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On some applications of the Clunie method

by Zbigniew Jerzy Jakubowski (Łódź)

1. The main purpose of the paper is to obtain on the basis of the Clunie method [2] the estimations of the coefficients of functions of some class of Carathéodory functions. The results obtained will also be applied to some families of functions generated by Carathéodory functions.

Let $\mathfrak{P}$ [1] denote the family of regular functions defined in the circle $K = \{z : |z| < 1\}$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \tag{1.1}$$

whose real part is positive in this circle. Denote by $\mathfrak{P}_M$, $M > \frac{1}{2}$, the class of functions of form (1.1) which satisfy in the region K the condition $|p(z) - M| < M$. Obviously $\mathfrak{P}_M \subset \mathfrak{P}$ and $\mathfrak{P}_\infty \equiv \mathfrak{P}$.

The class $\mathfrak{P}_M$ has been introduced by Goel [3]. He has also proved that for every function the following sharp estimation holds:

$$|p_n| \leqslant 2 - M^{-1}, \quad n = 1, 2, \ldots \tag{1.2}$$

Equality holds if and only if $p(z)$ is of the form

$$p(z) = (1 + \varepsilon z^n)[1 - \varepsilon(1 - M^{-1})z^n]^{-1}, \quad |\varepsilon| = 1.$$

In the case of the family $\mathfrak{P}$ we hence obtain the well-known result of Carathéodory [1]: $|p_n| \leqslant 2$, $n = 1, 2, \ldots$

Let m, M be two fixed numbers satisfying the condition $(m, M) \in D$, $D = D_1 \cup D_2$, where

$$D_1 = \{(m, M) : \tfrac{1}{2} < m < 1,\ 1 - m < M \leqslant m\},$$
$$D_2 = \{(m, M) : 1 \leqslant m,\ m - 1 < M \leqslant m\}.$$

Denote by $\mathfrak{P}_{m,M}$ the family of all functions of form (1.1) satisfying the condition

$$|p(z) - m| < M, \quad z \in K. \tag{1.3}$$

Obviously $\mathfrak{P}_{m,M} \subset \mathfrak{P}$ and $\mathfrak{P}_{M,M} \equiv \mathfrak{P}_M$. Assume

$$a = (M^2 - m^2 + m)M^{-1}, \quad b = (m - 1)M^{-1}. \tag{1.4}$$

We shall prove the following

THEOREM 1. *If $p(z)$ is an arbitrary function of the family* $\mathfrak{P}_{m,M}$, $(m, M)\in D$, *then*

$$|p_n| \leqslant a+b = [M^2-(m-1)^2]M^{-1}, \quad n = 1, 2, \ldots \tag{1.5}$$

and equality holds in (1.5) *if and only if*

$$p(z) = (1+a\varepsilon z^n)(1-b\varepsilon z^n)^{-1}, \quad |\varepsilon| = 1. \tag{1.6}$$

Moreover, we have

$$\sum_{k=1}^{\infty} |p_k|^2 \leqslant (a+b)^2(1-b^2)^{-1} = M^2-(m-1)^2. \tag{1.7}$$

Proof. Denote by Q the family of all regular functions of the form

$$q(z) = \sum_{k=1}^{\infty} q_k z^k \tag{1.8}$$

defined in the circle K which satisfy in this circle the condition

$$|q(z)| < 1. \tag{1.9}$$

Then $|q_k| \leqslant 1$, $k = 1, 2, \ldots$; $q(z) = \varepsilon z^k$, $|\varepsilon| = 1$ if $|q_n| = 1$. By condition (1.3) it can easily be proved that for every function $p(z)$ of the family $\mathfrak{P}_{m,M}$, $(m, M)\in D$ there exists a function $q(z)$ of the class Q such that

$$p(z) = [1+aq(z)][1-bq(z)]^{-1} \tag{1.10}$$

and conversely for every function $q\in Q$ the function (1.10) belongs to $\mathfrak{P}_{m,M}$. From formulas (1.10), (1.8), (1.1) we obtain the equality

$$\sum_{k=1}^{\infty} p_k z^k = \Big[a+b+b\sum_{k=1}^{\infty} p_k z^k\Big]\Big(\sum_{k=1}^{\infty} q_k z^k\Big), \quad z\in K. \tag{1.11}$$

Thus equating the coefficients at the corresponding powers of z, we have the relationships

$$p_1 = (a+b)q_1, \tag{1.12}$$

$$p_n = (a+b)q_n + b\sum_{k=1}^{n-1} p_k q_{n-k}, \quad n = 2, 3, \ldots \tag{1.13}$$

Since $|q_1| \leqslant 1$, by (1.12) inequality (1.5) is true for $n = 1$. Equality holds if and only if $q(z) = \varepsilon z$, i.e. if $p(z)$ is of form (1.6) for $n = 1$. Let $n \geqslant 2$: then we obtain from (1.11)

$$\sum_{k=1}^{n} p_k z^k + \sum_{k=n+1}^{\infty} s_k z^k = \Big[a+b+b\sum_{k=1}^{n-1} p_k z^k\Big]\Big(\sum_{k=1}^{\infty} q_k z^k\Big);$$

thus by (1.8), (1.9) we obtain in the circle K the inequality

$$\Big|\sum_{k=1}^{n} p_k z^k + \sum_{k=n+1}^{\infty} s_k z^k\Big|^2 < \Big|a+b+b\sum_{k=1}^{n-1} p_k z^k\Big|^2. \tag{1.14}$$

Assuming in (1.14) $z = re^{it}$, $0 < r < 1$, $0 \leqslant t \leqslant 2\pi$ and then integrating side-wise in the interval $\langle 0, 2\pi\rangle$, we get

$$\sum_{k=1}^{n} |p_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |s_k|^2 r^{2k} < (a+b)^2 + b^2 \sum_{k=1}^{n-1} |p_k|^2 r^{2k},$$

whence

$$\sum_{k=1}^{n} |p_k|^2 r^{2k} < (a+b)^2 + b^2 \sum_{k=1}^{n-1} |p_k|^2 r^{2k}. \tag{1.15}$$

Passing in (1.15) to the limit as $r \to 1$, we find the inequality

$$\sum_{k=1}^{n} |p_k|^2 \leqslant (a+b)^2 + b^2 \sum_{k=1}^{n-1} |p_k|^2,$$

equivalent to the following

$$|p_n|^2 \leqslant (a+b)^2 + (b^2-1) \sum_{k=1}^{n-1} |p_k|^2. \tag{1.16}$$

Since $(m, M) \in D$, $a+b > 0$ and $b^2 < 1$ (comp. (1.4)) and thus estimation (1.5) is true also for $n \geqslant 2$.

On the other hand, equality holds only if $p_1 = \ldots = p_{n-1} = 0$ (see (1.16)), which, because of (1.13), proves that $|q_n| = 1$, $q(z) = \varepsilon z^n$. Thus we obtain from (1.10) a function of form (1.6). It can easily be proved that this function belongs to the class $\mathfrak{P}_{m,M}$ for a and b defined by formulas (1.4). Inequality (1.7) is obtained by applying condition (1.9) immediately to identity (1.11). Then assuming $z = re^{it}$, $0 < r < 1$, $0 \leqslant t \leqslant 2\pi$ and integrating respectively, we get the inequality

$$\sum_{k=1}^{\infty} |p_k|^2 r^{2k} < (a+b)^2 + b^2 \sum_{k=1}^{\infty} |p_k|^2 r^{2k};$$

hence, because of the arbitrariness of r, we have

$$(1-b^2) \sum_{k=1}^{\infty} |p_k|^2 \leqslant (a+b)^2,$$

which ends the proof.

If $m = M$, we find from (1.4) $a = 1$, $b = 1 - M^{-1}$, and thus from estimation (1.5) we immediately get (1.2). On the other hand, inequality (1.7) implies

Corollary 1. *If* $p \in \mathfrak{P}_M$, $M > \frac{1}{2}$, *then* $\sum_{k=1}^{\infty} |p_k|^2 \leqslant 2M-1$.

2. Let $R_{m,M}$, $(m, M) \in D$ denote the family of regular functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{2.1}$$

defined in the circle K which satisfy in this circle the condition

$$|f'(z) - m| < M.$$

The family $R_{m,M}$ is a subclass of the family R of all functions of form (2.1) whose first derivative have positive real part in the circle K. It is also known that every function of the family R is univalent [5]. The class $R_M \equiv R_{M,M}$ [4] is also known. It is obvious that if $f \in R_{m,M}$, then the function $p(z) = f'(z)$ belongs to the class $\mathfrak{P}_{m,M}$ and conversely.

Thus $ka_k = p_{k-1}$, $k = 2, 3, \ldots$ Consequently we get from Theorem 1

Corollary 2. *If* $f(z)$ *is an arbitrary function of the family* $R_{m,M}$, *then*

$$|a_n| \leqslant (a+b)n^{-1}, \quad n = 2, 3, \ldots, \tag{2.2}$$

and

$$\sum_{k=1}^{\infty} [k+1]^2 |a_{k+1}|^2 \leqslant (a+b)^2 (1-b^2)^{-1}$$

with a *and* b *defined by formulas* (1.4). *Equality holds in estimation* (2.2) *if and only if*

$$f(z) = z + an^{-1}\varepsilon z^n, \quad |\varepsilon| = 1, \ b = 0, \tag{2.3}$$

$$f(z) = z + \sum_{k=1}^{\infty} \frac{(a+b)b^{k-1}\varepsilon^k}{k(n-1)+1} z^{k(n-1)+1}, \quad |\varepsilon| = 1, \ b \neq 0. \tag{2.4}$$

Corollary 3. *If* $f \in R_M$, *then the sharp estimations* $|a_n| \leqslant (2 - M^{-1})n^{-1}$, $n = 2, 3, \ldots$, *and* $\sum_{k=1}^{\infty} (k+1)^2 |a_{k+1}|^2 \leqslant 2M-1$.

Corollary 4. *If* $f \in R$, *then the sharp estimation* $|a_n| \leqslant 2n^{-1}$, $n = 2, 3, \ldots$, *holds.*

The extremal functions are here of forms (2.3) and (2.4) for $a = 1$, $b = 1 - M^{-1}$ and $a = b = 1$ respectively.

3. In 1932 Rogosinski [8] introduced the family T of functions of form (2.1), called *typically-real functions* ($f(z)$ is real in K if and only if z is real). It is known that $a_k = \operatorname{re} a_k$, $k = 2, 3, \ldots$, and $f \in T$ if and only if the function $p(z) = (1-z^2)z^{-1} f(z)$ is a function of the family $\mathfrak{P}$ with real coefficients [8].

Let $T_{m,M}$, $(m, M)\in D$, be a subclass of the family T of functions satisfying the condition

$$|(1-z^2)z^{-1}f(z)-m| < M \quad \text{for } z\in K.$$

Clearly $T_{M,M}\equiv T_M$ [4] and if $f\in T_{m,M}$, then $p(z) = (1-z^2)z^{-1}f(z)$ is a function of the class $\mathfrak{P}_{m,M}$. Thus by Theorem 1 we have

COROLLARY 5. *If $f\in T_{m,M}$, then*

$$-(a+b)\leqslant a_2\leqslant a+b,$$

$$1-(a+b)\leqslant a_3\leqslant a+b+1,$$

$$|a_{n+1}-a_{n-1}|\leqslant a+b, \quad n = 3, 4, \ldots,$$

the equality sign being realized by the function

$$f(z) = z(1-z^2)^{-1}(1+a\varepsilon z^n)(1-b\varepsilon z^n)^{-1}, \quad \varepsilon = \pm 1.$$

Moreover,

$$\sum_{k=1}^{\infty}|a_{k+1}-a_{k-1}|^2\leqslant (a+b)^2(1-b^2)^{-1}.$$

COROLLARY 6. *If $f\in T_M$, then the sharp estimations*

$$-2+M^{-1}\leqslant a_2\leqslant 2-M^{-1},$$

$$-1+M\leqslant a_3\leqslant 3-M^{-1},$$

$$|a_{n+1}-a_{n-1}|\leqslant 2-M^{-1}, \quad n = 3, 4 \ \ldots,$$

hold, and

$$\sum_{k=1}^{\infty}|a_{k+1}-a_{k-1}|^2\leqslant 2M-1.$$

COROLLARY 7 [8]. *If $f\in T$, then the sharp estimations*

$$-2k\leqslant a_{2k}\leqslant 2k, \quad 1-2k\leqslant a_{2k+1}\leqslant 2k+1, \quad k = 1\ 2, \ldots,$$

hold.

4. Let C [7] denote the family of functions of form (2.1) which are real on the real axis and which map the circle K onto a convex region in the direction of the imaginary axis. It is known that $f\in C$ if and only if $zf'(z)$ is a typically real function [7]; thus the function $p(z) = (1-z^2)f'(z)$ is a function of the family $\mathfrak{P}$ with real coefficients.

Let $C_{m,M}$, $(m, M)\in D$ the family of functions $f\in C$ which satisfy the condition

$$|(1-z^2)f'(z)-m| < M \quad \text{for } z\in K.$$

Obviously $C_{M,M}\equiv C_M$ [4] and if $f\in C_{m,M}$, then the function $p(z) = (1-z^2)f'(z)$ belongs to the class $\mathfrak{P}_{m,M}$.

COROLLARY 8. *If $f \in C_{m,M}$, then the sharp estimations*

$$-\tfrac{1}{2}(a+b) \leqslant a_2 \leqslant \tfrac{1}{2}(a+b),$$

$$-\tfrac{1}{3}(a+b-1) \leqslant a_3 \leqslant \tfrac{1}{3}(a+b+1),$$

$$|a_{n+1}(n+1)-(n-1)a_{n-1}| \leqslant a+b, \qquad n = 3, 4, \ldots,$$

hold. Moreover,

$$\sum_{k=1}^{\infty} |(k+1)a_{k+1}-(k-1)a_{k-1}|^2 \leqslant (a+b)^2(1-b^2)^{-1}.$$

COROLLARY 9. *If $f \in C$, then the sharp estimations*

$$-1 \leqslant a_{2k} \leqslant 1, \qquad -1+\frac{2}{2k+1} \leqslant a_{2k+1} \leqslant 1, \qquad k = 1, 2, \ldots$$

hold [7].

5. In 1936 Robertson [6] proved that if $f \in C$, then $\operatorname{re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2}$. Then Goel [3] introduced the class $\check{C}_M$ of all functions of form (2.1) which satisfy the condition $|[2f(z)z^{-1}-1]-M| < M$, $M \geqslant 1$, and obtained the sharp estimation $|a_n| \leqslant 1-(2M)^{-1}$, $n = 2, 3, \ldots$ Let $\check{C}_{m,M}$, $(m, M) \in D$ denote the family of all functions of form (2.1) which satisfy in the circle K the condition

$$|[2f(z)z^{-1}-1]-m| < M.$$

If $f \in \check{C}_{m,M}$, then the function $2f(z)z^{-1}-1$ belongs to the family $\mathfrak{P}_{m,M}$; thus by Theorem 1 we obtain the following generalization of the result of Goel:

COROLLARY 10. *If $f \in \check{C}_{m,M}$, then $|a_n| \leqslant \frac{1}{2}(a+b)$, $n = 2, 3, \ldots$, equality being realized by functions of the form*

$$f(z) = \tfrac{1}{2}z[2+(a-b)\varepsilon z^{n-1}][1-b\varepsilon z^{n-1}]^{-1}, \qquad |\varepsilon| = 1.$$

Moreover,

$$4\sum_{k=2}^{\infty} |a_k|^2 \leqslant (a+b)^2(1-b^2)^{-1}.$$

COROLLARY 11. *If $f \in \check{C}_{\infty}$, then $|a_n| \leqslant 1$, $n = 2, 3, \ldots$, where equality holds for functions of the form*

$$f(z) = z(1-\varepsilon z^{n-1})^{-1}, \qquad |\varepsilon| = 1.$$

6. Another class of functions which is closely connected with Carathéodory functions with positive real parts is the family U [7] of starlike functions in the direction of the real axis. It is known that $f \in U$ if and only if the function

(6.1)

$$p(z) = \frac{g_\beta(ze^{-ia})f(z) - e^{-ia}z\cos a}{ize^{-ia}\sin a}, \quad g_\beta(z) = 1 - 2\cos\beta\cdot z + z^2, \quad \sin a > 0,$$

belongs to $\mathfrak{P}$ [7]. Janowski [4] considered the family $U_M \equiv U_{M,M}$, where $U_{m,M}$, $(m, M) \in D$ is a family of functions of form (2.1) which satisfy the condition

$$\left| \frac{g_\beta(ze^{-ia})f(z) - e^{-ia}z\cos a}{ize^{-ia}\sin a} - m \right| < M$$

in the circle K.

If $f \in U_{m,M}$, then function (6.1) belongs to the class $\mathfrak{P}_{m,M}$. Thus we have

COROLLARY 12. *If* $f \in U_{m,M}$, *then*

$$|a_n - 2\cos\beta e^{-ia}a_{n-1} + e^{-2ia}a_{n-2}| \leqslant (a+b)\sin a, \quad n = 2, 3, \ldots,$$

the extremal function being of the form

$$f(z) = [g_\beta(ze^{-ia})]^{-1}\left[e^{-ia}z\cos a + ize^{-ia}\sin a \frac{1+a\varepsilon z^{n-1}}{1-b\varepsilon z^{n-1}}\right], \quad |\varepsilon| = 1.$$

COROLLARY 13. *If* $f \in U$, *then*

$$|a_n - 2\cos\beta e^{-ia}a_{n-1} + e^{-2ia}a_{n-2}| \leqslant 2\sin a, \quad n = 2, 3, \ldots$$

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MATHEMATICAL INSTITUTE, UNIVERSITY OF ŁÓDŹ

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