ANNALES
POLONICI MATHEMATICI
LII (1990)

Almost Hermitian structures on the frame bundle of an almost Hermitian manifold

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Abstract. Let (M, g, J) be an almost Hermitian manifold and consider on its frame bundle $\mathfrak{F}(M)$ the almost Hermitian structure (g^D, J^H). In this paper we study the type of almost Hermitian structure $\mathfrak{F}(M)$ acquires when we consider a particular one on M and we prove that there exists some type of almost Hermitian structure that $(\mathfrak{F}(M), g^D, J^H)$ cannot possess.

Introduction. Let M be an m-dimensional differentiable manifold of class C^∞ and $\mathfrak{F}(M)$ its frame bundle. Mok [5], [6], and Cordero & de León [2], [3], develop the general theory of lifts to $\mathfrak{F}(M)$ of tensor fields on M. In particular, in [5] Mok introduces a Riemannian metric g^D on the frame bundle of a Riemannian manifold (M, g), metric which is very similar to that defined by Sasaki [7] for the tangent bundle, and that we shall call the Sasaki–Mok metric induced on $\mathfrak{F}(M)$.

In this paper we suppose (M, g, J) an almost Hermitian manifold, we consider on $\mathfrak{F}(M)$ the almost Hermitian structure (g^D, J^H) and study, in a similar way as was done for $T(M)$, [1], the type of almost Hermitian structure $\mathfrak{F}(M)$ acquires when we consider a particular one on M. Moreover, we prove that there exists some type of almost Hermitian structure that $(\mathfrak{F}(M), g^D, J^H)$ cannot possess.

1. In this section, we summarize all the basic definitions and results that are needed later.

In the following all the manifolds, maps, connections in question are supposed to be differentiable of class C^∞.

Let M be an m-dimensional differentiable manifold, $\mathfrak{F}(M)$ its frame bundle and $\pi\colon \mathfrak{F}(M) \to M$ the projection map.

For the coordinate system (U, x^i) in M, we put $\mathfrak{F}(U) = \pi^{-1}(U)$ and the vector X_α of the frame $p_x \in \mathfrak{F}(U)$ can be uniquely expressed in the form $X_\alpha = X_\alpha^i(\partial/\partial x^i)_x$, so that $\{\mathfrak{F}(U), (x^i, X_\alpha^i)\}$ is a coordinate system in $\mathfrak{F}(M)$.

Let ∇ be a linear connection and X a vector field on M with local components Γ_{ij}^h and X^i, respectively. Then there is exactly one vector field X^H

on $\mathfrak{F}(M)$, called the *horizontal lift of* X, and exactly one vector field $X^{(\alpha)}$ on $\mathfrak{F}(M)$ for each $\alpha = 1, 2, \ldots, m$ called the α^{th}-*vertical lift of* X, [3].

If $X = X^i \dfrac{\partial}{\partial x^i}$ in U, then in $\mathfrak{F}(U)$,

$$X^H = X^i \frac{\partial}{\partial x^i} - X^i \Gamma^h_{ik} X^k_\alpha \frac{\partial}{\partial X^h_\alpha}, \qquad X^{(\alpha)} = X^i \frac{\partial}{\partial X^i_\alpha}.$$

If f is a differentiable function on M, $f^v = f \circ \pi$ denotes its canonical vertical lift to $\mathfrak{F}(M)$, and $f^H = 0$ its horizontal lift.

If F is a tensor field on M of type $(1, 1)$ with components F^h_j in U, then

$$F^H = F^h_j \frac{\partial}{\partial x^h} \otimes dx^j + X^k_\alpha (\Gamma^i_{jk} F^h_i - \Gamma^h_{ik} F^i_j) \frac{\partial}{\partial X^h_\alpha} \otimes dx^j + \delta^\alpha_\beta F^h_j \frac{\partial}{\partial X^h_\alpha} \otimes dX^j_\beta$$

in $\mathfrak{F}(U)$.

If one takes into account the different definitions of lifts, the following formulae are easily obtained:

$$X^H(f^V) = (X(f))^V, \qquad X^{(\alpha)}(f^V) = 0,$$
$$F^H(X^{(\alpha)}) = (F(X))^{(\alpha)}, \qquad F^H(X^H) = (F(X))^H.$$

The brackets of vertical and horizontal lifts are expressed by the following formulae:

$$[X^{(\alpha)}, Y^{(\beta)}] = 0, \qquad [X^H, Y^{(\alpha)}] = (\nabla_X Y)^{(\alpha)}, \qquad [X^H, Y^H] = [X, Y]^H - \gamma R(X, Y),$$

where $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Let now (M, g) be a Riemannian space, ∇ an arbitrary and not necessarily metric connection on M. The diagonal lift g^D of g to $\mathfrak{F}(M)$ with respect to connection ∇ is a Riemannian metric on $\mathfrak{F}(M)$ determined by the identities, [3],

$$g^D(X^H, Y^H) = \{g(X, Y)\}^V, \qquad g^D(X^H, Y^{(\alpha)}) = 0, \qquad g^D(X^{(\alpha)}, Y^{(\beta)}) = \delta^{\alpha\beta}\{g(X, Y)\}^V$$

for any vector fields X, Y on M and $\alpha, \beta = 1, 2 \ldots, m$.

In the sequel, ∇ shall represent the Levi-Civita connection on the Riemannian manifold (M, g) and will denote by $\tilde{\nabla}$ the Levi-Civita connection of $(\mathfrak{F}(M), g^D)$. This connection is determined by

$$\tilde{\nabla}_{X^{(\alpha)}} Y^{(\beta)} = 0, \qquad g^D(\tilde{\nabla}_{X^{(\alpha)}} Y^H, Z^{(\beta)}) = 0,$$
$$g^D(\tilde{\nabla}_{X^{(\alpha)}} Y^H, Z^H) = -\tfrac{1}{2} g^D(\gamma R(Z, Y), X^{(\alpha)}),$$
$$g^D(\tilde{\nabla}_{X^H} Y^{(\alpha)}, Z^{(\beta)}) = \delta^{\alpha\beta}\{g(\nabla_X Y, Z)\}^V,$$
$$g^D(\tilde{\nabla} \quad Y^{(\alpha)}, Z^H) = -\tfrac{1}{2} g^D(\gamma R(Z, X), Y^{(\alpha)}),$$
$$g^D(\tilde{\nabla}_{X^H} Y^H, Z^{(\alpha)}) = -\tfrac{1}{2} g^D(\gamma R(X, Y), Z^{(\alpha)}),$$
$$g^D(\tilde{\nabla}_{X^H} Y^H, Z^H) = \{g(\nabla_X Y, Z)\}^V$$

for all vector fields X, Y, Z on M, and $\alpha, \beta = 1, 2 \ldots, m$.

Furthermore, if (M, g, J) is an m-dimensional $(m = 2n)$ almost Hermitian manifold and we consider the horizontal lift of J, J^H, the Sasaki–Mok metric g^D is a Hermitian metric with respect to J^H and we may conclude that $(\mathfrak{F}(M), g^D, J^H)$ is an almost Hermitian manifold.

The derivative and coderivative of the Kähler form $\tilde{F}$ of $(\mathfrak{F}(M), g^D, J^H)$ and the Nijenhuis tensor of the almost complex structure J^H are given by the following theorems.

THEOREM 1.1. *The derivative of the Kähler form of the almost Hermitian manifold* $(\mathfrak{F}(M), g^D, J^H)$ *is given by*:

(i) $d\tilde{F}(X^H, Y^H, Z^H) = \{dF(X, Y, Z)\}^V$,

(ii) $d\tilde{F}(X^H, Y^H, Z^{(\alpha)}) = \frac{1}{3} g^D(\gamma R(X, Y), (JZ)^{(\alpha)})$,

(iii) $d\tilde{F}(X^H, Y^{(\alpha)}, Z^{(\beta)}) = \frac{1}{3}\delta^{\alpha\beta}\{(\nabla_X F)(Y, Z)\}^V$,

(iv) $d\tilde{F}(X^{(\alpha)}, Y^{(\beta)}, Z^{(\mu)}) = 0$

for any X, Y, Z *vector fields on* M *and* α, β, $\mu = 1, 2, \ldots, 2n$.

Proof. It is a straightforward computation just taking into account that $\tilde{F}(\tilde{X}, \tilde{Y}) = g^D(\tilde{X}, J^H \tilde{Y})$.

THEOREM 1.2. *The coderivative of the Kähler form* $\tilde{F}$ *of the almost Hermitian manifold* $(\mathfrak{F}(M), g^D, J^H)$ *is given by:*

$$\delta\tilde{F}(X^H) = \{\delta F(X)\}^V, \qquad \delta\tilde{F}(X^{(\beta)}) = \sum_{i=1}^{n} g^D(\gamma R(E_i, JE_i), X^{(\beta)})$$

with respect to a J-basis $\{E_1, \ldots, E_n, JE_1, \ldots, JE_n\}$ *of* (M, g, J), *and for any vector field* X *on* M.

Proof. Let us remark that if we consider the J-basis of (M, g, J) mentioned above, then $\{E_i^H, (JE_i)^H, E_i^{(\alpha)}, (JE_i)^{(\alpha)};\ i = 1, 2, \ldots, n,\ \alpha = 1, 2, \ldots, 2n\}$ is a J^H-basis for $(\mathfrak{F}(M), g^D, J^H)$.

THEOREM 1.3. *The Nijenhuis tensor of* J^H *is given by*:

(i) $\tilde{N}(X^{(\alpha)}, Y^{(\beta)}) = 0$,

(ii) $\tilde{N}(X^H, Y^H) = \{N(X, Y)\}^H - \gamma\{R(JX, JY) - JR(JX, Y) - JR(X, JY) - R(X, Y)\}$,

(iii) $\tilde{N}(X^H, Y^{(\alpha)}) = \{\nabla_{JX}(J)Y - J\nabla_X(J)Y\}^{(\alpha)}$,

where N *is the Nijenhuis tensor of the almost complex structure* J; X, Y *vector fields on* M *and* $\alpha, \beta = 1, 2, \ldots, 2n$.

2. In the sequel we shall consider the classification of Gray–Hervella [4] of almost Hermitian manifolds.

According to this classification and the fact of π: $(\mathfrak{F}(M), g^D, J^H) \to (M, g, J)$ being an almost Hermitian submersion with totally geodesic fibers, it is possible to deduce what kind of almost Hermitian structure (M, g, J) possesses when it is assumed that $(\mathfrak{F}(M), g^D, J^H)$ has a given one [8].

Moreover, in this paper we give the following improvements to these results:

THEOREM 2.1. *If $(\mathfrak{F}(M), g^D, J^H)$ is a G_1-manifold or a G_2-manifold, then (M, g, J) is a Hermitian manifold.*

Proof. If $(\mathfrak{F}(M), g^D, J^H)$ is a G_2-manifold, then

$$\underset{\tilde{X},\tilde{Y},\tilde{Z}}{\mathrm{S}} \{g^D(\tilde{\nabla}_{\tilde{X}}(J^H)\tilde{Y}, \tilde{Z}) - g^D(\tilde{\nabla}_{J^H\tilde{X}}(J^H)J^H\tilde{Y}, \tilde{Z})\} = 0$$

for any $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\mathfrak{F}(M))$, where S denotes the cyclic sum.

In particular, taking $\tilde{X} = X^{(\alpha)}$, $\tilde{Y} = Y^{(\alpha)}$, $\tilde{Z} = Z^H$, where X, Y, Z are vector fields on M, we obtain

$$\{g(\nabla_Z(J)X, Y) - g(\nabla_{JZ}(J)JX, Y)\}^V = 0$$

and the result follows.

An analogous method proves the case of G_1-manifold.

COROLLARY 2.1. *If $(\mathfrak{F}(M), g^D, J^H)$ is a $W_1 \oplus W_3$-manifold or a $W_2 \oplus W_3$-manifold, then (M, g, J) is a W_3-manifold.*

Proof. It follows taking into account Theorem 2.1, the Watson's results [8] and the inclusion relations among the almost Hermitian manifolds.

3. In order to study lifts of different almost Hermitian structures on (M, g, J) to $(\mathfrak{F}(M), g^D, J^H)$ we shall consider in some cases that the curvature tensor R of the connection ∇ on M satisfies the K_1-curvature identity, also known as Kähler identity, i.e., $R(X, Y, Z, W) = R(X, Y, JZ, JW)$ for any vector fields X, Y, Z, W on M.

In the sequel we shall represent with the subindex 1 the class of almost Hermitian structure that satisfies this identity.

Some relations among the almost Hermitian manifolds and those that verify the K_1-curvature identity are given in the following theorem [9].

THEOREM 3.1. *In the lattice of almost Hermitian structures, the following relationships are true:*

(i) $K = K_1 = NK_1 = AK_1$,

(ii) $K \subset H_1$,

(iii) $K \subset QK_1$

(NK = nearly-Kähler. AK = almost-Kähler. QK = quasi-Kähler. H = Hermitian).

THEOREM 3.2. *If (M, g, J) is a H_1-manifold, $(\mathfrak{F}(M), g^D, J^H)$ is a H-manifold.*

Proof. If M is a Hermitian manifold, then $N = 0$ and since M satisfies the K_1-curvature identity, the desired result is obtained as a consequence of Theorem 1.3.

THEOREM 3.3. *If* (M, g, J) *is a non-flat Kählerian manifold, then* $(\mathfrak{F}(M), g^D, J^H)$ *is a Hermitian manifold that is not a* W_4*-manifold.*

Proof. Trivially, $(\mathfrak{F}(M), g^D, J^H)$ is a Hermitian manifold. Moreover, it is not a W_4-manifold, because if it were we should have

$$g^D(\tilde{\nabla}_{X^H}(J^H)X^H, Y^{(\alpha)}) = \frac{1}{2(2n^2+n-1)} g^D(X^H, X^H)\delta\tilde{F}(Y^{(\alpha)})$$

for any vector fields X, Y on M, and using Theorem 1.2 we have

$$-\tfrac{1}{2}g^D(\gamma R(X, JX), Y^{(\alpha)}) = \frac{1}{2(2n^2+n-1)}\|X\|^2 \sum_{i=1}^{n} g^D(\gamma R(E_i, JE_i), Y^{(\alpha)}).$$

Setting $X = E_i$ and bearing in mind that $g^D(\gamma R(X, Y), Z^H) = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$, we obtain $R(X, JX) = 0$ and thus, since M is Kählerian, it follows that $R = 0$.

Let us recall now that the Ricci*-curvature tensor ϱ^* is defined by

$$\varrho^*(X, Y) = \sum_{i=1}^{n} R(X, JY, JE_i, E_i),$$

where $\{E_i, JE_i;\ i = 1, 2, \ldots, n\}$ is a local orthonormal J-basis for $\mathfrak{X}(M)$.

THEOREM 3.4. *If* (M, g, J) *is a semi-Kähler manifold with* $\varrho^* = 0$, *then* $(\mathfrak{F}(M), g^D, J^H)$ *is a semi-Kähler manifold.*

The result follows from Theorem 1.1 and the fact that, for an almost Hermitian manifold (M, g, J), $\varrho^* = 0$ if and only if

$$\sum_{i=1}^{n} R(E_i, JE_i) = 0.$$

COROLLARY 3.1. *If* (M, g, J) *is a Kählerian manifold with* $\varrho^* = 0$, *then* $(\mathfrak{F}(M), g^D, J^H)$ *is a* W_3*-manifold.*

This follows from Theorems 3.2 and 3.4.

THEOREM 3.5. *If* (M, g, J) *is a manifold belonging to one of the classes* $W_1 \oplus W_4$, $W_2 \oplus W_4$, $W_1 \oplus W_2 \oplus W_4$, $W_1 \oplus W_3 \oplus W_4$, $W_2 \oplus W_3 \oplus W_4$, *then* $(\mathfrak{F}(M), g^D, J^H)$ *is a* W*-manifold.*

Proof. Taking into account the inclusion relations among this type of manifolds, it will be sufficient to prove the result for the classes $W_1 \oplus W_4$ and $W_2 \oplus W_4$.

Since $\pi\colon \mathfrak{F}(M) \to M$ is an almost Hermitian submersion, if M were a $W_1 \oplus W_4$ or $W_2 \oplus W_4$-manifold, $\mathfrak{F}(M)$ could only be a $W_1 \oplus W_2 \oplus W_4$, $W_1 \oplus W_3 \oplus W_4$, $W_2 \oplus W_3 \oplus W_4$ or a W-manifold; but according to Theorem 2.1, $\mathfrak{F}(M)$ can be neither a $W_1 \oplus W_3 \oplus W_4$ nor $W_2 \oplus W_3 \oplus W_4$. Neither can $\mathfrak{F}(M)$ be a $W_1 \oplus W_2 \oplus W_4$, because if it were such a manifold, we should have

$$g^D(\tilde{\nabla}_{\tilde{X}}(J^H)\tilde{Y},\tilde{Z})+g^D(\tilde{\nabla}_{J^H\tilde{X}}(J^H)J^H\tilde{Y},\tilde{Z})$$
$$= -\frac{1}{2(2n^2+n-1)}\{g^D(\tilde{X},\tilde{Y})\delta\tilde{F}(\tilde{Z})-g^D(\tilde{X},\tilde{Z})\delta\tilde{F}(\tilde{Y})$$
$$-g^D(\tilde{X},J^H\tilde{Y})\delta\tilde{F}(J^H\tilde{Z})+g^D(\tilde{X},J^H\tilde{Z})\delta\tilde{F}(J^H\tilde{Y})\}$$

and then, in particular, taking $\tilde{X} = X^H$, $\tilde{Y} = Y^{(\alpha)}$, $\tilde{Z} = Z^{(\alpha)}$ we should obtain

$$g(\nabla_X(J)Y+\nabla_{JX}(J)JY,Z) = 0,$$

that is, M would be a quasi-Kähler manifold.

Remark. It is understood that (M,g,J) belongs to the kinds of almost Hermitian manifolds mentioned above and not to their proper subspaces.

THEOREM 3.6. *If (M,g,J) is a non-flat almost Hermitian manifold that satisfies the K_1-curvature identity, then $(\mathfrak{F}(M),g^D,J^H)$ cannot be a quasi-Kähler manifold.*

Proof. If $(\mathfrak{F}(M),g^D,J^H)$ were a quasi-Kähler manifold, then

$$g^D(\tilde{\nabla}_{\tilde{X}}(J^H)\tilde{Y},\tilde{Z})+g^D(\tilde{\nabla}_{J^H\tilde{X}}(J^H)J^H\tilde{Y},\tilde{Z}) = 0$$

for all $\tilde{X},\tilde{Y},\tilde{Z}\in\mathfrak{X}(\mathfrak{F}(M))$.

In particular, taking $\tilde{X} = X^H$, $\tilde{Y} = Y^{(\alpha)}$, $\tilde{Z} = Z^H$, with $X,Y,Z\in\mathfrak{X}(M)$ we obtain

$$\gamma R(X,JX) = 0$$

and by polarization

$$\gamma R(X,JY) = -\gamma R(Y,JX).$$

Hence, since M satisfies the K_1-curvature identity, it follows that $\gamma R(X,JY) = 0$ and then $R = 0$.

Finally, since we have obtained for $(\mathfrak{F}(M),g^D,J^H)$ analogous results to that obtained for $T(M)$ in [1], in a similar way to [1], we obtain the following theorem.

THEOREM 3.7. *The frame bundle $\mathfrak{F}(M)$ of an almost Hermitian non-flat manifold (M,g,J), with the Sasaki–Mok metric g^D and the almost complex structure J^H, is an almost Hermitian manifold that does not belong to any of the following classes: K, W_1, W_2, W_4, $W_1\oplus W_4$, $W_2\oplus W_4$.*

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Reçu par la Rédaction le 15.12.1987