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Poincaré's Recurrence Theorem for set-valued dynamical systems

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Abstract. The existence theorem of an invariant measure and Poincaré's Recurrence Theorem are extended to set-valued dynamical systems with closed graph on a compact metric space.

Introduction. Let X be a compact metric space and let $F\colon X \rightsquigarrow X$ be a closed set-valued map.

The purpose of this note is to extend to the set-valued case Poincaré's Recurrence Theorem: *Let X be a compact metric space, let $\mathscr{P}(X)$ denote the set of probability measures on X, and let $F\colon X \rightsquigarrow X$ be a closed*[1] *set-valued map and $\mu \in \mathscr{P}(X)$ an invariant measure for F. For any Borel subset B, let*

$$B_\infty := \bigcap_{N \geqslant 0} \bigcup_{n \geqslant N} F^{-n}(B)$$

be the subset of points x such that for all N, there exists $n \geqslant N$ such that $F^n(x) \cap B \neq \emptyset$. Then the measure of $B \cap B_\infty$ is equal to the measure of B.

The statement of this theorem is clear as soon as we have defined what is an invariant measure for a set-valued map F.

We denote by $\mathscr{A}$ the σ-algebra of Borel subsets of X. We recall that if $F \equiv f$ is single-valued, an invariant probability measure $\mu \in \mathscr{P}(X)$ is defined by:

$$\forall A \in \mathscr{A}, \quad \mu(A) = \mu(f^{-1}(A)),$$

When F is set-valued, we cannot extend this definition as it is because $A \mapsto \mu(F^{-1}(A))$ is no longer a measure. However, we shall introduce the following definition: We say that $\mu \in \mathscr{P}(X)$ is an *invariant measure* for a closed set-valued map $F\colon X \rightsquigarrow X$ if and only if

$$\forall A \in \mathscr{A}, \quad \mu(A) \leqslant \mu(F^{-1}(A)) \tag{1}$$

where $F^{-1}(A) := \{x \in X \mid F(x) \cap A \neq \emptyset\}$. Indeed, we see that for single-valued maps f, this definition coincides with the classical one by applying it to both

[1] This means that its graph is closed.

A and its complement. *Let X be a compact metric space and let $F\colon X \rightsquigarrow X$ be a closed set-valued map with nonempty values. Then there exists an invariant probability measure.*

As in the single-valued case, this theorem follows from the existence of an invariant measure for F, which can be regarded as a fixed point of the set-valued map $\mathscr{F}\colon \mathscr{P}(X) \rightsquigarrow \mathscr{P}(X)$ defined by

$$\nu \in \mathscr{F}(\mu) \Leftrightarrow \forall A \in \mathscr{A}, \quad \nu(A) \leqslant \mu(F^{-1}(A)).$$

This set-valued map can actually be extended to a set-valued analogue of a continuous linear operator from $\mathscr{C}^*(X)$ (the space of Radon measures) to itself, called a *closed convex process*. A closed convex process $\mathscr{F}$ is a set-valued map the graph of which is a closed convex cone, i.e., a closed map satisfying

(i) $$\forall \lambda > 0, \quad \mathscr{F}(\lambda\mu) = \lambda\mathscr{F}(\mu),$$

(ii) $$\forall \mu_1, \mu_2, \quad \mathscr{F}(\mu_1) + \mathscr{F}(\mu_2) \subset \mathscr{F}(\mu_1 + \mu_2).$$

This provides a global way to "linearize" a set-valued map, symmetric in some sense to the local linearization by using "graphical derivatives" at points (x, y) of the graph, which are also closed convex processes (see [1, Chapter 7] and [2] for instance).

1. Linear extension of a set-valued map. Let X and Y be two compact metric spaces and $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ their Borel σ-algebras. We recall that the dual $\mathscr{C}^*(X)$ of the space of continuous functions is isomorphic to the space of Radon measures on X and that a continuous single-valued $f\colon X \mapsto Y$ can be extended to a continuous linear operator $\mathscr{F}$ from $\mathscr{C}^*(X)$ to $\mathscr{C}^*(Y)$ by the formula

$$\forall \mu \in \mathscr{C}^*(X), \ \forall B \in \mathscr{B}(Y), \quad \mathscr{F}(\mu)(B) := \mu(f^{-1}(B)).$$

This fact can be extended to set-valued maps $F\colon X \rightsquigarrow Y$. We denote by $\mathscr{P}(X) \subset \mathscr{C}^*(X)$ the (weakly compact convex) set of probability measures on X.

DEFINITION 1.1. Let $F\colon X \rightsquigarrow Y$ be a set-valued map. Denote by $\mathscr{F}$ the *linear extension* of F, the set-valued map from $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ defined in the following way: $\nu \in \mathscr{P}(Y)$ belongs to $\mathscr{F}(\mu)$ if and only if

$$\forall B \in \mathscr{B}(Y), \quad \nu(B) \leqslant \mu(F^{-1}(B)).$$

We extend it to a set-valued map from $\mathscr{C}^*(X)$ to $\mathscr{C}^*(Y)$ by setting

$$\mathscr{F}(\mu) := \begin{cases} \emptyset & \text{if } \mu \text{ is nonpositive,} \\ \{0\} & \text{if } \mu = 0, \\ \mu(X)\mathscr{F}(\mu/\mu(X)) & \text{if } \mu \text{ is positive.} \end{cases}$$

PROPOSITION 1.2. *Consider compact metric spaces X, Y and a closed set-valued map $F\colon X \rightsquigarrow Y$ with nonempty values. Then $\mathscr{F}$ is a closed convex process with nonempty values.*

Furthermore, for any $\mu \in \mathscr{P}(X)$, ν belongs to $\mathscr{F}(\mu)$ if and only if

$$\text{for every open subset } \mathscr{O} \subset Y, \quad \nu(\mathscr{O}) \leqslant \mu(F^{-1}(\mathscr{O})).$$

The closed convex process $\mathscr{F}$ is a (set-valued linear) extension of F in the sense that for Dirac measures, $\delta_y \in \mathscr{F}(\delta_x)$ if and only if $y \in F(x)$.

If $G\colon Y \rightsquigarrow Z$ is a closed set-valued map with nonempty values from Y to a compact metric space Z, then the extension $\mathscr{H}$ of the product $H := G \circ F$ contains the product of the extensions: $\mathscr{G} \circ \mathscr{F} \subset \mathscr{H}$

Proof. Consider a measure $\mu \in \mathscr{P}(X)$. The image $\mathscr{F}(\mu)$ is not empty. Indeed, by the Measurable Selection Theorem, F, being upper semicontinuous with closed images, is measurable, so that there exists at least one measurable selection f of F. Define ν_f by the formula $\nu_f(A) := \mu(f^{-1}(A))$, which is a probability measure. Since $f^{-1}(A) \subset F^{-1}(A)$, we infer that $\mu(f^{-1}(A)) \leqslant \mu(F^{-1}(A))$ so that ν_f belongs to $\mathscr{F}(\mu)$.

We now prove that $\mathscr{F}(\mu)$ can be defined as the set of measures $\nu \in \mathscr{P}(X)$ satisfying

$$\text{for every open subset } \mathscr{O} \subset Y, \quad \nu(\mathscr{O}) \leqslant \mu(F^{-1}(\mathscr{O})).$$

We first extend this formula to compact subsets $K \subset Y$. Since the graph of F, and thus of F^{-1}, is closed and X is compact, F^{-1} is also upper semicontinuous. We then know that for any neighborhood $\mathscr{O}_n \supset F^{-1}(K)$, there exists an open neighborhood $\mathscr{M}_n \supset K$ satisfying $F^{-1}(\mathscr{M}_n) \subset \mathscr{O}_n$. Choose open subsets $\mathscr{O}_n$ such that $\mu(\mathscr{O}_n) \searrow \mu(F^{-1}(K))$. Hence the inequalities

$$\nu(K) \leqslant \nu(\mathscr{M}_n) \leqslant \mu(F^{-1}(\mathscr{M}_n)) \leqslant \mu(\mathscr{O}_n)$$

imply by going to the limit that $\nu(K) \leqslant \mu(F^{-1}(K))$.

Take now any measurable subset $B \in \mathscr{B}(Y)$. There exists a sequence of compact subsets $K_n \subset B$ such that $\nu(K_n) \nearrow \nu(B)$. Then the inequalities

$$\nu(K_n) \leqslant \mu(F^{-1}(K_n)) \leqslant \mu(F^{-1}(B))$$

imply that $\nu(B) \leqslant \mu(F^{-1}(B))$.

Assume that $y \in F(x)$. Then $\delta_y \in \mathscr{F}(\delta_x)$ since, for any open subset $\mathscr{O} \subset Y$, $\delta_y(\mathscr{O}) \leqslant \delta_x(F^{-1}(\mathscr{O}))$. This is obvious when $y \notin \mathscr{O}$. If not, the left-hand side is equal to 1, and so is the right-hand side, because $x \in F^{-1}(y) \subset F^{-1}(\mathscr{O})$. Conversely, if $y \notin F(x)$, there exists an open subset $\mathscr{O} \ni y$ such that $F(x) \cap \mathscr{O} = \varnothing$, i.e., $x \notin F^{-1}(\mathscr{O})$. Then $\delta_y(\mathscr{O}) = 1$ and $\delta_x((F^{-1}(\mathscr{O}))) = 0$, so that $\delta_y \notin \mathscr{F}(\delta_x)$.

The formula $\mathscr{G} \circ \mathscr{F} \subset \mathscr{H}$ is obvious as well as the convexity of the graph of $\mathscr{F}$.

It remains to prove that Graph$(\mathscr{F})$ is closed when the spaces of Radon measures are supplied with the weak-$*$ topology.

For that purpose, consider a sequence of measures $(\mu_n, \nu_n) \in$ Graph$(\mathscr{F})$ converging to (μ, ν) in the weak-$*$ topologies of the duals $\mathscr{C}^*(X)$ and $\mathscr{C}^*(Y)$ respectively.

It is sufficient to prove that the graph of the restriction of $\mathscr{F}$ to $\mathscr{P}(X)$ is weakly closed. Indeed, when the measures μ_n and $\nu_n \in \mathscr{F}(\mu_n)$ are positive and converge weakly to μ and ν, some subsequences of the probability measures $\bar{\mu}_n := \mu_n/\mu_n(X)$ and $\bar{\nu}_n := \nu_n/\mu_n(X) \in \mathscr{F}(\bar{\mu}_n)$ converge weakly to probability measures $\bar{\mu}$ and $\bar{\nu} \in \mathscr{F}(\bar{\mu})$ respectively, because the sets of probability measures are weakly compact and the graph of the restriction of $\mathscr{F}$ to $\mathscr{P}(X)$ is assumed to be weakly closed. Since the measures $\mu_n(X)$ converge to $\mu(X)$, we deduce that $\mu = \mu(X)\bar{\mu}$ and $\nu = \mu(X)\bar{\nu}$. Then $\nu = 0$ when $\mu = 0$ and otherwise,

$$\nu = \mu(X)\bar{\nu} \in \mu(X)\mathscr{F}(\mu/\mu(X)).$$

Hence we consider a sequence $(\mu_n, \nu_n) \in \mathscr{P}(X) \times \mathscr{P}(Y)$ in the graph of $\mathscr{F}$ converging to (μ, ν). In order to prove that $\nu \in \mathscr{F}(\mu)$, it is enough to check that for any open subset $\mathcal{O} \subset X$, the inequality

$$\nu(\mathcal{O}) \leqslant \mu(F^{-1}(\mathcal{O}))$$

holds true thanks to the first part of the proposition.

Fix an open subset $\mathcal{O} \subset X$. Since X is metric, $\mathcal{O}$ is the union of an increasing sequence of open subsets $\mathcal{O}_p \subset \mathcal{O}$, $p \geqslant 1$, such that for every $p \geqslant 1$, $\overline{\mathcal{O}_p} \subset \mathcal{O}$. Fix p and observe that $F^{-1}(\overline{\mathcal{O}_p})$ is compact, as the image of a compact set by the upper semicontinuous set-valued map F^{-1}.

The inequalities

$$\forall n \geqslant 1, \quad \nu_n(\mathcal{O}_p) \leqslant \mu_n(F^{-1}(\mathcal{O}_p)) \leqslant \mu_n(F^{-1}(\overline{\mathcal{O}_p}))$$

imply, thanks to a version of Alexandrov's Theorem recalled just after the end of the proof, that

$$\nu(\mathcal{O}_p) \leqslant \liminf_{n\to\infty} \nu_n(\mathcal{O}_p) \leqslant \limsup_{n\to\infty} \mu_n(F^{-1}(\overline{\mathcal{O}_p})) \leqslant \mu(F^{-1}(\overline{\mathcal{O}_p})) \leqslant \mu(F^{-1}(\mathcal{O})).$$

It remains to observe that $\nu(\mathcal{O}) = \sup_{p\geqslant 1} \nu(\mathcal{O}_p)$. ∎

We now prove the version of Alexandrov's Theorem (see [4, Theorem 4.9.15]) we needed:

THEOREM 1.3 (Alexandrov). *Let X be a compact metric space. Consider a sequence of Radon probability measures $\mu_n \in \mathscr{P}(X)$ converging weakly to μ. Then, for any open subset $\mathcal{O} \subset X$,*

$$\mu(\mathcal{O}) \leqslant \liminf_{n\to\infty} \mu_n(\mathcal{O}),$$

and for any closed subset $K \subset X$,

$$\limsup_{n\to\infty} \mu_n(K) \leqslant \mu(K).$$

Proof. Let $\mathcal{O} \subset X$ be an open subset. Since a Radon measure μ is regular, we can associate with any $\varepsilon > 0$ a compact subset $K \subset \mathcal{O}$ such that $\mu(\mathcal{O}\setminus K) \leqslant \varepsilon$.

Let $f \in \mathscr{C}(X)$ be a continuous function equal to 1 on the closed subset K and to 0 on the complement of $\mathcal{O}$. We observe that $\chi_K \leqslant f \leqslant \chi_{\mathcal{O}}$ and thus

$$\mu(\mathcal{O}) \leqslant \mu(K)+\varepsilon \leqslant \int f\,d\mu+\varepsilon.$$

Therefore, the inequalities $\int f\,d\mu_n \leqslant \mu_n(\mathcal{O})$ imply that

$$\mu(\mathcal{O}) \leqslant \int f\,d\mu+\varepsilon \leqslant \liminf_{n\to\infty} \mu_n(\mathcal{O})+\varepsilon.$$

We conclude by letting ε converge to 0.

The proof of the second inequality for compact subsets K is analogous, since for any $\varepsilon > 0$, there exists an open $\mathcal{O} \supset K$ such that $\mu(\mathcal{O}\backslash K) \leqslant \varepsilon$. Define the continuous function f as above. The inequalities $\mu_n(K) \leqslant \int f\,d\mu_n$ imply that

$$\limsup_{n\to\infty} \mu_n(K) \leqslant \int f\,d\mu \leqslant \mu(\mathcal{O}) \leqslant \mu(K)+\varepsilon. \ \blacksquare$$

Remark. We deduce that for any open subset $\mathcal{O}$ such that $\mu(\mathcal{O}) = \mu(\overline{\mathcal{O}})$, weak convergence of probability measures μ_n to μ implies that $\mu_n(\mathcal{O})$ converges to $\mu(\mathcal{O})$. The converse is also true. (See the proof of [4, Theorem 4.9.15] for instance.)

2. Invariant measures. Taking $X = Y$, we are able to derive the existence of invariant measures by showing that the extension $\mathscr{F}$ has fixed points on $\mathscr{P}(X)$.

THEOREM 2.1. *Let X be a compact metric space and let $F\colon X \rightsquigarrow X$ be a closed set-valued map with nonempty values. Then there exists an invariant probability measure μ, i.e., one satisfying*

$$(2) \qquad \forall A \in \mathscr{B}(X), \quad \mu(A) \leqslant \mu\big(F^{-1}(A)\big).$$

Proof. Let $\mathscr{F}$ be the set-valued map from $\mathscr{P}(X)$ to itself associating with any $\mu \in \mathscr{P}(X)$ the (nonempty) set of probability measures $\nu \in \mathscr{F}(\mu)$. Then the above proposition states that $\mathscr{F}$ is a map with closed graph and convex values from the convex compact subset $\mathscr{P}(X) \subset \mathscr{C}^*(X)$ to itself, and thus upper semicontinuous with convex compact values. Kakutani–Fan's Fixed Point Theorem [6](2) implies the existence of a fixed point $\mu \in \mathscr{P}(X)$ of $\mathscr{F}$, which is a measure satisfying

$$(3) \qquad \forall A \in \mathscr{B}(X), \quad \mu(A) \leqslant \mu\big(F^{-1}(A)\big).$$

Therefore, μ is invariant under F. $\blacksquare$

Naturally, we can derive most of the properties of invariant measures enjoyed by single-valued maps. Let us show for instance that Poincaré's Recurrence Theorem holds true for closed set-valued dynamical systems.

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(2) Recall that the proof we provided, based on the Ky Fan inequality, showed that Ky Fan's theorem remains true in locally convex Hausdorff topological vector spaces.

THEOREM 2.2 (Poincaré's Recurrence). *Let X be a compact metric space, $F\colon X \rightsquigarrow X$ a closed set-valued map and $\mu \in \mathcal{P}(X)$ an invariant measure of F. For any Borel subset $B \subset X$, let*

$$B_\infty := \bigcap_{N\geq 0} \bigcup_{n\geq N} F^{-n}(B)$$

be the subset of points x such that for all N, there exists $n \geq N$ such that $F^n(x) \cap B \neq \emptyset$. Then the measure of $B \cap B_\infty$ is equal to the measure of B.

Proof. The proof is a straightforward extension of the proof in the single-valued case: We introduce the subsets

$$B_N := \bigcup_{n\geq N} F^{-n}(B)$$

and we observe that $B \subset B_0$, that $B_N \subset B_{N-1} \subset \ldots \subset B_0$ and $B_N = F^{-N}(B_0)$. Since μ is invariant, we deduce that

$$\mu(B) \leq \mu(F^{-1}(B)) \leq \ldots \leq \mu(F^{-N}(B)) \leq \ldots$$

and thus,

$$\mu(B_0) \leq \mu(F^{-N}(B_0)) = \mu(B_N) \leq \mu(B_0).$$

Since the sequence B_N is not increasing and since $\mu(B_N) = \mu(B_0)$, we infer that $\mu(B_\infty) = \mu(B_0)$. Therefore,

$$\mu(B \cap B_\infty) = \mu(B \cap B_0) = \mu(B). \quad \blacksquare$$

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