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ONE-PARAMETER SUBMONOIDS IN LOCALLY COMPLETE DIFFERENTIABLE MONOIDS

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Suppose X is a Banach space, D is a closed admissible subset of X containing 0, and V is an associative multiplication from $D \times D$ into X which is strongly differentiable at (0, 0) satisfying $V(x, 0) = V(0, x) = x$ for each x in D. Suppose further that there is a positive number b such that if each of x and y is in D and has norm less than b, then $V(x, y)$ is in D. If there exists a function f: $[0, 1] \to D$ such that $f(0) = 0$, f is strongly differentiable at 0, and $f'(0) \neq 0$, then there exists a function T: $[0, 1] \to D$ and a number s in $(0, 1]$ satisfying $T(0) = 0$, $V(T(x), T(y)) = T(x+y)$ for each x, y, and $x+y$ in $[0, 1]$, T is strongly differentiable at 0, and $T'(0) = sf'(0)$. This result sheds some light on a question posed by Graham in [G2], regarding the existence of one-parameter subsemigroups.

Before proceeding to the main theorem we will indicate some background.

The definitions in this paragraph are due to Graham in [G2] and [G3]. A subset D of the Banach space X is said to be *admissible* provided that each point of D is a limit point of the interior of D. A function f, with domain the admissible subset D of the Banach space X and codomain contained in the Banach space Y, is *strongly differentiable at the point p* in D provided there is a continuous linear map T from X to Y such that for each positive number c there is a positive number d such that if each of x and y is in D and within d of p, then

$$|f(x)-f(y)-T(x-y)| \leqslant c|x-y|.$$

In this case T is unique and is denoted by $f'(p)$.

The statement that the function f from D into the Banach space Y is C^1 means that f is strongly differentiable at each point of D and the function f' is continuous as a function from D into $L(X, Y)$. The statement that f is C^k means that $f^{(k-1)}$ is C^1. A Hausdorff topological space S is a *C^k manifold based on the Banach space X* provided that for each point p of S there is a homeomorphism g_p from a neighborhood U of p onto an admissible subset D of X containing 0 so that $g_p(p) = 0$ and the composition $g_p \circ g_q^{-1}$ is C^k on its domain for each choice of p and q in S. Finally, according to Graham, a topological semigroup is said to be C^k provided that it is based on a C^k manifold and the multiplication is C^k as a function from $S \times S$ into S.

Much of the calculus on C^k manifolds mimics the standard theory. Most of the difference is due to the possible non-convexity of admissible sets. This non-convexity also implies that a C^k monoid need have no non-trivial one-parameter subsemigroups. For example, by [G2], the subset of the plane to which (x, y) belongs only in case x is positive and y is between 0 and x^2 or $(x, y) = (0, 0)$ forms a C^k monoid under vector addition and contains no non-trivial one-parameter subsemigroups.

A question Graham asks in [G1] is: Under what hypothesis does a C^∞ monoid contain a non-trivial one-parameter subsemigroup? He answers this question, in [G1], in certain finite dimensional C^∞ monoids with smooth boundary.

In 1987, in [H2] Holmes shows that if S is a locally compact connected C^k monoid, then S contains a non-trivial C^k one-parameter subsemigroup. Holmes shows in [H1], in 1987, that if S is a locally complete C^k monoid, $k \geqslant 2$, which contains a C^2 curve starting at 1, then S must contain non-trivial C^k one-parameter subsemigroups. By using a much different approach, Theorem 2 in this paper improves on this result by requiring only that S be a monoid with multiplication strongly differentiable at (0, 0) and that S contain a curve starting at 1 which is strongly differentiable at 0. We now proceed with Theorem 1.

Let D be a closed admissible subset of the Banach space X, containing 0. Let V be an associative multiplication from $D \times D$ into X which is strongly differentiable at (0, 0) satisfying $V(x, 0) = V(0, x) = x$ for each x in D. Suppose there is a positive number b such that if each of x and y is in D and has norm less than b, then $V(x, y)$ is in D. Such a function is called a *strongly differentiable local monoid.*

THEOREM 1. *Suppose V is a strongly differentiable local monoid. If there exists a function f: $[0, 1] \to D$ such that $f(0) = 0$, f is strongly differentiable at* 0, *and $f'(0) \neq 0$, then there is a function T: $[0, 1] \to D$ and a number s in* $(0, 1]$ *satisfying $T(0) = 0$, $T(x+y) = V(T(x), T(y))$ whenever each of x, y, and $x+y$ is in* $[0, 1]$, *T is strongly differentiable at* 0, *and $T'(0) = sf'(0)$.*

Theorem 1 will follow from a sequence of lemmas. Lemmas 1.1 and 1.4 were suggested from arguments in [B].

LEMMA 1.1. *If c is a positive number there is a positive number d such that if each of $x_1, x_2, \ldots, x_n$ is in D and $\sum_{i=1}^{n} |x_i| < d$, then $\prod_{i=1}^{n} x_i$ is in D and*

$$\left|\prod_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i\right| \leqslant c \sum_{i=1}^{n} |x_i|.$$

Here $\prod_{i=1}^{n} x_i$ denotes $V(x_n, V(x_{n-1}, \ldots, V(x_2, x_1) \cdots)$.

Proof. Choose a positive number b so that if each of x and y is in D and within b of 0, then $V(x, y)$ is in D. Suppose c is a positive number less than 1. Using

$$V'(0, 0)(x, y) = x+y \quad \text{and} \quad |V(x, y)-x-y| = |V(x, y)-V(x, 0)-y|,$$

choose a positive number $d_1 < b$ so that if each of x and y is in D and has norm less than d_1, then $|V(x, y)-x-y| \leqslant c|y|$. Let d be a positive number less than $d_1/2$. The proof is by induction on n. If each of x_1 and x_2 is in D and $|x_1|+|x_2| < d$, then $V(x_2, x_1)$ is in D by the choice of b, and

$$|V(x_2, x_1)-x_2-x_1| \leqslant c|x_1| \leqslant c(|x_1|+|x_2|)$$

by the choice of d_1. Next, suppose each of $x_1, x_2, \ldots, x_n$ is in D and $\sum_{i=1}^{n} |x_i| < d$. If $\prod_{i=1}^{n-1} x_i$ is in D and

$$\Big|\prod_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} x_i\Big| \leqslant c \sum_{i=1}^{n-1} |x_i|,$$

then

$$\Big|\prod_{i=1}^{n-1} x_i\Big| \leqslant 2 \sum_{i=1}^{n-1} |x_i| < d_1 < b.$$

Therefore, since $|x_n|$ is also less than b,

$$\prod_{i=1}^{n} x_i = V(x_n, \prod_{i=1}^{n-1} x_i)$$

is in D. Furthermore, since each of $|x_n|$ and $\Big|\prod_{i=1}^{n-1} x_i\Big|$ is less than d_1, it follows that

$$\Big|\prod_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i\Big| \leqslant \Big|V(x_n, \prod_{i=1}^{n-1} x_i)-(x_n+\prod_{i=1}^{n-1} x_i)\Big| + \Big|\prod_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} x_i\Big| \leqslant c \sum_{i=1}^{n} |x_i|.$$

Lemma 1.1 now follows from induction. Note that associativity is not used in the proof of Lemma 1.1.

Suppose x is in D and has norm less than b. Let $x^0 = 0$, and if n is a positive integer so that x^{n-1} is defined and has norm less than b, let $x^n = V(x, x^{n-1})$. Let $f\colon [0, 1] \to D$ be such that $f(0) = 0$, f is strongly differentiable at 0, and $f'(0) \neq 0$. Let $M = |f'(0)|+1$. The corollary below follows immediately from Lemma 1.1.

COROLLARY 1.2. *There is a positive number d such that if x is in $(0, d)$ and each of n and m is a positive integer, then $\big(f(x/nm)\big)^n$ is in D and*

$$\big|\big(f(x/nm)\big)^n\big| \leqslant 2Mx/m.$$

LEMMA 1.3. *If c is a positive number there is a positive number d such that if each of x, y, a, and e is in D and $|x|+|y|+|a| < d$ and $|x|+|y|+|e| < d$, then each of $V(x, V(a, y))$ and $V(x, V(e, y))$ is in D and*

$$|V(x, V(a, y)) - V(x, V(e, y)) - (a-e)| \leqslant c|a-e|.$$

The proof follows directly from strong differentiability of V at (0, 0) and the chain rule. Lemma 1.4, unlike Lemma 1.1 which is true for non-associative multiplications, relies heavily on the hypothesis that V is an associative multiplication.

LEMMA 1.4. *If c is a positive number there is a positive number d such that if each of $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ is in D, $\sum_{i=1}^{n} |x_i| < d$, and $\sum_{i=1}^{n} |y_i| < d$, then each of $\prod_{i=1}^{n} x_i$ and $\prod_{i=1}^{n} y_i$ is in D and*

$$(1) \qquad |\prod_{i=1}^{n} x_i - \prod_{i=1}^{n} y_i - \sum_{i=1}^{n} (x_i - y_i)| \leqslant c \sum_{i=1}^{n} |x_i - y_i|.$$

Proof. Suppose c is a positive number less than 1. Let d_1 be a positive number less than b satisfying Lemma 1.3. Using Lemma 1.1, assume that d_2 is a positive number less than $d_1/6$ such that if each of $x_1, x_2, \ldots, x_n$ is in D and $\sum_{i=1}^{n} |x_i| < d$, then $\prod_{i=1}^{n} x_i$ is in D and

$$|\prod_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i| \leqslant c \sum_{i=1}^{n} |x_i|.$$

Finally, let d be a positive number less than $d_2/2$. Suppose each of $x_1, \ldots, x_n, y_1, \ldots, y_n$ is in D, and

$$\sum_{i=1}^{n} |x_i| < d, \qquad \sum_{i=1}^{n} |y_i| < d.$$

It then follows from the choice of d_2 that if k is a positive integer less than $n+1$, then each of

$$V(\prod_{i=k+1}^{n} x_i, V(x_k, \prod_{i=1}^{k-1} y_i)) \quad \text{and} \quad V(\prod_{i=k+1}^{n} x_i, \prod_{i=1}^{k} y_i)$$

is defined. Moreover, using the choice of d_1 and the fact that

$$V(\prod_{i=k+2}^{n} x_i, V(x_{k+1}, \prod_{i=1}^{k} y_i)) = V(\prod_{i=k+1}^{n} x_i, \prod_{i=1}^{k} y_i) = V(\prod_{i=k+1}^{n} x_i, V(y_k, \prod_{i=1}^{k-1} y_i)),$$

we see that (1) holds true.

We now have the following corollary to Lemma 1.4:

COROLLARY 1.5. *There is a positive number d such that if x is in (0, d) and n and m are positive integers, then*

(2) $$|(f(x/n))^n-(f(x/mn))^{mn}| \leqslant 2n\,|f(x/n)-(f(x/mn))^m|.$$

LEMMA 1.6. *Suppose F is a function from the subset U of the Banach space X to the Banach space Y such that F(0) = 0 and F is strongly differentiable at 0. If c is a positive number there is a positive number d such that if each of* $x_1, x_2, \ldots, x_n, \sum_{i=k}^{n} x_i$ *is in U for each* $k = 1, 2, \ldots, n$, *and* $\sum_{i=1}^{n} |x_i| < d$, *then*

$$|F(\sum_{i=1}^{n} x_i)-\sum_{i=1}^{n} F(x_i)| \leqslant c\sum_{i=1}^{n} |x_i|.$$

Proof. Suppose c is a positive number and let d be a positive number less than 1 such that if each of x and y is in U and within d of 0, then

$$|F(x)-F(y)-F'(0)(x-y)| \leqslant \frac{c}{2}|x-y|.$$

This implies that if each of x, y, and $x+y$ is in U and within d of 0, then

$$|F(x+y)-(F(x)+F(y))| \leqslant |F(x+y)-F(x)-F'(0)(y)|+|F'(0)(y)-F(y)| \leqslant c|y|.$$

Therefore, if each of $x_1, x_2, \ldots, x_n$ and $\sum_{i=k}^{n} x_i$ is in U for each $k = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} |x_i| < d$, then

$$\begin{aligned}|F(\sum_{i=1}^{n} x_i)-\sum_{i=1}^{n} F(x_i)| &\leqslant \sum_{k=1}^{n-1} |F((\sum_{i=k+1}^{n} x_i)+x_k)-(F(\sum_{i=k+1}^{n} x_i)+F(x_k))| \\ &\leqslant c\sum_{k=1}^{n-1} |x_k| \leqslant c\sum_{i=1}^{n} |x_i|.\end{aligned}$$

Notice, since [0, 1] is a subset of the real numbers and f is strongly differentiable at 0, that if c is a positive number, there is a positive number d such that if each of $x_1, x_2, \ldots, x_n, \sum_{i=1}^{n} x_i$ is in $[0, d)$, then

$$|f(\sum_{i=1}^{n} x_i)-\sum_{i=1}^{n} f(x_i)| \leqslant c\sum_{i=1}^{n} x_i.$$

We are now in a position to show that if s is sufficiently small, then sequences of the form $\{(f(s/2^n))^{2^n}\}_{n=1}^{\infty}$ converge in D. Furthermore, they converge to non-zero elements.

LEMMA 1.7. *There is a positive number B such that if s is in (0, B), then the sequence* $\{(f(s/2^n))^{2^n}\}_{n=1}^{\infty}$ *is Cauchy in D and converges to a non-zero element.*

Proof. As before, let $M = |f'(0)|+1$. Choose B in $(0, 1]$ so that

(i) if x is in $(0, B)$ and n is a positive integer, then $(f(x/n))^n$ is in D and $|(f(x/n))^n| \leqslant 2Mx$ by Corollary 1.2;

(ii) if x is in $(0, B)$ and n and m are positive integers, then inequality (2) holds by using Corollary 1.5;

(iii) if x is in $(0, B)$ and n is a positive integer, then

$$\left|\left(f\left(\frac{x}{n}\right)\right)^n - nf\left(\frac{x}{n}\right)\right| \leqslant \frac{n}{2}\left|f\left(\frac{x}{n}\right)\right|$$

by Lemma 1.1;

(iv) if each of $x_1, x_2, \ldots, x_n$ is in D and $\sum_{i=1}^{n} |x_i| < d$, then $\prod_{i=1}^{n} x_i$ is in D and

$$\left|\prod_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i\right| \leqslant \sum_{i=1}^{n} |x_i|$$

by Lemma 1.1; and

(v) if each of $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ is in D, and

$$\sum_{i=1}^{n} |x_i| < 2MB, \qquad \sum_{i=1}^{n} |y_i| < 2MB,$$

then

$$\left|\prod_{i=1}^{n} x_i - \prod_{i=1}^{n} y_i\right| \leqslant 2\sum_{i=1}^{n} |x_i - y_i|$$

by Lemma 1.4.

Now suppose s is in $(0, B)$ and c is a positive number. Let d_1 be a positive number less than 1 such that if x is in $[0, d_1)$, then $|f(x)| \leqslant Mx$. Let d_2 be a positive number less than d_1 such that if each of $x_1, x_2, \ldots, x_n$ is in D and $\sum_{i=1}^{n} |x_i| < d_2$, then $\prod_{i=1}^{n} x_i$ is in D and

$$\left|\prod_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i\right| \leqslant \frac{c}{4M}\sum_{i=1}^{n} |x_i|.$$

Let P be a positive integer such that $Ms/2^P < d_2$. Then, if n and m are positive integers such that $P \leqslant n \leqslant m$, it follows that

$$2^{m-n}|f(s/2^m)| \leqslant 2^{m-n}Ms/2^m = Ms/2^n < d_2,$$

which implies

$$\left|2^{m-n}f\left(\frac{s}{2^m}\right) - \left(f\left(\frac{s}{2^m}\right)\right)^{2^{m-n}}\right| \leqslant \frac{c}{4M}2^{m-n}\left|f\left(\frac{s}{2^m}\right)\right| \leqslant \frac{cs}{4\cdot 2^n}.$$

By Lemma 1.6, let N be a positive integer greater than P such that if n and m are positive integers with $N \leqslant n \leqslant m$, then

$$\left|f\left(\frac{s}{2^n}\right)-2^{m-n}f\left(\frac{s}{2^m}\right)\right| = \left|f\left(\sum_{m=1}^{2^{m-n}}\frac{s}{2^m}\right)-\sum_{m=1}^{2^{m-n}}f\left(\frac{s}{2^m}\right)\right| \leqslant \frac{cs}{4\cdot 2^n}.$$

Therefore, if $N \leqslant n \leqslant m$, then by the choice of B, N, and P it follows that

$$\left|\left(f\left(\frac{s}{2^n}\right)\right)^{2^n}-\left(f\left(\frac{s}{2^m}\right)\right)^{2^m}\right| \leqslant 2^{n+1}\left|f\left(\frac{s}{2^n}\right)-\left(f\left(\frac{s}{2^m}\right)\right)^{2^{m-n}}\right|$$

$$\leqslant 2^{n+1}\left(\left|f\left(\frac{s}{2^n}\right)-2^{m-n}f\left(\frac{s}{2^m}\right)\right|+\left|2^{m-n}f\left(\frac{s}{2^m}\right)-\left(f\left(\frac{s}{2^m}\right)\right)^{2^{m-n}}\right|\right)$$

$$\leqslant 2^{n+1}\left(\frac{cs}{4\cdot 2^n}+\frac{cs}{4\cdot 2^n}\right) = cs < c.$$

In order to complete the proof, we next show that there is a positive number r such that $|(f(s/n))^n| > r$ for sufficiently large n. There is a positive number e less than 1 such that if x is in $[0, e)$, then

$$|f'(0)(x)-f(x)| \leqslant \frac{|f'(0)(1)|\,x}{2}.$$

Therefore, if n is sufficiently large, then

$$\left|f\left(\frac{s}{n}\right)\right| \geqslant \left|f'(0)\left(\frac{s}{n}\right)\right| - \frac{|f'(0)(1)|\,s}{2n} = \frac{|f'(0)(1)|\,s}{2n}.$$

It then follows from property (iii) in the choice of B that if n is sufficiently large, then

$$\left|\left(f\left(\frac{s}{n}\right)\right)^n\right| \geqslant \frac{n}{2}\left|f\left(\frac{s}{n}\right)\right| \geqslant \frac{|f'(0)(1)|\,s}{4},$$

which is positive by the hypothesis on f. Setting

$$r = \frac{|f'(0)(1)|\,s}{4}$$

concludes the proof of Lemma 1.7.

Denote by Q the set of dyadic rational numbers in $[0, 1]$. For each positive integer n, let

$$T(1/2^n) = \lim_{m\to\infty} (f(s/2^{n+m}))^{2^m}.$$

For each pair (m, n) of positive integers such that $m \leqslant 2^n$, let

$$T(m/2^n) = (T(1/2^n))^m$$

and let $T(0) = 0$. The existence of T on Q is shown in Lemma 1.8. Since

$$(T(1/2^n))^2 = T(1/2^{n-1})$$

for each positive integer n, it follows that T is well defined on Q. It is also clear from Lemma 1.7 that T is non-trivial. The next lemma shows that T has a unique continuous extension to $[0, 1]$.

LEMMA 1.8. *T is uniformly continuous on Q.*

Proof. Suppose n is a positive integer and m is a non-negative integer less than 2^n. Notice that if k is a positive integer, then

$$|T(1/2^k)| = \lim_{m\to\infty} |(f(s/2^{k+m}))^{2^m}| \leqslant 2Ms/2^k$$

by the choice of B in (i). Therefore,

$$m\,|T(1/2^n)| \leqslant 2Msm/2^n.$$

Thus, by the choice of B in (iv), $T(m/2^n)$ exists. Moreover, the choice of B in (v) yields

$$|T(m/2^n) - T((m+1)/2^n)| = |(T(1/2^n))^m - (T(1/2^n))^{m+1}| \leqslant 2\,|T(1/2^n)| \leqslant 4Ms/2^n.$$

This implies that T is Lipschitz on Q.

LEMMA 1.9. *If p is in Q, then*

$$T(p) = \lim_{n\to\infty} (f(sp/2^n))^{2^n}.$$

Proof. Suppose c is a positive number, k is a positive integer, and y is a non-negative integer less than or equal to 2^k. Choose a positive integer N such that if n is a positive integer greater than N, then

$$\left|\left(f\left(\frac{s}{2^k}\frac{1}{2^n}\right)\right)^y - yf\left(\frac{s}{2^k}\frac{1}{2^n}\right)\right| < \frac{c}{2^{n+2}}$$

by Lemma 1.1 and the continuity of f at 0; and

$$\left|yf\left(\frac{s}{2^k}\frac{1}{2^n}\right) - f\left(\frac{sy}{2^k}\frac{1}{2^n}\right)\right| < \frac{c}{2^{n+2}}$$

by Lemma 1.6. Hence, if n is a positive integer greater than N, it follows from the choice of B in (ii) and the choice of N that

$$\left|\left(\left(f\left(\frac{s}{2^k}\frac{1}{2^n}\right)\right)^y\right)^{2^n} - \left(f\left(\frac{sy}{2^k}\frac{1}{2^n}\right)\right)^{2^n}\right| \leqslant 2^{n+1}\left|\left(f\left(\frac{s}{2^k}\frac{1}{2^n}\right)\right)^y - f\left(\frac{sy}{2^k}\frac{1}{2^n}\right)\right|$$

$$\leqslant 2^{n+1}\left(\left|\left(f\left(\frac{s}{2^k}\frac{1}{2^n}\right)\right)^y - yf\left(\frac{s}{2^k}\frac{1}{2^n}\right)\right| + \left|yf\left(\frac{s}{2^k}\frac{1}{2^n}\right) - f\left(\frac{sy}{2^k}\frac{1}{2^n}\right)\right|\right)$$

$$< 2^{n+1}\left(\frac{c}{2^{n+2}} + \frac{c}{2^{n+2}}\right) < c.$$

Extend T to be continuous on $[0, 1]$. It is clear by construction that $T(0) = 0$ and $T(x+y) = V(T(x), T(y))$ for each x, y, and $x+y$ in $[0, 1]$. It only remains to be seen that T is strongly differentiable at 0 and $T'(0) = sf'(0)$. With this goal in mind we first show that if c is a positive number there is a positive number d less than 1 such that if x is in $(0, d)$, then

$$|T(x)-f'(0)(sx)| \leqslant cx.$$

Suppose c is a positive number. Let d_1 be a positive number less than 1 such that if x is in $[0, d_1)$, then $|f(x)| \leqslant Mx$. Let d_2 be a positive number such that if each of $x_1, x_2, \ldots, x_n$ is in D and $\sum_{i=1}^{n} |x_i| < d$, then $\prod_{i=1}^{n} x_i$ is in D and

$$\left|\prod_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i\right| \leqslant \frac{c}{5Ms} \sum_{i=1}^{n} |x_i|.$$

Let d be a positive number less than d_1 and less than d_2/Ms. Suppose x is in $(0, d)$. Let p be in Q such that

$$0 < p < x, \quad |T(x)-T(p)| < \frac{cx}{5}, \quad \text{and} \quad |p-x| < \frac{cx}{5|f'(0)(s)|}.$$

By Lemma 1.9, let n be a positive integer such that

$$\left|T(p)-(f(sp/2^n))^{2^n}\right| < cx/5, \quad |2^n f(sp/2^n)-f'(0)(sp)| < cx/5.$$

Hence

$$\begin{aligned} |T(x)-f'(0)(sx)| \leqslant\ & |T(x)-T(p)| + \left|T(p)-(f(sp/2^n))^{2^n}\right| \\ & + \left|(f(sp/2^n))^{2^n} - 2^n f(sp/2^n)\right| + |2^n f(sp/2^n) - f'(0)(sp)| \\ & + |f'(0)(sp)-f'(0)(sx)| \leqslant cx, \end{aligned}$$

since each of these summands is less than $cx/5$. This shows that T is differentiable at 0 and $T'(0) = sf'(0)$. Since $V'(0, 0)(x, y) = x+y$, there is a positive number e such that if each of x and y is in. D and within e of $(0, 0)$, then

$$|V(x, y)-x-y| = |V(x, y)-V(0, y)-x| \leqslant cx.$$

Therefore, the fact that T is strongly differentiable at 0 follows from the inequality

$$\begin{aligned} |T(x)-T(y)-f'(0)(s(x-y))| &= |V(T(x-y), T(y))-T(y)-f'(0)(s(x-y))| \\ &\leqslant |V(T(x-y), T(y))-T(x-y)-T(y)| + |T(x-y)-f'(0)(s(x-y))| \\ &\leqslant c(x-y) \end{aligned}$$

for sufficiently small x, y, and $x-y$ in $[0, 1]$. This completes a proof of Theorem 1.

Locally complete monoids. A C^k monoid in which $k \geqslant 1$ and which has a neighborhood U of 1 so that $g_1(U)$ is a closed subset of X is called a *locally complete monoid*. It is clear, since $g_1(U)$ is closed in X and each C^k monoid is strongly differentiable at its identity, that Theorem 1 can be applied to the setting of differentiable semigroups as defined by Graham. Thus we have the following theorem:

THEOREM 2. *Suppose* $k \geqslant 1$, *S is a* C^k *locally complete monoid, and there is a function* $h: [0, 1] \to S$ *such that* $h(0) = 1$, *h is strongly differentiable at* 0, *and* $h'(0) \neq 0$. *Then S has a* C^k *one-parameter submonoid T and* $T'(0) = sh'(0)$ *for some s in* $(0, 1]$.

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