations of N_n to SL(2), B and N_m such that $n|m$. Examples:

(a) $N_n \to \mathrm{SL}(2)$. Take

$$C = \{p_t = tx^n \colon t \neq 0\} \subset k(\varrho_n).$$

For $t \neq 0$, $G_{p_t} = N_n$, $p_0 = 0 \in \bar{C}$ and $G_{p_0} = \mathrm{SL}(2)$.

(b) $N_n \to B$. Take

$$C = \{p_t = t \oplus x^n \colon t \neq 0\} \subset P(\varrho_0 \oplus \varrho_n).$$

Then, for $t \neq 0$, $G_{p_t} = N_n$, $p_0 = 0 \oplus x^n \in \bar{C}$ and $G_{p_0} = B$.

(c) $N_n \to N_{rn}$. Take

$$C = \{p_t = tx^n \oplus x^{rn} \colon t \neq 0\} \subset k(\varrho_n \oplus \varrho_{rn}).$$

Then, for $t \neq 0$, $G_{p_t} = N_n$, $p_0 = 0 \oplus x^{rn} \in \bar{C}$ and $G_{p_0} = N_{rn}$.

THEOREM 3. *Let X be an irreducible* SL(2)*-variety and let $U \subset X$ be a non-empty open subset such that, for $x \in U$, $k^+ = N_1 \subsetneqq G_x$. Then, for any $x \in X$, $k^+ \subsetneqq G_x$.*

Proof. Let $\Gamma = \{(g, x) \in G \times X \colon gx = x\}$ $(G = \mathrm{SL}(2))$ and let $\pi \colon \Gamma \to X$ be the projection. Let G^u be the set of all unipotent elements in G. Then G^u is closed in G and $G^u \times X \cap \Gamma = Y$ is a closed subset of Γ. For $x \in U$,

$$\dim \pi^{-1}(x) \cap Y \geqslant 1.$$

We denote by $Y_1, \ldots, Y_r$ all irreducible components of Y, and by $Y_1, \ldots, Y_s$ all components such that $\{e\} \times X \subset Y_i$ $(i = 1, \ldots, s)$. Then there exists a non-empty open subset $V \subset X$ such that $\{e\} \times V \cap Y_j = \emptyset$ for $j > s$. Let

$$Y_0 = \bigcup_{i=1}^{s} Y_i \quad \text{and} \quad \pi_0 = \pi | Y_0.$$

For $x \in U \cap V$, $\dim \pi_0^{-1}(x) \geqslant 1$ since $\dim(G_x^0 \times \{x\} \cap Y_0) \geqslant 1$. There exists a non-empty open subset $U_1 \subset X$ such that, for $x \in U_1$ and for a fixed component of Y_0, e.g., Y_1,

$$\dim(\pi_0^{-1}(x) \cap Y_1) \geqslant 1.$$

Since $\{e\} \times X \subset Y_1$, $\pi_0 | Y_1 = \pi_1 \colon Y_1 \to X$ is surjective. Since U_1 is dense in X, for any $x \in X$ we have $\dim \pi_1^{-1}(x) \geqslant 1$ ([2], Part II, Ex. 3.22). Moreover, $\pi_1^{-1}(x) \subset G_x \times \{x\}$. Hence there exists $H \subset G$, $H \cong k^+$ such that $H \times \{x\} \subset \pi_1^{-1}(x)$.

As an immediate consequence of Theorem 3 we get

COROLLARY 2. *There is no deformation in* SL(2) *of N_n $(n = 1, 2, \ldots)$ to the torus T or to its normalizer $N(T)$.*

THEOREM 4. *Let n and r be positive integers. If there exists a deformation in* SL(2) *of N_n to N_r, then $n | r$.*

Proof. Fix positive integers n and r. Suppose that there exists a deformation of N_n to N_r on an irreducible G-variety X $(G = \mathrm{SL}(2))$, i.e., there exist a non-empty open subset $U \subset X$ and a point $z_0 \in X$ such that, for $z \in U$, $G_z \cong N_n$ and $G_{z_0} = N_r$.

We may assume that X is normal. In fact, let ν: $X^\nu \to X$ be the normalization of X and let ψ^ν: $G \times X^\nu \to X^\nu$ be the action of G on X^ν, induced by the G-action ψ: $G \times X \to X$ on X. The set

$$V = \{x \in X\colon X \text{ is normal at } x\}$$

is dense in X and $V \subset X^\nu$. For $y \in U \cap V$,

$$G_y = G_{\nu(y)} \cong N_n.$$

For $y \in X^\nu$,

$$G_y^0 = G_{\nu(y)}^0 \quad \text{and} \quad G_y \subset G_{\nu(y)}.$$

Let, for $y_0 \in X^\nu$, $\nu(y_0) = z_0$. Then $G_{y_0} \subset G_{z_0} = N_r$, so $G_{y_0} = N_s$ with $s|r$.

Therefore we may assume that X is normal. Let C be a curve such that $C \subset U$, $z_0 \in \bar{C}$ and $\bar{C}$ is irreducible. There exist a G-stable quasi-projective open neighbourhood W of z_0 ([10], Lemma 8) and a projective G-embedding f: $W \to P^N$, where G acts linearly on P^N ([10], Theorem 1).

Replacing $f(W)$ by W, $f(C \cap W)$ by C, and $f(z)$ by z, we have $z_0 \in \bar{C}$, $G_{z_0} = N_r$ and $G_z \cong N_n$ for $z \in C$, where $\bar{C}$ is the irreducible curve in P^N.

P^N with a linear SL(2)-action on it is of the following form:

$$P^N = P(\varrho_0^{l_0} \oplus \ldots \oplus \varrho_k^{l_k}),$$

where $l_i \geq 0$ and $\varrho_i^{l_i} = \varrho_i \oplus \ldots \oplus \varrho_i$ (l_i times) for $i = 0, 1, \ldots, k$.

Let $z \in P^N$. Then $z = [z^0; \ldots; z^k]$, where $z^i \in \boldsymbol{k}(\varrho_i^{l_i})$. One has $G_z = N_n$ iff there exist an integer s and a sequence $0 \leq i_1 < \ldots < i_s \leq k$ with the following properties:

(*) $s \geq 2$ and $\mathrm{GCD}\{(i_l - i_p)\colon l, p = 1, \ldots, s\} = n$, where GCD denotes the greatest common divisor;

(a) for any $i_j \in \{i_1, \ldots, i_s)$ there exists

$$(\alpha_1^{(i_j)}, \ldots, \alpha_{l_{i_j}}^{(i_j)}) \in \boldsymbol{k}^{l_{i_j}} \setminus \{(0, \ldots, 0)\}$$

such that

$$z^{i_j} = \alpha_1^{(i_j)} x^{i_j} \oplus \ldots \oplus \alpha_{l_{i_j}}^{(i_j)} x^{i_j};$$

(b) for any $j \notin \{i_1, \ldots, i_s\}$, $z^j = 0 \oplus \ldots \oplus 0$.

For a fixed sequence $\{i_1, \ldots, i_s\}$ with property (*) let

$$X(i_1, \ldots, i_s) = \{z \in P^N\colon z \text{ has properties (a) and (b)}\}.$$

Let $Y(i_1, \ldots, i_s) = G(X(i_1, \ldots, i_s))$. Then

$$\{z \in P^N\colon G_z \cong N_n\} = \bigcup_{\{i_1, \ldots, i_s\}} Y(i_1, \ldots, i_s),$$

where $\{i_1, \ldots, i_s\}$ has property (*). For $z \in Y(i_1, \ldots, i_s)$ and $j \notin \{i_1, \ldots, i_s\}$,

$$z^j = 0 \oplus \ldots \oplus 0.$$

We have

$$C \subset \bigcup_{\{i_1,\ldots,i_s\}} Y(i_1, \ldots, i_s) \subset \bigcup_{\{i_1,\ldots,i_s\}} \overline{Y(i_1, \ldots, i_s)}.$$

From the irreducibility of $\bar{C}$ it follows that there exists $\{i_1, \ldots, i_s\}$ with property (*) such that

$$\bar{C} \subset \overline{Y(i_1, \ldots, i_s)},$$

so $z_0 \in \overline{Y(i_1, \ldots, i_s)}$. •

On the other hand, $G_{z_0} = N_r$, which implies that there exist an integer $t \geqslant 2$ and a sequence $0 \leqslant j_1 < \ldots < j_t \leqslant k$ such that

$$\mathrm{GCD}\{(j_l - j_p)\colon\ l, p = 1, \ldots, t\} = r$$

and, for $j \notin \{j_1, \ldots, j_s\}$,

$$z_0^j = 0 \oplus \ldots \oplus 0.$$

Hence $\{j_1, \ldots, j_t\} \subset \{i_1, \ldots, i_s\}$ and $n|r$. The proof is complete.

4. Deformations of T. For any subgroup $H \subset \mathrm{SL}(2)$ such that $\dim H \geqslant 1$ there exists a deformation of T to H. Examples:

(a) $T \to \mathrm{SL}(2)$. Take

$$C = \{p_t = txy\colon\ t \neq 0\} \subset k(\varrho_2).$$

For $t \neq 0$, $G_{p_t} = T$, $p_0 = 0 \in \bar{C}$ and $G_{p_0} = \mathrm{SL}(2)$.

In examples (b), (d), (e) below we use the notation

$$g_t = \begin{pmatrix} 1/t & 1/2 \\ -1 & t/2 \end{pmatrix} \quad \text{for } t \neq 0.$$

(b) $T \to B$. For $p = x^2 y \in P(\varrho_3)$, $G_p = T$. Let

$$C = \{p_t = g_t(p) = x^3 - tx^2 y - t^2 xy^2 + t^3 y^3\colon\ t \neq 0\} \subset P(\varrho_3).$$

Then $G_{p_t} \cong T$ for $t \neq 0$, $p_0 = x^3 \in \bar{C}$ and $G_{p_0} = B$.

(c) $T \to N(T)$. Take

$$C = \{p_t = t \oplus xy\colon\ t \neq 0\} \subset P(\varrho_0 \oplus \varrho_2).$$

Then, for $t \neq 0$, $G_{p_t} = T$, $p_0 = 0 \oplus xy \in \bar{C}$ and $G_{p_0} = N(T)$.

(d) $T \to N_{2k}$ for $k = 1, 2, \ldots$ For $t \neq 0$ let

$$p_t = t^{k+1} x^{k+1} y^{k-1} \oplus t^{2k+1} x^{2k+1} y^{2k-1} \in P(\varrho_{2k} \oplus \varrho_{4k}).$$

Then $G_{p_t} = T$. Take

$$C = \{w_t = g_t(p_t) = (x-ty)^{k+1}(x+ty)^{k-1} \oplus 2^{-k}(x-ty)^{2k+1}(x+ty)^{2k-1}: t \neq 0\} \subset P(\varrho_{2k} \oplus \varrho_{4k}).$$

Then $G_{w_t} \cong T$, $w_0 = x^{2k} \oplus 2^{-k}x^{4k} \in \bar{C}$ and $G_{w_0} = N_{2k}$.

(e) $T \to N_n$ for $n = 2k+1$ and $k = 0, 1, 2, \ldots$ For $t \neq 0$ let

$$p_t = t^{3k+2}x^{3k+2}y^{3k+1} \oplus t^{5k+3}x^{5k+3}y^{5k+2} \in P(\varrho_{3n} \oplus \varrho_{5n}).$$

Then $G_{p_t} = T$. Take

$$C = \{w_t = g_t(p_t) = (x-ty)^{3k+2}(x+ty)^{3k+1} \oplus 2^{-(2k+1)}(x-ty)^{5k+3}(x+ty)^{5k+2}: t \neq 0\} \subset P(\varrho_{3n} \oplus \varrho_{5n}).$$

Then $G_{w_t} \cong T$, $w_0 = x^{3n} \oplus 2^{-(2k+1)}x^{5n} \in \bar{C}$ and $G_{w_0} = N_n$.

5. Deformations of $N(T)$. There exist deformations of $N(T)$ to SL(2), B and N_{2k} for $k = 1, 2, \ldots$ Examples:

(a) $N(T) \to \mathrm{SL}(2)$. Take

$$C = \{p_t = tx^2y^2: t \neq 0\} \subset k(\varrho_4).$$

Then, for $t \neq 0$, $G_{p_t} = N(T)$, $p_0 = 0 \in \bar{C}$ and $G_{p_0} = \mathrm{SL}(2)$.

In the examples below g_t, $t \neq 0$, is the matrix defined in the previous section.

(b) $N(T) \to B$. For $p = xy \in P(\varrho_2)$, $G_p = N(T)$. Let

$$C = \{p_t = g_t(p) = (x-ty)(x+ty): t \neq 0\} \subset P(\varrho_2).$$

Then $G_{p_t} \cong N(T)$ for $t \neq 0$, $p_0 = x^2 \in \bar{C}$ and $G_{p_0} = B$.

(c) $N(T) \to N_{2k}$ for $k = 1, 2, \ldots$ For $t \neq 0$, let

$$p_t = t^k x^k y^k \oplus t^{3k}x^{3k}y^{3k} \in P(\varrho_{2k} \oplus \varrho_{6k}).$$

Then $G_{p_t} = N(T)$. Take

$$C = \{w_t = g_t(p_t) = (x-ty)^k(x+ty)^k \oplus 2^{-2k}(x-ty)^{3k}(x+ty)^{3k}: t \neq 0\} \subset P(\varrho_{2k} \oplus \varrho_{6k}).$$

Then $G_{w_t} \cong N(T)$, $w_0 = x^{2k} \oplus 2^{-2k}x^{6k} \in \bar{C}$ and $G_{w_0} = N_{2k}$.

THEOREM 5. *There is no deformation of* $N(T)$ *to* N_{2r+1} *for any non-negative integer* r.

Proof. Suppose that for some integer $r \geqslant 0$ there exists a deformation of $N(T)$ to N_{2r+1} on an irreducible G-variety X. Analogously as in the proof of Theorem 4 we may assume that there exist a projective space P^N with a linear SL(2)-action on it, a curve $C \subset P^N$ and a point $x_0 \in \bar{C} \setminus C$ such that $\bar{C}$ is irreducible and, for any $x \in C$, $G_x \cong N(T)$ and $G_{x_0} = N_{2r+1}$.

Let us use the notation of the proof of Theorem 4. Let

$$Z = \{w \in P^N:\ G_w \cong N(T)\}.$$

Then

$$Z \subset Y = \{w = [w^0; \ldots; w^k] \in P^N:$$
$$w^i = 0 \oplus \ldots \oplus 0 \text{ for } i = 2p+1, \text{ where } p = 0, 1, \ldots, E(k/2)\}.$$

On the other hand,

$$x_0 \in \{w \in P^N:\ G_w = N_{2r+1}\} = \bigcup_{\{i_1, \ldots, i_s\}} X(i_1, \ldots, i_s),$$

where $\{i_1, \ldots, i_s\}$ has property (*) with $n = 2r+1$ (see the proof of Theorem 4). It is easy to see that

$$\bar{Y} \cap \Big(\bigcup_{\{i_1, \ldots, i_s\}} X(i_1, \ldots, i_s)\Big) = \emptyset.$$

Hence

$$\bar{Z} \cap \Big(\dot{\bigcup_{\{i_1, \ldots, i_s\}}} X(i_1, \ldots, i_s)\Big) = \emptyset,$$

$$\bar{C} \cap \Big(\bigcup_{\{i_1, \ldots, i_s\}} X(i_1, \ldots, i_s)\Big) = \emptyset,$$

a contradiction.

THEOREM 6. *There is no deformation of $N(T)$ to T.*

Proof. Suppose that there exists a deformation of $N(T)$ to T on an irreducible G-variety X ($G = \mathrm{SL}(2)$), i.e., there exist a non-empty open subset $U \subset X$ and a point $x_0 \in X$ such that, for any $x \in U$, $G_x \cong N(T)$ and $G_{x_0} = T$.

The plan of the proof is as follows: First we show that we may assume that X is normal, affine and all G-orbits are closed. Then we apply the "étale-slice" theorem to obtain a subset Y which is transversal to the orbit Gx_0 at x_0. There exist $x_n \in Y$ ($n = 1, 2, \ldots$) such that

$$\lim_{n \to \infty} x_n = x_0$$

in the complex topology and $G_{x_n} = N(T)$. Since $G_{x_0} = T$, this gives a contradiction.

As easy consequences of Theorems 1 and 3 we infer that there is no 3-dimensional orbit in X and that a subset

$$Z = \{x \in X:\ k^+ \subsetneq G_x\}$$

is closed in X. Hence taking $X := X \setminus Z$ we may assume that $Z = \emptyset$. We may also assume that X is normal (see the proof of Theorem 4).

Under these assumptions, for any $x \in X$, $G_x \cong N(T)$ or $G_x \cong T$, and thus any G-orbit in X is closed. Let

$$U' = \{x \in X\colon\ G_x \cong N(T)\};$$

then U' is a G-stable, dense subset of X and $U \subset U'$.

By Theorem 5 of [1] there exists a geometric quotient of X by SL(2), i.e., there exists an affine open surjective morphism $\psi\colon X \to X/G$ such that, for any $y \in X/G$, $\psi^{-1}(y)$ is exactly one G-orbit and, for any open subset $V \subset X/G$, the induced homomorphism

$$k[U] \to k[\psi^{-1}(V)]^G$$

is an isomorphism.

Let V be an open affine neighbourhood of the point $\psi(x_0)$ in X/G. Then $\psi^{-1}(V)$ is a G-stable irreducible neighbourhood of the point x_0 in X. Replacing X by $\psi^{-1}(V)$, and U' by $U' \cap \psi^{-1}(V)$, we may assume that X is a normal affine irreducible G-variety with a G-stable dense subset $U' \subset X$ such that, for any $x \in U'$, $G_x \cong N(T)$.

The isotropy subgroup at x_0 is a reductive group ([7], Definition 1.4) $G_{x_0} = T$ and the G-orbit of x_0 is closed. By Luna's "étale-slice" theorem (see [4]) there exists a locally closed affine subset $Y \subset X$ such that $x_0 \in Y$ and a G-action on X induces an étale G-morphism

$$f\colon\ G \times_T Y \to X,$$

where $G \times_T Y$ denotes a quotient of $G \times Y$ by T, and a T-action on $G \times Y$ is given by

$$t(g, y) = (gt^{-1}, ty) \quad \text{for } t \in T,\ g \in G,\ y \in Y.$$

A G-action on $G \times_T Y$ is induced by the following G-action on $G \times Y$:

$$g_1(g, y) = (g_1 g, y) \quad \text{for } g,\ g_1 \in G,\ y \in Y.$$

By Lemma 1 in [5], if X is normal, then Y may be chosen irreducible.

$f\colon G \times_T Y \to X$ is given by the equality

$$f(\overline{(g, y)}) = gy,$$

where $\overline{(g, y)}$ denotes the image of $(g, y) \in G \times Y$ by the quotient morphism. Since f is étale, $\operatorname{im}(f)$ is an open subset of X ([6], Theorem I.2.12), whence $\operatorname{im}(f) \cap U' \neq \emptyset$. On the other hand,

$$\operatorname{im}(f) = f(G \times_T Y) \subset G(Y),$$

so $G(Y) \cap U' \neq \emptyset$. This implies that $Y \cap U' \neq \emptyset$ (U' is G-stable). Let $V = Y \cap U'$. By the density of U' in X the set V is dense in Y, so $x_0 \in Y \subset \bar{V}$, the closure is taken in X.

Since all varieties and morphisms are defined over the field $\boldsymbol{k}$ of characteristic zero, we may assume that $\boldsymbol{k} = \boldsymbol{C}$, where $\boldsymbol{C}$ is the field of complex numbers. Consequently, X is a complex space and there exists a sequence of points $x_n \in V$ ($n = 1, 2, \ldots$) such that $x_n \to x_0$ in the complex topology. We will show that $G_{x_n} = N(T)$ for $n = 1, 2, \ldots$

For $n = 1, 2, \ldots$ we have $(e, x_n) \in G \times Y$ (e is the identity) and

$$f(\overline{(e, x_n)}) = x_n.$$

Since f is a G-morphism,

$$G_{\overline{(e,x_n)}} \subset G_{x_n}.$$

Moreover, $G_{x_n}/G_{\overline{(e,x_n)}}$ is finite (f is étale) and $G_{x_n} \cong N(T)$. Hence

$$T \stackrel{\subset}{\simeq} G_{\overline{(e,x_n)}} \subset G_{x_n}.$$

We will show that $T \subset G_{x_n}$.

Let $g \in G_{\overline{(e,x_n)}}$. Then

$$\overline{(e, x_n)} = \overline{(g, x_n)},$$

i.e., there exists $t \in T$ such that

$$(gt^{-1}, tx_n) = (e, x_n),$$

so $gt^{-1} = e$ and $g \in T$. Hence $G_{\overline{(e,x_n)}} \subset T$. We know that

$$T \stackrel{\subset}{\simeq} G_{\overline{(e,x_n)}},$$

so

$$T = G_{\overline{(e,x_n)}}.$$

This equality implies that $T \subset G_{x_n} \cong N(T)$ and, by easy computations, $G_{x_n} = N(T)$ for any positive integer n. Hence we have

$$G_{x_0} \supset N(T)$$

since

$$x_0 = \lim_{n \to \infty} x_n,$$

a contradiction.

I wish to thank Professor Andrzej Białynicki-Birula for his help in the preparation of this paper.

REFERENCES

[1] E. Duma, *On* SL(2)*-actions without 3-dimensional orbits*, this fascicle, pp. 233–241.
[2] R. Hartshorne, *Algebraic Geometry*, Springer, New York–Heidelberg–Berlin 1977.
[3] J. Konarski, *Properties of projective orbits of actions of affine algebraic groups*, pp. 79–91 in: *Group Actions and Vector Fields*, Lecture Notes in Math. 956, Springer, 1982.
[4] D. Luna, *Slices étales*, Bull. Soc. Math. France, Suppl., Mém. 33 (1973), pp. 81–105.
[5] — et Th. Vust, *Un théorème sur les orbites affines des groupes algébriques semi-simples*, Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat. III. Ser. 27 (1974), pp. 527–535.
[6] J. S. Milne, *Étale Cohomology*, Princeton Univ. Press, Princeton, New Jersey, 1980.
[7] D. Mumford, *Geometric Invariant Theory*, Springer, New York–Berlin–Heidelberg 1965.
[8] V. L. Popov, *Quasi-homogeneous affine algebraic varieties of* SL(2) (in Russian), Izv. Akad. Nauk USSR 37 (1973), pp. 799–832.
[9] R. W. Richardson, *Deformations of Lie subgroups and the variation of isotropy subgroups*, Acta Math. 129 (1972), pp. 35–73.
[10] H. Sumihiro, *Equivariant completions*, J. Math. Kyoto Univ. 14 (1974), pp. 1–28.

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
00-901 WARSZAWA
PKiN IX p., POLAND

Reçu par la Rédaction le 10.5.1984;
en version définitive le 6.6.1988