COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

GRAPHS OF DIAMETER 2 WITH CYCLIC DEFECT

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Let G be a finite simple regular graph with n vertices. Let A be the adjacency matrix of G and d the degree of G. The $n \times n$ matrix whose every entry is 1 will be denoted by J. A matrix C is the *defect* of G if C satisfies the equation

$$A^2 + A - (d-1)I = J + C. \qquad (*)$$

If C is the zero matrix then $(*)$ is the equation of the Moore graph (of diameter 2). With the exception of the case $d = 57$, all graphs satisfying this equation were found in [5]. The case of C being the adjacency matrix of a matching with n vertices was solved in [4].

Alan J. Hoffman told me that the Möbious ladder M_4 satisfies $(*)$ with C being the adjacency matrix of the 8-cycle and he asked if there are any other such graphs. We shall show here that M_4 is the only solution.

THEOREM. *If C is the adjacency matrix of the n-cycle then $d = 3$.*

This paper was essentially written during my sabbatical stay at the IBM Research Center in Yorktown Heights. I would like to thank IBM and particularly Alan J. Hoffman for the hospitality. I later discussed the proof with Dr. Itô and that helped me make some simplifications.

Proof. Since J commutes with A and C therefore C commutes with A and hence all three matrices can be diagonalized so that corresponding eigenvalues have a common eigenvector.

Corresponding to the eigenvector $(1, 1, \ldots, 1)$ the matrices A, J and C have respectively eigenvalues d, n, and 2 and thus

$$n = d^2 - 1. \qquad (1)$$

Because G has diameter 2, G must be triangle-free; otherwise G would have at most $1 + (d-2)(d-1) + 2(d-2) = d^2 - d - 1$ vertices. Similarly we can see that every vertex of G must be contained in exactly two 4-cycles of G.

This implies that the number of 4-cycles of $G = 2n/4$, i.e. n is even. Thus (1) implies that

(2) $$d \text{ is odd and } n \text{ is divisible by } 4.$$

Since n is even -2 is a simple eigenvalue of C. Let β be the corresponding eigenvalue of A. Corresponding to -2, C has the eigenvector $(1, -1, 1, -1, \ldots)$ and since -2 is a simple eigenvalue this vector is also an eigenvector of A corresponding to β. Thus we have

(3) $$\beta \text{ is an integer}.$$

Because n is divisible by 4, 0 is an eigenvalue of C and since n is the only nonzero eigenvalue of J the corresponding eigenvalues of A satisfy the equation

$$\lambda^2+\lambda-(d-1)=0,$$

i.e. they are

$$\lambda_1 = \frac{-1+s}{2} \quad \text{and} \quad \lambda_2 = \frac{-1-s}{2} \quad \text{where } s = 4d-3.$$

We shall prove now

LEMMA. *s is a rational.*

Proof. Because vectors $(1, 1, -1, -1, 1, 1, -1, -1, \ldots)$ and $(1, -1, -1, 1, 1, -1, -1, 1, \ldots)$ are linearly independent eigenvectors of C we can assume that λ_1 and λ_2 have respectively the eigenvectors $P = (1, p, -1, p, \ldots)$ and $Q = (1, q, -1, -q, \ldots)$. Since P and Q are orthogonal it follows that

(4) $$pq = -1.$$

We also have integers k, l, x and y such that

(5) $$\frac{-1+s}{2} = k+lp$$

and

(6) $$\frac{-1+s}{2}p = x+yp.$$

Let us suppose now that s is irrational. Then there are unique rationals a and b such that

(7) $$p = a+bs.$$

Because $\phi(a+bs) = a-bs$ is an automorphism of $Q(s)$ such that $\phi(\lambda_1)$

$= \lambda_2$ and because coefficients of A are rational it follows that

(8) $$q = a - bs.$$

Multiplying (6) by q and using (8) we get

$$\frac{s-1}{2} = -y + x(a-bs) = xa - y - xbs.$$

Substituting (7) into (6) we have

$$\frac{-1+s}{2} = k + la + lbs$$

and thus we can conclude that

(9) $$x = l$$

and hence also that

(10) $$y = k + 2la.$$

From (5) and (7) we have that $k + la = -\frac{1}{2}$. Therefore because y and k are integers (10) implies

(11) $$2la \text{ is odd.}$$

Let v be the vertex of G corresponding to the first component of P, and let γ_1, γ_2, γ_3 and γ_4 be numbers of neighbors of v such that corresponding components of P are respectively 1, -1, p and $-p$. Then

$$\frac{-1+s}{2} = (\gamma_1 - \gamma_2) + (\gamma_3 - \gamma_4)p = k + lp.$$

Thus l is even iff γ_3 and γ_4 have the same parity. Because $d = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ and by (3) d is odd we can conclude that k and l have opposite parities. The same argument shows that x and y have opposite parities. Moreover, also (10) and (11) imply that y and k have opposite parities. But this contradicts (9) and thus proves the Lemma.

We can now prove the Theorem. Since s is rational both λ_i's are integers. Hence the equation $x^2 + x - (d-1) = 0$ has an integer solution α. But (3) implies that the equation $x^2 + x - (d-1) = -2$ has an integer solution β. Letting k be $\beta - \alpha$ we get that

$$2\alpha k + k^2 + k = -2.$$

Thus $k = \pm 1$, or $k = \pm 2$. Therefore $\alpha = 1$ or -2 and in both cases $d = 3$.

This proves our Theorem.

It is easy to verify that the Möbious ladder M_4 is the unique solution.

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Reçu par la Rédaction le 02. 05. 1984