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Generic finiteness in solving optimal control problems

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Abstract. One proves that the finiteness for the set of optimal controls is generic with respect to parameters entering smoothly the different optimal control problems. The genericity is obtained from an abstract result concerning minimization problems which depend smoothly on parameters. Our approach is based on the transversality theory.

1. Introduction. In [7], [8], J. C. Saut and R. Temam showed that the property of a nonlinear elliptic boundary value problem to admit finitely many solutions is generic with respect to one of the data. This is deduced from an abstract result ensuring that, generically relative to a parameter α, a nonlinear equation

$$A(u, \alpha) = 0, \tag{1.1}$$

with the unknown u, has a finite set of solutions. The key hypothesis is the transversality of A to the one-point submanifold $\{0\}$.

The aim of the present paper is to apply the method of J. C. Saut and R. Temam in optimal control problems.

We start with the abstract minimization problem

$$\inf_{u \in M} J(u, \alpha), \tag{1.2}$$

where M is a compact smooth submanifold of a Banach space B, α is a parameter belonging to a smooth Banach manifold E and $J\colon B \times E \to R$ is a smooth function. Notice that if $u \in M$ solves (1.2), then

$$J'_u(u, \alpha)|_{T_u M} = O_u \in T_u^* M, \tag{1.3}$$

i.e., the partial derivative $J'_u(u, \alpha)$ vanishes on the vectors of B which are tangent to M at u. Thus problem (1.2) yields equation (1.3) which is only in a formal manner of type (1.1). Nevertheless, this fact suggests to proceed along the lines of [7], [8] for obtaining the generic finiteness with respect to $\alpha \in E$ for the number of solutions in (1.2). In contrast with [7], [8], we impose the transversality of $(J|_{M \times E})'_u\colon M \times E \to T^* M$ to the zero-section $\{O_u \in T_u^* M;\ u \in M\}$ of $T^* M$.

The abstract result concerning problem (1.2) is applied in the following control problems: (a) minimization of a cost functional over initial states x (i.e., control via initial conditions) taking as parameter a control term $u = u(t)$ entering the state equation $y' = f(t, y, u)$, $y(0) = x$; (b) minimization of a cost functional over controls $u = u(t)$ (i.e., optimal control problem) choosing as parameter the control system f defining the state equation $y' = f(t, y, u)$. For the control problems (a) and (b) we obtain computable criteria implying the generic finiteness of the set of optimal controls.

2. A generic result for minimization. Throughout this paper, if M is a differentiable manifold, TM and $T^* M$ denote the tangent bundle and the cotangent bundle of M, respectively. For a differentiable mapping F, the notations F' and F'' mean the first derivative and the second one of F, respectively, the eventual subscripts indicating the variables with respect to which one differentiates. The necessary prerequisites of transversality theory can be found in [1] and [2].

THEOREM 2.1. *Let M be a compact C^{r+1} manifold, E a C^{r+1} Banach manifold and J: $M \times E \to R$ a function satisfying the following conditions*:

(i) *E is a second countable, complete metric space and J is differentiable of class C^{r+1} with $r \geq 1$;*

(ii) *for every (u, α) in $M \times E$ with $J'_u(u, \alpha) = O_u \in T^*_u M$ and every ξ in the vertical space $T_{O_u}(T^*_u M) \cong T^*_u M$, there exists (v, a) in $T_u M \times T_\alpha E$ such that*

$$J''_{uu}(u, \alpha)(\cdot, v) + J''_{u\alpha}(u, \alpha)(\cdot, a) = \xi.$$

Then there exists a dense open subset G of E such that for every α in G the minimization problem

$$(\mathrm{P}_\alpha) \qquad \inf_{u \in M} J(u, \alpha)$$

has a finite set of solutions. If $u \in M$ is a solution of (P_α) with $\alpha \in G$, then $J''_{uu}(u, \alpha)$ is an isomorphism of $T_u M$ onto $T_{O_u}(T^*_u M) \cong T^*_u M$. Furthermore, on each component G_0 of G, the dependence on $\alpha \in G_0$ of the solutions to (P_α) is C^r differentiable in the following sense: there exists a finite set $\{\varphi_i$: $G_0 \to M$; $1 \leq i \leq k\}$ of C^r mappings such that if $\alpha \in G_0$ the set of solutions to (P_α) is contained in $\{\varphi_i(\alpha)$; $1 \leq i \leq k\}$. In particular, every problem (P_α) with α in G_0 has at most k solutions.

Proof. Let $n = \dim M$ which is finite because the manifold M is compact. Define the mapping F: $M \times E \to T^* M$ by

$$(2.1) \qquad F(u, \alpha) = J'_u(u, \alpha), \quad (u, \alpha) \in M \times E.$$

Condition (i) implies that F is C^r differentiable. Let S denote the zero-section of the cotangent bundle $T^* M$, that is

$$(2.2) \qquad S = \{O_u \in T^*_u M;\ u \in M\}.$$

It is well known that S is a C^r submanifold of $T^* M$ with

$$\dim S = n \tag{2.3}$$

and, for every point $u \in M$, the following splitting decomposition holds

$$T_{O_u}(T^* M) = T_{O_u} S \oplus T_{O_u}(T_u^* M). \tag{2.4}$$

In view of (2.4), condition (ii) expresses the transversality of the mapping F to the submanifold S of $T^* M$, so one can apply Abraham's transversality theorem [1], Theorem 19.1 and Theorem 18.2 (see also [2], p. 53). Hence

$$G = \{\alpha \in E;\ F(\cdot, \alpha)\colon M \to T^* M \text{ is transversal to } S\} \tag{2.5}$$

is a dense open set in E. By (2.2), (2.3) and (2.5) it follows that, if $\alpha \in G$,

$$\{u \in M;\ J'_u(u, \alpha) = O_u\} = F(\cdot, \alpha)^{-1}(S) \tag{2.6}$$

is a C^r submanifold of dimension 0 in M. Taking into account the compactness of M we deduce that the set (2.6) is finite for every $\alpha \in G$. Since for each solution u of (P_α) the pair (u, α) belongs to the set (2.6), G is the required dense open subset of E.

If $u \in M$ is a solution to (P_α) with $\alpha \in G$, by (2.1), (2.4) and (2.5) we derive that $J''_{uu}(u, \alpha) = F'_u(u, \alpha)$ is an isomorphism of $T_u M$ onto the vertical space $T_{O_u}(T_u^* M) \cong T_u^* M$.

Because the transversality reduces to a submersion condition (see [2], p. 52), the last assertion of Theorem is a straightforward consequence of the implicit function theorem combined with the first part of Theorem and the connectedness of G_0.

3. A generic property of control problems via initial conditions. In this section we apply Theorem 2.1 to control problems via initial conditions, i.e., optimal control problems in which the admissible controls are initial states. This type of problems is considered, e.g., in Lions [6], p. 213.

Let us fix a compact interval $I = [0, T]$ of $R = (-\infty, \infty)$ and an open subset U of R^n. Suppose $f\colon I \times U \times R^m \to R^n$ and $g\colon I \times U \times R^m \to R$ are functions verifying the hypothesis

(H_1) (i) for every $t \in I$, the mappings $f(t, \cdot, \cdot)$ and $g(t, \cdot, \cdot)$ are C^{r+1} differentiable with $r \geqslant 1$;

(ii) for each $i = 0, 1, \ldots, r+1$, the i-th derivatives $f^{(i)}_{(x,w)}$ and $g^{(i)}_{(x,w)}$ of f and g with respect to the last two variables (x, w) in $U \times R^m$ are locally bounded on $I \times U \times R^m$;

(iii) for every (x, w) in $U \times R^m$ and each $i = 0, 1, \ldots, r+1$, the mappings $t \to f^{(i)}_{(x,w)}(t, x, w)$ and $t \to g^{(i)}_{(x,w)}(t, x, w)$ are measurable on I.

Remark. In the terminology of Grasse [3], [4], the mappings f and g satisfying condition (H_1) are called *quasi-C^{r+1}*.

Corresponding to a compact C^{r+1} submanifold M of R^n, $r \geqslant 1$, with $M \subset U$, consider now the control problem via initial conditions

$$(\mathrm{P}_\alpha) \qquad \text{minimize } \int_I g(t, y(t), \alpha(t))\,dt$$

over initial states $x \in M$ subject to

$$(3.1) \qquad y' = f(t, y(t), \alpha(t)) \quad \text{a.e. } t \in I, \; y(0) = x,$$

where the parameter α belongs to a second countable, closed C^{r+1} submanifold E of $L^m_\infty(I) = L^\infty(I; R^m)$. Here $L^m_\infty(I)$ denotes the Banach space of equivalence classes of essentially bounded measurable mappings of I into R^m with the essential-supremum norm.

Under hypothesis (H_1), for every (x, α) in $U \times L^m_\infty(I)$, the Cauchy problem (3.1) admits a unique absolutely continuous solution $t \to y(t) = y(t, x, \alpha)$.

Assume further that the mapping f satisfies the condition

(H_2) For every $\alpha \in E$ the trajectory $y(\cdot, x, \alpha)$ of (3.1) starting from a point x of M is defined on the entire interval I.

Remark. Condition (H_2) holds if, for example, the mapping f defines a control vector field on the manifold M in the sense of Grasse [3], [4], i.e., $f(t, x, w) \in T_x M$ for all $(t, x, w) \in I \times M \times R^m$. If $U = R^n$, another condition implying (H_2) is the global boundedness of f.

Hypothesis (H_2) yields the existence of a neighbourhood V of $M \times E$ in $U \times L^m_\infty(I)$ such that the domain of the trajectory $y(\cdot, x, \alpha)$ is I provided $(x, \alpha) \in V$.

For the solution $y(\cdot, x, \alpha)$ of the Cauchy problem (3.1) with (x, α) in V and for an arbitrary $a \in L^m_\infty(I)$ consider the linear variational control system of f along the response $t \to y(t, x, \alpha)$,

$$(3.2) \quad z'(t) = f'_x(t, y(t, x, \alpha), \alpha(t))z(t) + f'_w(t, y(t, x, \alpha), \alpha(t))a(t), \qquad t \in I.$$

Denote by $X(t, x, \alpha)$ the fundamental matrix of (3.2) such that $X(0, x, \alpha)$ is the identity matrix.

Then the partial derivatives $y'_x(t, x, \alpha)$ and $y'_\alpha(t, x, \alpha)$ of $y = y(t, x, \alpha)$ are given by

$$(3.3) \qquad y'_x(t, x, \alpha) = X(t, x, \alpha)$$

and

$$(3.4) \quad y'_\alpha(t, x, \alpha)(a) = X(t, x, \alpha) \int_0^t X(s, x, \alpha)^{-1} f'_w(s, y(s, x, \alpha), \alpha(s))a(s)\,ds$$

for all $t \in I$, $(x, \alpha) \in V$ and $a \in L^m_\infty(I)$ (see Grasse [4], Corollary 2.13, or Lee and Markus [5], p. 379–380).

Similarly, one obtains the partial derivatives $X'_x(t, x, \alpha)$ and $X'_\alpha(t, x, \alpha)$ of $X(t, x, \alpha)$

$$(3.5) \quad X'_x(t, x, \alpha)(h)(\cdot) = X(t) \int_0^t X(s)^{-1} f''_{xx}(s, y(s), \alpha(s))(X(s)(\cdot), X(s)(h))\,ds$$

and

(3.6) $$X'_\alpha(t, x, \alpha)(a)(\cdot) = X(t)\int_0^t X(s)^{-1}\big[f''_{xx}(s, y(s), \alpha(s))(X(s)(\cdot), y'_\alpha(s, x, \alpha)(a)) + f''_{xw}(s, y(s), \alpha(s))(X(s)(\cdot), a(s))\big]\,ds$$

for all $t \in I$, $(x, \alpha) \in V$, $h \in R^n$, $a \in L^m_\infty(I)$, where $y'_\alpha(s, x, \alpha)(a)$ is written in (3.4), while $y(t)$ and $X(t)$ denote for simplicity $y(t, x, \alpha)$ and $X(t, x, \alpha)$ respectively.

In addition to (H_1) and (H_2) the following hypothesis will be imposed.

(H_3) For every (x, α) in $M \times L^m_\infty(I)$ such that the vector

(3.7) $$\int_0^T X(t, x, \alpha)^* \operatorname{grad}_x g(t, y(t, x, \alpha), \alpha(t))\,dt$$

is orthogonal in R^n to $T_x M$ and for every $\xi \in T^*_x M$, there exist $h \in T_x M$ and $a \in T_\alpha E$ such that

(3.8) $$\xi = \int_0^T \big[g''_{xx}(t, y(t), \alpha(t))(X(t)(\cdot), X(t)(h) + y'_\alpha(t, x, \alpha)(a)) + g''_{xw}(t, y(t), \alpha(t))(X(t)(\cdot), a(t)) + g'_x(t, y(t), \alpha(t))(X'_x(t, x, \alpha)(h)(\cdot) + X'_\alpha(t, x, \alpha)(a)(\cdot))\big]\,dt,$$

where $y(t)$ and $X(t)$ mean $y(t, x, \alpha)$ and $X(t, x, \alpha)$ respectively, and $y'_\alpha(t, x, \alpha)(a)$, $X'_x(t, x, \alpha)(h)(\cdot)$, $X'_\alpha(t, x, \alpha)(a)(\cdot)$ are determined explicitly in (3.4)–(3.6).

In (3.7), $X(t, x, \alpha)^*$ denotes the transpose of $X(t, x, \alpha)$ and $\operatorname{grad}_x g$ is the gradient of g with respect to the second variable.

THEOREM 3.1. *Let M be a compact C^{r+1} submanifold of R^n, let U be an open subset of R^n containing M and let E be a second countable closed C^{r+1} submanifold of $L^m_\infty(I)$ with $r \geq 1$ and $I = [0, T]$. Suppose that the mappings $f\colon I \times U \times R^m \to R^n$ and $g\colon I \times U \times R^m \to R$ satisfy conditions* (H_1), (H_2) *and* (H_3). *Then there exists an open dense set G of E such that, for every α in G, problem (P_α) has a finite set of solutions. If $x \in M$ is a solution to (P_α) with $\alpha \in G$, then the map*

$$h \to \int_0^T \big[g''_{xx}(t, y(t, x, \alpha), \alpha(t))(X(t, x, \alpha)(\cdot), X(t, x, \alpha)(h)) + g'_x(t, y(t, x, \alpha), \alpha(t))\,X'_x(t, x, \alpha)(h)(\cdot)\big]\,dt$$

is an isomorphism of $T_x M$ onto $T_{O_x}(T^*_x M) \cong T^*_x M$. Furthermore, for each *component* G_0 *of* G *there exist finitely many* C^r *mappings* $\varphi_i\colon G_0 \to M$, $1 \leqslant i \leqslant k$, *such that if* $\alpha \in G_0$ *every solution to* (P_α) *is an element of* $\{\varphi_i(\alpha)\colon 1 \leqslant i \leqslant k\}$. *In particular, for any component* G_0 *of* G *there is a finite number* k *such that, if* $\alpha \in G_0$, (P_α) *has at most* k *solutions.*

Proof. Let $J\colon M \times E \to R$ denote the function

$$J(x, \alpha) = \int_0^T g\big(t, y(t, x, \alpha), \alpha(t)\big)\,dt \quad \text{for } (x, \alpha) \in M \times E. \tag{3.9}$$

We introduce the mappings $F\colon C(I, U) \times L^m_\infty(I) \to R$ and $Y\colon M \times E \to C(I, U)$ as follows

$$F(c, \alpha) = \int_0^T g\big(t, c(t), \alpha(t)\big)\,dt \quad \text{for } (c, \alpha) \in C(I, U) \times L^m_\infty(I)$$

and

$$Y(x, \alpha) = y(\cdot, x, \alpha) \quad \text{for } (x, \alpha) \in M \times E.$$

In view of (H_2), Y takes values in $C(I, U)$, and by (H_1), both F and Y are C^{r+1} mappings (see Grasse [3], Theorem 3.3.5 and Theorem 3.3.11). Noting from (3.9) that J is the composite map

$$J = F \circ (Y, \mathrm{pr}_2), \tag{3.10}$$

where $\mathrm{pr}_2\colon M \times E \to E$ is the projection onto the second factor, it follows that J is C^{r+1} differentiable, so condition (i) in Theorem 2.1 is verified.

From (3.10) and (3.3) one finds the partial derivative J'_x of J

$$\begin{aligned} J'_x(x, \alpha)(h) &= F'\big(Y(x, \alpha), \alpha\big)\big(Y'_x(x, \alpha)(h), 0\big) \\ &= \int_0^T g'_x\big(t, y(t, x, \alpha), \alpha(t)\big) X(t, x, \alpha)(h)\,dt \\ &\qquad\qquad \text{for } (x, \alpha) \in M \times E \text{ and } h \in T_x M. \end{aligned}$$

Therefore J'_x vanishes at a point (x, α) in $M \times E$ if and only if the vector defined in (3.7) is orthogonal to $T_x M \subset R^n$. A similar argument shows the equivalence between (3.8) and the surjectivity of the mapping

$$(h, a) \in T_x M \times T_\alpha E \to J''_{xx}(x, \alpha)(\cdot, h) + J''_{x\alpha}(x, \alpha)(\cdot, a) \in T^*_x M.$$

Consequently hypothesis (H_3) is just condition (ii) in Theorem 2.1 for the functional J of (3.9). To complete the proof it suffices to apply Theorem 2.1.

Remark. Hypotheses (H_1)–(H_3) take simpler forms for particular cases of control problem (P_α). For example, let us consider the same cost functional defined by a function $g\colon I \times U \times R^m \to R$ satisfying (H_1), but replace (3.1) by a linear process in R^n

$$y' = A(t)y + B(t)\alpha, \quad \text{a.e. } t \in I = [0, T], \; y(0) = x \in M, \tag{3.1$'$}$$

depending on the parameter $\alpha \in E \subset L^m_\infty(I)$, where A and B denote bounded measurable matrix functions. In this case hypothesis (H_2) is verified and, by (3.2), $X(t, x, \alpha)$ with $X(0, x, \alpha) = \mathrm{id}_{R^n}$ is the fundamental matrix of (3.1)'. Hypothesis (H_3) reduces to the following

(H_3)' For every (x, α) in $M \times E$ such that the vector (3.7) is orthogonal in R^n to $T_x M$ and for every $\xi \in T_x^* M$ there exist $h \in T_x M$ and $a \in T_\alpha E$ such that

$$\xi = \int_0^T \big[g''_{xx}(t, y(t, x, \alpha), \alpha(t))\big(X(t, x, \alpha)(\cdot),$$
$$X(t, x, \alpha)\big(h + \int_0^T X(s, x, \alpha)^{-1} B(s) a(s)\,ds\big)\big) +$$
$$+ g''_{xw}(t, y(t, x, \alpha), \alpha(t))\big(X(t, x, \alpha)(\cdot), a(t)\big)\big]\,dt.$$

4. A generic property of optimal control problems. For some integer $r \geq 1$, denote by $E = B^{r+1}(R \times R^n \times R^m, R^n)$ the separable Banach space of all C^{r+1} mappings from $R \times R^n \times R^m$ into R^n whose derivatives up to order $r+1$ are bounded, with the norm

$$\|f\| = \sup\{\|(f(p), f'(p), \ldots, f^{(r+1)}(p))\|;\ p = (t, x, w) \in R \times R^n \times R^m\}, \quad f \in E.$$

Corresponding to the compact interval $I = [0, T]$ in R and to the fixed data $x_0 \in R^n$, $f \in E$ and the control $u \in L^m_\infty(I) = L^\infty(I, R^m)$, let us consider the initial value problem

$$y' = f(t, y, u(t)) \quad \text{a.e. } t \in I,\ y(0) = x_0. \tag{4.1}$$

By the global boundedness of $f \in E$, the integral curve $t \to y(t) = y(t, u, f)$ of (4.1) is defined everywhere on I.

Consider the linear variational control system of (4.1) along the response $t \to y(t, u, f)$,

$$z'(t) = f'_x(t, y(t, u, f), u(t))z(t) + f'_w(t, y(t, u, f), u(t))v(t) \tag{4.2}$$

for any $v \in L^m_\infty(I)$, and denote by $X(t, u, f)$ the fundamental matrix of (4.2) such that $X(0, u, f) = \mathrm{id}_{R^n}$.

The linear differential system (4.2) is useful for studying the differentiability properties of the mapping $Y\colon L^m_\infty(I) \times E \to C(I, R^n)$ defined by

$$Y(u, f)(t) = y(t, u, f) \quad \text{for all } (u, f) \in L^m_\infty(I) \times E \text{ and } t \in I. \tag{4.3}$$

Lemma 4.1. *The mapping Y of* (4.3) *is C^{r+1} differentiable. If u, v, $q \in L^m_\infty(I)$, f, $h \in E$ and $t \in I$, the following formulae hold*:

$$Y'_u(u, f)(v)(t) = y'_u(t)(v) = X(t) \int_0^t X(s)^{-1} f'_w(s, y(s), u(s)) v(s)\,ds,$$

$$Y'_f(u,f)(h)(t) = y'_f(t)(h) = X(t)\int_0^t X(s)^{-1}\, h(s, y(s), u(s))\,ds,$$

$$\begin{aligned} Y''_{uu}(u,f)(q,v)(t) &= y''_{uu}(t)(q,v) \\ &= X(t)\int_0^t X(s)^{-1}\big[f''_{xx}(s, y(s), u(s))(y'_u(s)(q), y'_u(s)(v)) + \\ &\quad + f''_{xw}(s, y(s), u(s))(y'_u(s)(q), v(s)) + f''_{xw}(s, y(s), u(s))(y'_u(s)(v), q(s)) + \\ &\quad + f''_{ww}(s, y(s), u(s))(q(s), v(s))\big]\,ds, \end{aligned}$$

$$\begin{aligned} &Y''_{uf}(u,f)(v,h)(t) \\ &= y''_{uf}(t)(v,h) = X(t)\int_0^t X(s)^{-1}\big[f''_{xx}(s, y(s), u(s))(y'_u(s)(v), y'_f(s)(h)) + \\ &\quad + f''_{xw}(s, y(s), u(s))(y'_f(s)(h), v(s)) + h'_x(s, y(s), u(s))\, y'_u(s)(v) + \\ &\quad + h'_w(s, y(s), u(s))\, v(s)\big]\,ds, \end{aligned}$$

where we denoted

$$y(s) = y(s, u, f), \quad y'_u(s) = y'_u(s, u, f), \quad y'_f(s) = y'_f(s, u, f),$$

$$y''_{uu}(s) = y''_{uu}(s, u, f), \quad y''_{uf}(s) = y''_{uf}(s, u, f) \quad \textit{and} \quad X(s) = X(s, u, f).$$

Proof. Since the Banach space $E = B^{r+1}(R \times R^n \times R^m, R^n)$ is separable, by choosing a sequence $\{E_i\}_{i \geq 1}$ of vector subspaces of E such that

$$E = \bigcup_{i \geq 1} E_i, \quad E_i \subset E_{i+1} \quad \text{and} \quad \dim E_i = i \quad \text{for every } i \geq 1,$$

it is sufficient to prove the C^{r+1} differentiability of the restriction of Y to $L^m_\infty(I) \times E_i$ (denoted again by Y) for every $i \geq 1$. The mapping Q: $I \times R^n \times E_i \times \times R^m \to R^n \times E_i$ given by $Q(t, x, f, w) = (f(t, x, w), 0)$, $(t, x, f, w) \in I \times R^n \times E_i \times \times R^m$ is differentiable of class C^{r+1} (cf. Abraham–Robbin [1], Theorem 10.3). Then we can apply Theorem 3.3.11 in Grasse [3] (see also [4], Theorem 2.9, and [5], p. 379–380) to the control system Q on the finite dimensional space $R^n \times E_i$ to deduce that the solution $x(\cdot, f, u) \in C(I, R^n \times E_i)$ of the Cauchy problem $x' = Q(t, x, u(t))$, $x(0) = (x_0, f)$, depends C^{r+1} differentiably on $(f, u) \in E_i \times L^m_\infty(I)$. Because $x(t, f, u) = (y(t, f, u), f)$, it follows from (4.3) that Y is a C^{r+1} mapping. The formulae for the partial derivatives of the mapping Y are direct consequences of (4.2), (4.3) and the variation of parameters formula. For example, the derivative $Y'_u(u,f)(v)$ is obtained from the fact that $t \to y'_u(t, u, f)(v)$ is the unique solution of (4.2) vanishing at $t = 0$ (see Grasse [4], Corollary 2.13). This completes the proof.

We are concerned with the optimal control problem

$$(\mathrm{P}_f) \qquad \text{minimize} \int_0^T g(t, y(t, u, f), u(t))\,dt$$

over all controls $u \in M$ subject to (4.1), where M denotes a compact C^{r+1} submanifold of the Banach space $L_\infty^m(I)$, g: $I \times R^n \times R^m \to R$ is a function which satisfies condition (H_1) of Section 3 (with $U = R^n$) and $t \to y(t, u, f)$ is the unique absolutely continuous solution of (4.1) starting from the fixed point $x_0 \in R^n$. Here $f \in E$ is viewed as a parameter.

The following hypothesis will be made.

(H) If (u, f) is a point in $M \times E$ such that the following orthogonality condition holds

$$(4.4) \qquad \int_0^T \langle f_w'(s, y(s), u(s))^* X(s)^{*-1} \int_s^T X(t)^* \operatorname{grad}_x g(t, y(t), u(t))\,dt + \\ + \operatorname{grad}_w g(s, y(s), u(s)), q(s) \rangle ds = 0 \quad \text{for all } q \in T_u M \subset L_\gamma^m(I)$$

and if $\xi \in T_u^* M$, then there exist $v \in T_u M$ and $h \in E$ such that

$$(4.5) \qquad \xi = \int_0^T \big[g_{xx}''(t, y(t), u(t))(y_u'(t)(\cdot), y_u'(t)(v) + y_f'(t)(h)) + \\ + g_{xw}''(t, y(t), u(t))(y_u'(t)(\cdot), v(t)) + \\ + g_{xw}''(t, y(t), u(t))(y_u'(t)(v) + y_f'(t)(h), (\cdot)(t)) + \\ + g_{ww}''(t, y(t), u(t))((\cdot)(t), v(t)) + \\ + g_x'(t, y(t), u(t))(y_{uu}''(t)(\cdot, v) + y_{uf}''(t)(\cdot, h)) \big]\,dt.$$

In (4.4) and (4.5) we used the notations $y(t) = y(t, u, f)$, $X(t) = X(t, u, f)$ and similar abreviations for y_u', y_f', y_{uu}'' and y_{uf}'' evaluated at (t, u, f). The partial derivatives appearing in (4.5) are described by explicit formulae in Lemma 4.1. In (4.4) the superscript $*$ means the transpose of a matrix and $\langle\ ,\ \rangle$ denotes the canonical scalar product in R^m.

THEOREM 4.2. *For every $f \in E = B^{r+1}(R \times R^n \times R^m, R)$ consider the control problem (P_f) with fixed data g: $I \times R^n \times R^m \to R$, $M \subset L_\infty^m(I)$ and $x_0 \in R^n$ as above. Assume that hypothesis* (H) *is verified. Then there exists an open dense subset G of E such that, for every $f \in G$, problem (P_f) has only a finite set of solutions. Moveover, for each component G_0 of G there exists a finite set of C^r mappings φ_i: $G_0 \to M$, $1 \leqslant i \leqslant k$, such that if $f \in G_0$ every solution to (P_f) is an element of $\{\varphi_i(f);\ 1 \leqslant i \leqslant k\}$. In particular, for any component G_0 of G there is a finite number k such that, if $f \in G_0$, problem (P_f) has at most k solutions.*

Proof. We proceed as in the proof of Theorem 3.1, First, we introduce the function J: $M \times E \to R$ by

$$J(u,f) = \int_0^T g(t, y(t, u, f), u(t))\,dt \quad \text{for } (u,f) \in M \times E. \tag{4.6}$$

It follows that J is the composition

$$J = F \circ (Y, \mathrm{pr}_1), \tag{4.7}$$

where pr_1: $M \times E \to M$ is the projection onto the first factor, Y is the restriction to $M \times E$ of the mapping Y of (4.3) and F: $C(I, R^n) \times L^m_\infty(I) \to R$ is the C^{r+1} mapping defined in the proof of Theorem 3.1 (with $U = R^n$). Then Lemma 4.1 ensures the C^{r+1} differentiability of the function J, and because E is a separable Banach space, it follows that condition (i) in Theorem 2.1 is verified. By (4.7), Lemma 4.1 and the chain rule, one finds

$$\begin{aligned} J'_u(u,f)(v) &= F'(Y(u,f), u)(Y'_u(u,f)(v), v) \\ &= \int_0^T \Big[g'_x(t, y(t), u(t))X(t)\int_0^t X(s)^{-1} f'_w(s, y(s), u(s))v(s)\,ds + \\ &\qquad\qquad + g'_w(t, y(t), u(t))v(t)\Big]\,dt \\ &= \int_0^T \Big\langle f'_w(s, y(s), u(s))^* X(s)^{*-1} \int_s^T X(t)^* \operatorname{grad}_x g(t, y(t), u(t))\,dt + \\ &\qquad\qquad + \operatorname{grad}_w g(s, y(s), u(s)), v(s)\Big\rangle\,ds \\ &\qquad\qquad \text{for all } (u,f) \in M \times E \text{ and } v \in T_u M, \end{aligned}$$

the final equality being obtained by interchanging the order of integration. It turns out that the orthogonality condition (4.4) means the vanishing of $J'_u(u,f) \in T^*_u M$. Similarly, using (4.7) and Lemma 4.1, one sees that relation (4.5) expresses the surjectivity of the mapping

$$(v, h) \in T_u M \times E \to J''_{uu}(u,f)(\cdot, v) + J''_{uf}(u,f)(\cdot, h) \in T^*_u M.$$

This shows, in the case of the functional J of (4.6), the equivalence between hypothesis (H) and condition (ii) in Theorem 2.1. It suffices to apply Theorem 2.1 for ending the proof.

Remark From Theorem 2.1 it follows also that, if $u \in M$ is a solution to (P_f) with $f \in G$, then the partial derivative $J''_{uu}(u,f)$ of the mapping J (introduced in (4.6)) is an isomorphism of $T_u M$ onto $T_{0_u}(T^*_u M) \cong T^*_u M$.

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