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SCHAUDER BASES AND BEST APPROXIMATION

BY

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1. Introduction. Let Z denote a closed linear subspace of a Banach space E and let $x \in E$. An element $z_0 \in Z$ is a *best approximation* of x *from* Z provided

$$b(x) = \inf\{\|x-z\| : z \in Z\} = \|x-z_0\|.$$

Let $B_Z(x) = \{z_0 \in Z : \|x-z_0\| = b(x)\}$.

It is well known and easy to prove that $B_Z(x)$ is bounded, closed and convex. It can be empty. Indeed let E be the continuous functions on $[0,1]$ with

$$\|x\| = \sup\{|x(t)| : 0 \leqslant t \leqslant 1\} + \int_0^1 |x(t)|\,dt;$$

$$x_0(t) \equiv 1 \quad \text{and} \quad Z = \{x \in E : x(0) = 0\};$$

then $B_Z(x_0) = \emptyset$.

A subspace Z in E is a *Haar subspace* provided $B_Z(x) \neq \emptyset$ for each $x \in E$; and a subspace Z in E is a *Čebyšev subspace* provided $B_Z(x)$ is one pointed for each $x \in E$. A simple argument shows that each finite-dimensional subspace F of a Banach space E is a Haar subspace. It is also easy to show that in a strictly convex space a Haar subspace is Čebyšev.

In order to state the problems considered in this paper we need the following notations. In general, we will use $\{\cdot\}$ to denote sets, $[\ ]$ to denote closed linear spans of the indicated sets, and $(\cdot)$ to denote sequences. For a sequence (x_i) in E let

$$L_n = [x_i : i \leqslant n] \quad \text{and} \quad L^n = [x_i : i > n].$$

If Σ denotes the family of finite subsets of the set of positive integers, ω, directed by inclusion and if $\sigma \in \Sigma$, we similarly define

$$L_\sigma = [x_i : i \in \sigma] \quad \text{and} \quad L^\sigma = [x_i : i \in \omega \setminus \sigma].$$

We also let $B_n(x)$, $B^n(x)$, $B_\sigma(x)$, and $B^\sigma(x)$ denote $B_{L_n}(x)$, $B_{L^n}(x)$, $B_{L_\sigma}(x)$ and $B^{L_\sigma}(x)$ respectively. We say that a sequence (x_n) in E is *fundamental* provided $[x_n : n \in \omega] = E$. Finally $\mathscr{L}(X, Y)$ denotes the set of all continuous linear operators from X to Y.

In this paper we are concerned with the following general problems:

Under what conditions on a fundamental sequence (x_n) in E does there exist a sequence (u_n) [a net (u_σ)], $u_n \in \mathscr{L}(E, L_n)$ $[u_\sigma \in \mathscr{L}(E, L_\sigma)]$, such that $u_n(x) \in B_n(x)$ for each n $[u_\sigma(x) \in B_\sigma(x)$ for each $\sigma \in \Sigma]$? Similarly for L^n and L^σ. What about uniqueness?

We will examine these questions by means of the notions of monotone sequences (§ 2) and orthogonal sequences (§ 5). This latter notion is closely related to the notion of an (unconditional) Schauder basis, which we now define. A system (x_i, f_i), $(x_i) \subset E$, $(f_i) \subset E^*$ is *biorthogonal* provided $f_i(x_j) = \delta_{ij}$. If (x_i, f_i) is a biorthogonal system with (x_i) fundamental in E, then (x_i, f_i) is a *Schauder basis for E* provided, for each $x \in E$,

$$x = \sum_{i=1}^{\infty} f_i(x) x_i. \tag{1}$$

We say that (x_i) is a *basis with coefficient functionals* (f_i).

A sequence (x_i) is *basic* in E if (x_i) is a basis for $[x_i : i \in \omega]$. Moreover, a basis (x_i) is *unconditional* provided the convergence of (1) is unconditional for each $x \in E$.

The following internal characterization of basic sequences in E will be referred to throughout this paper as the *Grinblyum K-condition*:

A sequence (x_n) is basic in E [an unconditional basic sequence in E] provided there is a K such that for every $p \leqslant q$ $[\sigma \subset \sigma', \sigma' \in \Sigma]$ and arbitrary scalars $a_1 \dots a_q$ $[(a_i)_{i \in \sigma'}]$

$$\Big\| \sum_{n=1}^{p} a_i x_i \Big\| \leqslant K \Big\| \sum_{n=1}^{q} a_i x_i \Big\| \qquad \Big[\Big\| \sum_{i \in \sigma} a_i x_i \Big\| \leqslant K \Big\| \sum_{i \in \sigma'} a_i x_i \Big\|\Big].$$

Associated with a basis [unconditional basis] (x_n, f_n) are the operators S_n and R_n $[S_\sigma$ and $R_\sigma]$ given by

$$S_n(x) = \sum_{i=1}^{n} f_i(x) x_i \quad \text{and} \quad R_n(x) = x - S_n(x),$$

$$\Big[S_\sigma(x) = \sum_{i \in \sigma} f_i(x) x_i \quad \text{and} \quad R_\sigma(x) = x - S_\sigma(x)\Big].$$

In § 3 we examine the fundamental work of Nikol'skiĭ [9] (see also [16]) Our proofs are new and simpler.

In § 4, following the expository paper [15] we give some characterizations of bases in terms of best approximation.

Finally, in § 6, we consider the notions discussed in the first five sections in the space $C[0, 1]$.

Except for the material discussed in the introduction and the results of § 6 the paper is self-contained.

The author is indebted to Professor I. Singer for many long conversations concerning the material in this paper and to Professor R. C. James for the pleasant collaboration on the paper [13].

2. Monotone sequences. Following [15] we say that a sequence (x_i) in a Banach space E is (2.1)

(a) *monotone* provided $\|\sum_{i=1}^{n} a_i x_i\| \leqslant \|\sum_{i=1}^{n+1} a_i x_i\|$ for all n and all choices of scalars $a_1, \ldots, a_{n+1}$;

(b) *strictly monotone* provided strict inequality holds in (a) whenever $a_{n+1} \neq 0$;

(c) *co-monotone* provided $\|\sum_{i=n}^{\infty} a_i x_i\| \leqslant \|\sum_{i=n-1}^{\infty} a_i x_i\|$ whenever $\sum_{i=1}^{\infty} a_i x_i$ converges; and,

(d) *strictly co-monotone* provided strict inequality holds in (c) whenever $a_{n-1} \neq 0$.

It is clear that the above conditions on a sequence (x_n) in E are rather strong. Indeed,

(2.2) PROPOSITION. *If* (x_i) *is a sequence in* E, $x_n \neq 0$ *for each* n *and* $[x_n : n \in \omega] = E$, *satisfying any of* (2.1), (a)-(d), *then* (x_n) *is a Schauder basis for* E.

Proof. We show that in all cases (x_i) satisfies Grinblyum's K-condition.

Of course, this is immediate if (x_n) satisfies (a) or (b). If (x_n) satisfies (c) or (d), then

$$\Big\|\sum_{i=p+1}^{q} a_i x_i\Big\| \leqslant \Big\|\sum_{i=p}^{q} a_i x_i\Big\| \leqslant \ldots \leqslant \Big\|\sum_{i=1}^{q} a_i x_i\Big\|.$$

Thus

$$\Big\|\sum_{i=1}^{p} a_i x_i\Big\| = \Big\|\sum_{i=1}^{q} a_i x_i - \sum_{i=p+1}^{q} a_i x_i\Big\| \leqslant 2\Big\|\sum_{i=1}^{q} a_i x_i\Big\|$$

proving (2.2).

Obviously, we could consider basic sequences and delete the hypothesis that (x_n) is fundamental. We choose not to do this, however, and so for the remainder of this work we assume that (x_n) is a basis for E.

(2.3). Example. (i) The unit vector basis (e_n) of (c_0) is both monotone and co-monotone, but strict in neither case.

(ii) The unit vector basis of l^1 is both strictly monotone and strictly co-monotone.

Of course, it is easy to construct bases that satisfy none of the conditions of (2.1).

(2.4) Example. Let (e_n) denote the unit vector basis of (c_0) and consider the following basis: $x_1 = e_1$, $x_2 = -\frac{1}{2}e_1+\frac{1}{2}e_2$, $x_3 = -\frac{1}{2}e_1 - \frac{1}{2}e_2+\frac{1}{2}e_3$ and $x_n = e_n$, $n \geqslant 4$. With respect to the usual norm on (c_0), (x_n) is neither monotone nor co-monotone. Indeed, $\|x_1\| = 1$, $\|x_1+x_2\| = \frac{1}{2}$ and so (x_n) is not monotone; and, $\|x_1+x_2+x_3\| = \frac{1}{2}$, $\|x_2+x_3\| = 1$ and so (x_n) is not co-monotone.

In [18] an example is given in l^1 showing the above. However, there is an error in this example. It is easy to check that the sequence given in [18] is not a basis for l^1.

A natural question arises:

Can a basis be (strictly) monotone without being (strictly) co-monotone? Conversely?

(2.5) Example. Consider the space $(bv_0) = \{x = (x_i) \in (c_0): \sum_{i=1}^{\infty} |x_i - x_{i+1}| < +\infty\}$ with norm given by

$$\|(x_i)\| = \sum_{i=1}^{\infty} |x_i - x_{i+1}| + \sum_{i=1}^{\infty} |x_i| 2^{-i}.$$

It is easy to check that the unit vector basis (e_n) is strictly monotone with respect to this norm. But $\|e_2+e_3\| = 2+2^{-2}+2^{-3}$ and $\|e_1+e_2+e_3\| = 1+2^{-2}+2^{-2}+2^{-3}$ and so (e_n) is not co-monotone.

(2.6) Example. Consider (c_0) but with norm given by

$$\|(x_i)\| = \sup_n \left\{ \frac{1}{n} \left[\sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |x_i - x_{i+1}| \right] + \sup_{m \geqslant n} |x_m| \right\}.$$

One can show by an argument similar to that given in § 3, Theorem (3.3), that the unit vector basis is strictly co-monotone with respect to this norm. However, $\|e_1\| = 3$ and $\|e_1+e_2\| = 2+2^{-1}$ and so (e_n) is not strictly monotone.

We now prove the fundamental theorem (cf. [18]).

(2.7) Theorem. *Let E be a Banach space with a basis (x_i, f_i). Then*

(a) *(x_i) is monotone if and only if $R_n(x) \in B^n(x)$ for all n and all $x \in E$;*

(b) *(x_i) is strictly monotone if and only if $\{R_n(x)\} = B^n(x)$ for all n and all $x \in E$;*

(c) *(x_i) is co-monotone if and only $S_n(x) \in B_n(x)$ for all n and all $x \in E$; and*

(d) (x_i) *is strictly co-monotone if and only if* $\{S_n(x)\} = B_n(x)$ *for all* n *and all* $x \in E$.

Proof. We prove (a) and (b); the others are similar.

(a) Let

$$x = \sum_{i=1}^{\infty} f_i(x) x_i \in E \quad \text{and} \quad y = \sum_{i=n+1}^{\infty} b_i x_i \in L^n.$$

If (x_i) is monotone, then

$$\|x - R_n(x)\| = \|S_n(x)\| \leqslant \|S_n(x) + [f_{n+1}(x) - b_{n+1}] x_{n+1}\| \leqslant \ldots \leqslant \|x - y\|,$$

i.e.

$$R_n(x) \in B^n(x).$$

Conversely, if $R_n(x) \in B^n(x)$ for all n and x, then, since $L^{n-1} \supset L^n$,

$$\|S_{n-1}(x)\| = \|x - R_{n-1}(x)\| \leqslant \|x - R_n(x)\| = \|S_n(x)\|$$

and it follows that (x_i) is monotone.

(b) If (x_i) is strictly monotone, then by what was just proved $R_n(x) \in B^n(x)$ for all n and x . Suppose there is an n and x and $y_0 \in B^n(x)$ with $y_0 \neq R_n(x)$. Since

$$R_n(x) - y_0 = \sum_{i=n+1}^{\infty} f_i(x - y_0) x_i \neq 0,$$

there is an $N \geqslant n+1$ such that

$$\sum_{n=1}^{N} |f_i(x - y_0)| \neq 0.$$

Let

$$z = S_n(x) + \sum_{i=n+1}^{\infty} f_i(x - y_0) x_i.$$

Then

$$\|S_n(x)\| = \|S_n(z)\| < \|S_N(z)\| \leqslant \ldots \leqslant \|z\| = \|x - y_0\| = \|x - R_n(x)\| = \|S_n(x)\|.$$

This contradiction shows that $\{R_n(x)\} = B^n(x)$ for all n and x.

Conversely, if $\{R_n(x)\} = B^n(x)$ for all n and x, then, again by the above, (x_i) is monotone. If (x_i) is not strictly monotone, then there is an integer N and scalars $a_1, \ldots, a_{N+1} \neq 0$, such that

$$\Big\| \sum_{i=1}^{N} a_i x_i \Big\| = \Big\| \sum_{i=1}^{N+1} a_i x_i \Big\|.$$

If $x = \sum_{i=1}^{N} a_i x_i$, then $R_N(x) = 0$ and so, by hypothesis, $B^N(x) = \{0\}$.

In particular, $\inf\{\|x-y\|: y \in L^N\} = \|x\|$. But $\|x\|$ is attained for $y - a_{N+1}x_{N+1} \in L^N$ and so $y = 0$, i.e. $a_{N+1} = 0$.

The interpretation of examples (2.3)-(2.6) in terms of theorem (2.7) is immediate.

It is well known that if (x_i) is a basis for a Banach space E and (f_i) its sequence of coefficients functionals, then (f_i) is a basis for $[f_i: i \in \omega]$. In view of (2.7) the following duality theorem is interesting:

(2.8) THEOREM. *If (x_n, f_n) is a basis for a Banach space E and (x_n) is monotone, then (f_n) is monotone. The converse is false.*

Proof. If (x_n) is monotone, then clearly $\|S_n(x)\| \leqslant \|x\|$ for every $x \in E$ and for every n. Thus for $\|x\| \leqslant 1$ and scalars $a_1, \ldots, a_{n+1}$ we have

$$\Big\|\sum_{i=1}^{n+1} a_i f_i\Big\| \geqslant \Big|\sum_{i=1}^{n+1} a_i f_i\big(S_n(x)\big)\Big| = \Big|\sum_{i=1}^{n} a_i f_i(x)\Big|.$$

Since this is true for each x in the unit ball of E, it follows that

$$\Big\|\sum_{i=1}^{n} a_i f_i\Big\| \leqslant \Big\|\sum_{i=1}^{n+1} a_i f_i\Big\|.$$

To see that the converse is false consider the following basis for (c_0):

$$x_n = \sum_{i=1}^{n} (-1)^{n+i} e_i, \qquad f_n = e_n + e_{n+1},$$

where e_n denotes the n^{th} unit vector in (c_0) and l^1, respectively (this basis was constructed by B. R. Gelbaum for a different purpose).

Now $1 = \|x_1\| > \frac{1}{2} = \|x_1 + \frac{1}{2}x_2\|$ and so (x_n) is not monotone. However, (f_n) is a monotone basic sequence in l^1. To see this observe that for scalars a and b,

$$\|af_1\| = 2|a| \leqslant |a| + |a+b| + |b| = \|af_1 + bf_2\|.$$

That (f_n) is monotone follows by induction.

The above example shows that the assertion of proposition 2.3, Part I, of [15] is not valid in general.

3. K-, T- and KT-norms. In this section we present the fundamental work of V. N. Nikol'skiĭ. The proofs, however, are new and because of (2.7) simpler than Nikol'skiĭ's original proofs (see [9] and [16]).

(3.1) Definition. Let E be a Banach space with a basis (x_n). The norm on E is said to be

(i) a *T-norm* (*Čebyšev norm*) *with respect to* (x_n) if and only if $B_n(x) = \{S_n(x)\}$ for all n and all $x \in E$;

(ii) a *K-norm* (*Canonical norm*) *with respect to* (x_n) if and only if $B^n(x) = \{R_n(x)\}$ for all n and all $x \in E$; and

(iii) a *KT-norm with respect to* (x_n) if it is simultaneously a T-norm and a K-norm.

Nikol'skiĭ [8], [9] has shown that in any Banach space with a basis one can introduce K-, T- and KT-norms equivalent to the original norm of E. To show that B_n and B^n had the desired form, Nikol'skiĭ used the rather laborious arguments referred to above. In view of (2.7) we will need only show that the norms introduced below are strictly monotone, strictly co-monotone or both.

The Nikol'skiĭ norms. If (x_i, f_i) is a basis for a Banach space E with norm $\|\cdot\|$ we introduce the following norms in E:

$$\|x\|_K = \sup_n \Big\{\Big\|\sum_{i=1}^{n} f_i(x)x_i\Big\|\Big\} + \sum_{i=1}^{\infty} \|f_i(x)x_i\| 2^{-i},$$

$$\|x\|_T = \sup_n \Big\{\frac{1}{n}\sum_{i=1}^{n} \|f_i(x)x_i\| + \Big\|\sum_{i=n+1}^{\infty} f_i(x)x_i\Big\|\Big\},$$

$$\|x\|_{KT} = \sup_n \Big\{\Big\|\sum_{1}^{n} f_i(x)x_i\Big\|_T + \Big\|\sum_{i=n+1}^{\infty} f_i(x)x_i\Big\|_K\Big\}.$$

(3.2) THEOREM. *The norms* $\|\cdot\|_K, \|\cdot\|_T$ *and* $\|\cdot\|_{KT}$ *are all equivalent to* $\|\cdot\|$.

Proof. It is clear that all three expressions are indeed norms. Let K be the number guaranteed by Grinblyum's K-condition. First observe that

$$\|f_n(x)x_n\| = \|x-(S_{N-1})(x)+R_n(x)\| < 2(K+1)\|x\|$$

since $\|S_{n-1}(x)\| \leqslant K\|x\|$ and $\|R_n(x)\| \leqslant (K+1)\|x\|$ for all n and x.
Thus

$$\|x\|_T = \sup_n \Big\{\frac{1}{n}\sum_{i=1}^{n} \|f_i(x)x_i\| + \|R_n(x)\|\Big\} \leqslant 3(K+1)\|x\|.$$

Letting $n = 1$ we see that $\|x\|_T \geqslant \|x\|$ and so $\|\cdot\|_T$ is equivalent to $\|\cdot\|$.
Now

$$\|x\|_K = \sup_n \{\|S_n(x)\|\} + \sum_{i=1}^{\infty} \frac{\|f_i(x)x_i\|}{2^i} \leqslant (3K+2)\|x\|$$

and by definition $\|x\| \leqslant \|x\|_K$. Hence $\|\cdot\|_K$ is equivalent to $\|\cdot\|$.

From the above if follows that $\|x\|_{KT} \leqslant 3K(K+1)(K+2)\|x\|$ and, by definition, $\|x\| \leqslant \|x\|_{KT}$. Hence $\|\cdot\|_{KT}$ is equivalent to $\|\cdot\|$.

(3.3) THEOREM. *The basis* (x_n) *is*

(a) *strictly monotone with respect to* $\|\cdot\|_K$;

(b) *strictly co-monotone with respect to* $\|\cdot\|_T$; *and,*

(c) *both strictly monotone and strictly co-monotone with respect to* $\|\cdot\|_{KT}$.

Proof. The proof of (a) is trivial and (c) will follow once we prove (b). To prove (b) observe that the supremum in the definition of $\|\cdot\|_T$ is actually a maximum. Thus for (a_i) such that $\sum_{i=1}^{\infty} a_i x_i$ converges and $a_{n-1} \neq 0$ there are integers n_0 and m_0 such that

$$\left\|\sum_{i=n-1}^{\infty} a_i x_i\right\|_T = \frac{1}{m_0}\sum_{i=n-1}^{m_0} \|a_i x_i\| + \left\|\sum_{i=m_0+1}^{\infty} a_i x_i\right\|,$$

and

$$\left\|\sum_{i=n}^{\infty} a_i x_i\right\|_T = \frac{1}{n_0}\sum_{i=n}^{n_0} \|a_i x_i\| + \left\|\sum_{i=n_0+1}^{\infty} a_i x_i\right\|.$$

Thus,

$$\begin{aligned}\left\|\sum_{n-1}^{\infty} a_i x_i\right\|_T &\geqslant \frac{1}{n_0}\sum_{i=n-1}^{n_0} \|a_i x_i\| + \left\|\sum_{i=n_0+1}^{\infty} a_i x_i\right\| \\ &> \frac{1}{n_0}\sum_{i=n}^{n_0} \|a_i x_i\| + \left\|\sum_{i=n_0+1}^{\infty} a_i x_i\right\| = \left\|\sum_{i=n}^{\infty} a_i x_i\right\|_T,\end{aligned}$$

with "greater than" holding since $a_{n-1} \neq 0$.

Singer [17] has recently introduced the following T- and KT-norms for a Banach space E with basis (x_i, f_i) (let us remark that the KT-norm was given earlier by Vaniček [18] and by the author in his dissertation):

$$|x|_T = \sup_n \left\|\sum_{i=n}^{\infty} f_i(x) x_i\right\| + \sum_{i=1}^{\infty} \|f_i(x) x_i\| 2^{-i}$$

and

$$|x|_{KT} = \sup_{m,n} \left\|\sum_{i=m}^{n} f_i(x) x_i\right\| + \sum_{i=1}^{\infty} \|f_i(x) x_i\| 2^{-i}.$$

We omit the proofs, which are similar to (3.2) and (3.3), that these norms have the asserted properties. These norms are simpler for some applications than the norms of Nikol'skiĭ.

Let us also mention that if $\|\cdot\|$ is a K-, T- or KT-norm with respect to a basis (x_n, f_n) and if one introduces the norm of Day [1],

$$|x| = \left[\|x\|^2 + \sum_{i=1}^{\infty} \|f_i(x) x_i\|^2 2^{-2i}\right]^{1/2},$$

then one obtains a strictly convex K-, T- or KT-norm. This fact was observed by Istrăţescu [6].

For a sequence (y_n) in a Banach space E let $P_n = [y_i : i \leqslant n]$ and let $P = \bigcup_{n=1}^{\infty} P_n$.

For $p \in P$, $p = \sum_{i=1}^{n} a_i y_i$ let

$$S_m(p) = \begin{cases} \sum_{i=1}^{m} a_i y_i & \text{if } m < n, \\ p & \text{if } m \geqslant n \end{cases}$$

and

$$R_m(p) = p - S_m(p).$$

We return to the work of Nikol'skiĭ.

(3.4) Definition. The norm $\|\cdot\|$ of E is a

(i) *weak T-norm relative to (y_n)* provided that for each "polynomial" $p \in P$, $p = \sum_{i=1}^{n} a_i y_i$ and each $m \leqslant n$, the polynomial $\sum_{i=1}^{m} a_i y_i$ is a best approximation to p from $[y_i : i \leqslant m]$;

(ii) *weak K-norm relative to (y_n)* provided that for each $p \in P$, $p = \sum_{i=1}^{n} a_i y_i$ and each $m \leqslant n$ the "complementary polynomial" $\sum_{i=m+1}^{n} a_i y_i$ is a best approximation to p from $[y_i : m+1 \leqslant i \leqslant n]$; and,

(iii) *weak KT-norm relative to (y_n)* if it is simultaneously a weak K-norm and a weak T-norm relative to (y_n).

It is clear that if (x_n) is a basis for E, then a K-, T- or KT-norm with respect to (x_n) is a weak K-, weak T-, or weak KT-norm relative to (x_n). Example (2.3), (i), shows that the converse is false.

(3.5) THEOREM. *Let (y_n) be a non-zero sequence in a Banach space E with norm $\|\cdot\|$. Then*

(a) *For the norm to be a weak T-norm relative to (y_n) it is necessary and sufficient that*

(a₁) $$\sup_n \sup \{\|R_n(p)\| : p \in P, \|p\| \leqslant 1\} = 1;$$

(b) *For the norm to be a weak K-norm relative to (y_n) it is necessary and sufficient that*

(b₁) $$\sup_n \sup \{\|S_n(p)\| : p \in P, \|p\| \leqslant 1\} = 1;$$

and,

(c) *For the norm to be a weak KT-norm relative to (y_n) it is necessary and sufficient that*

(c₁) $$\operatorname{Max}[(a_1), (b_1)] = 1.$$

Proof. The proof of (c) follows trivially from (a) and (b). Since the proofs of (a) and (b) are similar, we prove (a). Suppose

$$p = \sum_{i=1}^{n} a_i y_i \in P$$

and (a_1) holds. Let

$$\gamma = \sum_{i=1}^{m} b_i y_i \in P_m.$$

If $\|p-\gamma\| \neq 0$, let $p' = \|p-\gamma\|^{-1}(p-\gamma)$. From ($a_1$) it follows that $\|p'-S_m(p')\| \leqslant 1$ and so $\|(p-\gamma)-S_m(p-\gamma)\| \leqslant \|p-\gamma\|$. Since $S_m(\gamma) = \gamma$, we have $\|p-S_m(p)\| \leqslant \|p-\gamma\|$ and $S_m(p)$ is a best approximation to p from $[y_i: i \leqslant m]$. If $\|p-\gamma\| = 0$, then $\gamma = p = S_m(p)$ and the result is trivial.

Conversely, if $S_m(p)$ is a best approximation to $p = \sum_{i=1}^{n} a_i y_i$ for $m \leqslant n$, then for $\|p\| \leqslant 1$ we have $\|R_m(p)\| = \|p-S_m(p)\| \leqslant \|p\| \leqslant 1$; since we are working only with finite sums, we can clearly find $p \in P$ and n such that $\|R_n(p)\|$ is near 1 as we please. Thus (a_1) holds.

There is an analog of (2.7) for weak T- and weak K- norms. We state the theorem for weak T-norms. The result for weak K-norms is then obvious.

(3.6) THEOREM. *For a norm to be a weak T-norm relative to (y_n) it is necessary and sufficient that*

$$\Big\|\sum_{i=m}^{n} a_i y_i\Big\| \leqslant \Big\|\sum_{i=m-1}^{n} a_i y_i\Big\|$$

for arbitrary scalars $a_{m-1}, \ldots, a_n$, $a_{m-1} \neq 0$ and arbitrary m, n.

Moreover, if the above inequality is strict for $a_{m-1} \neq 0$, then for $p = \sum_{i=1}^{n} a_i y_i \in P$, best approximation to $[y_i: i \leqslant m]$ is unique for all $m \leqslant n$.

The proof is quite similar to that of (2.7).

In a discussion with I. Singer the following questions arose:

(1) Is the condition

$$\Big\|\sum_{i=m}^{n} a_i x_i\Big\| < \Big\|\sum_{i=m-1}^{n} a_i x_i\Big\| \qquad (\text{all } a_{m-1}, \ldots, a_n \text{ with } a_{m-1} \neq 0)$$

sufficient in order that the norm be a T-norm?

(2) Is the condition

$$\Big\|\sum_{i=m}^{n} a_i x_i\Big\| < \Big\|\sum_{i=m-1}^{n+1} a_i x_i\Big\| \qquad (\text{all } a_{m-1}, \ldots, a_{n+1} \text{ with } |a_{m-1}| + |a_{n+1}| \neq 0)$$

sufficient for the norm to be KT-norm?

It is somewhat surprising that the answer to both (1) and (2) is *no*. We postpone the example until § 5.

4. Characterization of bases in terms of best approximation. In this section we assume that (x_n) is a sequence in a Banach space E with $[x_n: n \in \omega] = E$ and $x_n \neq 0$ for all n.

Again, we let $L_n = [x_i: i \leqslant n]$ and $\mu_n(x) = \inf\{\|x-z\|: z \in L_n\}$.

Our first theorem is due to Nikol'skiĭ:

(4.1) THEOREM. *The following statements about (x_n) are equivalent:*

(i) *(x_n) is a basis for E;*

(ii) *one can introduce a weak K-norm relative to (x_n) equivalent to the original norm;*

(iii) *one can introduce a weak T-norm relative to (x_n) equivalent to the original norm; and,*

(iv) *one can introduce a weak KT-norm relative to (x_n) equivalent to the original norm.*

Proof. That (i) implies the other three properties has been observed in the stronger form (3.2) and (3.3). That (iv) implies (i) is trivial. One has to show that (ii) implies (i) and (iii) implies (i). Since the proofs of those latter two implications are similar, we prove that (ii) implies (i). Suppose $p \leqslant q$, $\sum_{i=1}^{q} a_i x_i \neq 0$; then by (3.5)(b)

$$\Big\| S_p\Big(\Big\|\sum_{i=1}^{q} a_i x_i\Big\|^{-1} \sum_{i=1}^{q} a_i x_i\Big)\Big\| \leqslant 1,$$

i.e.

$$\Big\|\sum_{i=1}^{p} a_i x_i\Big\| \leqslant \Big\|\sum_{i=1}^{q} a_i x_i\Big\|.$$

If $\sum_{i=1}^{q} a_i x_i = 0$, then, since the norm is a weak K-norm,

$$\Big\|\sum_{i=1}^{p} a_i x_i\Big\| = \Big\|\sum_{i=1}^{q} a_i x_i - \sum_{i=p+1}^{q} a_i x_i\Big\| \leqslant \Big\|\sum_{i=1}^{q} a_i x_i\Big\| = 0.$$

Thus Grinblyum's K-condition is satisfied with $K = 1$.

(4.2). THEOREM. *The following statements about (x_n) are equivalent:*

(i) *(x_n) is a basis for E;*

(ii) *there is a constant $C \geqslant 1$ such that*

$$\|p - S_n(p)\| \leqslant C\mu_n(p)$$

for all $p \in P$.

Moreover, if (i) *and* (ii) *hold, then* $\|x - S_n(x)\| \leqslant C\mu_n(x)$ *for all* $x \in E$.

Proof. If (x_n) is a basis, then by (2.7) and (3.3) we may renorm E so that with respect to this new norm, $|||\cdot|||$, $B_n(x) = \{S_n(x)\}$ for all $x \in E$ and for all n. Let C be such that $\|x\| \leqslant |||x||| \leqslant C\|x\|$. Then

$$\|x - S_n(x)\| \leqslant |||x - S_n(x)||| = \inf\{|||x-z|||: z \in L_n\} \leqslant C\mu_n(x).$$

Thus (i) implies (the strong form of) (ii).

Conversely, if (ii) holds and $p = \sum_{i=1}^{q} a_i x_i \in P$, then for $n \leqslant q$,

$$\|S_n(p)\| \leqslant \|p - S_n(p)\| + \|p\| \leqslant C\mu_n(p) + \|p\| \leqslant (C+1)\|p\|,$$

i.e., Grinblyum's K-condition holds with $K = C+1$.

Let us recall the following simple fact:

(4.3) Proposition. *If* (x_n) *is a fundamental sequence in* E, *then* $\lim_{n\to\infty} \mu_n(x) = 0$.

Proof. Clearly, $\mu_n(x) \geqslant \mu_{n+1}(x) \geqslant 0$ and so $(\mu_n(x))$ converges for each $x \in E$. Since (x_n) is fundamental in E, there is a $y_{m_n} \in L_{m_n}$ such that $\lim_{n\to\infty}\|x - y_{m_n}\| = 0$. Since $0 \leqslant \mu_{m_n}(x) \leqslant \|x - y_{m_n}\|$, it follows that $\lim_{n\to\infty}\mu_n(x) = 0$.

Let us place for the remainder of this section an additional assumption on (x_n); namely, there exists a sequence $(f_n) \subset E^*$ such that $f_n(x_m) = \delta_{nm}$. Now

$$S_n(x) = \sum_{i=1}^{n} f_i(x)x_i.$$

(4.4) Theorem. *The sequence* (x_i) *is a basis for* E *if and only if*

$$\lim_{n\to\infty}\|S_n\|\mu_n(x) = 0 \quad \textit{for every } x \in E.$$

Proof. If (x_n) is a basis for E, then there is a K such that $\sup_n\|S_n\| = K < +\infty$. Thus $0 \leqslant \|S_n\|\mu_n(x) \leqslant K\mu_n(x)$ and the latter tends to 0 by (4.3).

For the converse, let $z_n \in B_n(x)$. Then $\|z_n - S_n(x)\| = \|S_n(z_n - x)\| \leqslant \|S_n\|\mu_n(x)$. Thus,

$$\|x - S_n(x)\| \leqslant \|x - z_n\| + \|S_n\|\mu_n(x) \leqslant [\|S_n\|+1]\mu_n(x)$$

and the latter expression tends to zero by (4.3) and the hypothesis.

Theorems 4.2 and 4.4 are stated in [15].

We conclude § 4 with a theorem of Foguel [2]. Let

$$|||x||| = \sup_n\Big\|\sum_{i=1}^{n} f_i(x)x_i\Big\| = \sup_n\|S_n(x)\|.$$

Of course, this may be infinite for some $x \in E$. An equivalent formulation of the Grinblyum K-condition is that (x_i, f_i) is a basis if and only if $|||x||| < +\infty$ for each $x \in E$. Foguel's theorem weakens this requirement in the following way:

(4.5) THEOREM. *The biorthogonal sequence* (x_i, f_i) *is a basis for* E *if and only if for each* $x \in E$ *there is a* $z_n \in B_n(x)$ *such that* $(|||z_n|||)_{n=1}^{\infty}$ *is bounded.*

The proof of (4.5) is an application of the Baire category theorem. We first prove a lemma.

(4.6) LEMMA. *Let* $S \subset E$ *be such that each* $x \in S$ *is the limit of a sequence* (y_n) *and* $(|||y_n|||)$ *is bounded. If* (x_n, f_n) *is not a basis for* E, *then* S *is a first category set.*

Proof. Let $E_0 = \{x \in E: |||x||| = +\infty\}$. Then

$$E_0 = E \setminus \bigcup_{K=1}^{\infty} \bigcap_{n=1}^{\infty} \{x \in E: \|S_n(x)\| \leqslant K\}.$$

If (x_n, f_n) is not a basis for E, then (using the Banach-Steinhaus theorem) E_0 is dense and second category in E.

Let

$$v_n(x) = \frac{|||S_n(x)|||}{1 + |||S_n(x)|||}.$$

Then $0 \leqslant v_1(x) \leqslant v_2(x) \leqslant \ldots \leqslant 1$. If $v(x) = \lim_{n\to\infty} v_n(x)$ and $x \in E_0$, then $v(x) = 1$. Since E_0 is dense in E, $v(x) = 1$ at every point of continuity of v. It follows that if x is a point of continuity of v and $\lim_{n\to\infty} y_n = x$, then $\lim_{n\to\infty} v(y_n) = v(x) = 1$ and so $(|||y_n|||)$ is unbounded. Clearly, S is a subset of the set of points of discontinuity of v, which is a first category set.

Proof of (4.5). If (x_i, f_i) is a basis, then by Grinblyum's K-condition there is a K such that $|||x||| \leqslant K\|x\|$. If $z_n \in B_n(x)$, then $|||z_n||| \leqslant K\|z_n\| \leqslant K\{\|x\| + \|x - z_n\|\} \leqslant 2K\|x\|$.

Conversely, suppose $x \in E$ and $z_n \in B_n(x)$ with $(|||z_n|||)$ bounded. By (4.3), $\lim_{n\to\infty} \|x - z_n\| = 0$ and so $x \in S$. Thus $S = E$ and, by (4.6), (x_i, f_i) is a basis for E.

5. Orthogonal sequences; NK-, NT- and NKT-norms. Following [14] we say that

(5.1) a sequence (x_i) in a Banach space E is

(a) *orthogonal* provided $\|\sum_{i \in \alpha} a_i x_i\| \leqslant \|\sum_{i \in \beta} a_i x_i\|$ for arbitrary $\alpha, \beta, \alpha \subset \beta$, $\beta \in \Sigma$ and arbitrary scalars $(a_i)_{i \in \beta}$;

(b) *strictly orthogonal* provided the inequality of (a) is strict whenever $\sum_{i\in\beta\setminus\alpha} |a_i| \neq 0$;

(c) *co-orthogonal* provided $\|\sum_{i\in\omega\setminus\beta} a_i x_i\| \leqslant \|\sum_{i\in\omega\setminus\alpha} a_i x_i\|$ for arbitrary α, β, $\alpha \subset \beta$, $\beta \in \Sigma$ and arbitrary scalars (a_i) for which $\sum_{i\in\omega} a_i x_i$ converges;

(d) *strictly co-orthogonal* provided the inequality of (c) is strict whenever $\sum_{i\in\beta\setminus\alpha} |a_i| \neq 0$.

There is a theory for orthogonal sequences analogous to that developped in § 2, 3 and 4. Because of the similarity of proofs we will only state results.

(5.2) Remarks. (i) A sequence (x_n), $x_n \neq 0$, $[x_n : n \in \omega] = E$, satisfying any of (5.1), (a)-(d), is an unconditional basis for E.

(ii) The unit vector basis of c_0 satisfies (a) and (c) but not (b) and (d). The unit vector basis of l^1 satisfies (b) and (d).

In § 2 examples were given showing that the notions of strictly monotone and strictly co-monotone are completely different.

This is not quite the case with orthogonal sequences.

(5.3) Proposition. (i) *The notions of orthogonal and co-orthogonal sequences are equivalent.*

(ii) *A (strictly) orthogonal sequence is co-orthogonal.*

(iii) *A strictly co-orthogonal sequence is strictly orthogonal.*

Proof. (i) It is clear that a co-orthogonal sequence is orthogonal; the converse follows by passing to the limit.

(ii) and (iii) are immediate.

That the converse of (iii) is not valid is much more difficult. The example given below was first given in [13].

(5.4) Example. Let E have the same members as (c_0) but with norm given by

$$\|(x_1, x_2, \ldots)\| = \sup |x_1| n^{-1} 2^{-n} + \sum_{\substack{i=2 \\ i\neq n}}^{\infty} |x_{p_i}| 2^{-i},$$

where the sup is for all $n \geqslant 2$ and all permutations (p_i) of $\omega \setminus \{1, n\}$. If $|x|$ denotes the usual supremum norm of (c_0), then it is easy to see that

$$\frac{1}{8} |x| \leqslant \|x\| \leqslant \frac{3}{8} \|x\|$$

and so E is isomorphic to (c_0).

1. The unit vector basis (e_i) of E is strictly orthogonal. To see this, let $\beta \in \Sigma$, $\alpha \subset \beta$. The norm of $\sum_{i\in\alpha} a_i e_i$ is attained for a particular value of

n and a particular permutation (p_i). A larger sum must be attained when this same n and permutation (p_i) are used for $\sum_{i\in\beta} a_i e_i$, provided $\sum_{i\in\beta\setminus\alpha} |a_i| \neq 0$.

2. $\left\| \sum_{m=1}^{\infty} \frac{1}{m} e_m \right\| = \left\| \sum_{m=2}^{\infty} \frac{1}{m} e_m \right\| = \sum_{m=2}^{\infty} \frac{1}{m} 2^{-m}$ and so (e_i) is not strictly co-orthogonal. By definition

$$\left\| \sum_{m=1}^{\infty} \frac{1}{m} e_m \right\| \geqslant \left\| \sum_{m=2}^{\infty} \frac{1}{m} e_m \right\|.$$

Also,

$$\left\| \sum_{m=2}^{\infty} \frac{1}{m} e_m \right\| = \sup \sum_{\substack{m=2\\ m\neq n}}^{\infty} \frac{1}{p_m} 2^{-m} = \sum_{m=2}^{\infty} \frac{1}{m} 2^{-m}.$$

Since

$$\frac{1}{n} 2^{-n} + \sum_{\substack{m=2\\ m\neq n}}^{\infty} \frac{1}{p_m} 2^{-m} \leqslant \frac{1}{n} 2^{-n} + \sum_{\substack{m=2\\ m\neq n}}^{\infty} \frac{1}{m} 2^{-m} = \sum_{m=2}^{\infty} \frac{1}{m} 2^{-m},$$

the result follows. Example (5.4) is the promised counter-example to (1) and (2) of § 3.

We now give the definition analogous to (3.1).

(5.5) Definition. Let E be a Banach space with an unconditional basis (x_n). The norm on E is said to be

(i) an *NT-norm with respect to* (x_n) if and only if $B_\sigma(x) = \{S_\sigma(x)\}$ for all $\sigma\in\Sigma$ and $x\in E$;

(ii) an *NK-norm with respect to* (x_n) if and only if $B^\sigma(x) = \{R_\sigma(x)\}$ for all $\sigma\in\Sigma$ and $x\in E$, and

(iii) an *NKT-norm with respect to* (x_n) if and only if it is simultaneously an *NT*- and *NK*-norm.

The above notions were introduced by Singer [14].

The theorem analogous to (2.7) is valid.

(5.6) Theorem. *Let E be a Banach space with an unconditional basis (x_n). Then norm on E is*

(i) *an NT-norm with respect to (x_n) if and only if (x_n) is strictly co-orthogonal;*

(ii) *an NK-norm with respect to (x_n) if and only if (x_n) is strictly orthogonal; and*

(iii) *an NKT-norm with respect to (x_n) if and only if (x_n) is both strictly orthogonal and strictly co-orthogonal.*

Thus by (5.3), (iii), and (5.6) we see that an NT-norm is always an NK-norm and example (5.4) shows that there are NK-norms which are not NT.

Let σ be any collection of positive integers (not necessarily finite) and for an unconditional basis (x_n, f_n) for E let $S_\sigma(x) = \sum_{i\epsilon\sigma} f_i(x)x_i$. We now show that it is always possible to give E an equivalent norm making $S_\sigma(x)$ the unique best approximation to x from $[x_i: i\epsilon\sigma]$. This construction was first given in [12].

If $\|\cdot\|$ is the original norm on E let

$$|x| = \sup\Big\{\sum_{i=1}^{\infty} |f_i(x)f(x_i)|: f\epsilon E^*,\ \|f\|\leqslant 1\Big\} + \sum_{i=1}^{\infty} \|f_i(x)x_i\| 2^{-i}.$$

(5.7) THEOREM. *Let K be a constant guaranteed by the unconditional form of Grinblyum's K-condition. Then for any $x\epsilon E$*

$$\|x\| \leqslant |x| \leqslant 6K\|x\|$$

and so $|\cdot|$ is equivalent to $\|\cdot\|$ on E.

Proof. It is not difficult to see that $|\cdot|$ is well-defined and is clearly a norm on E. Also

$$\begin{aligned}\|x\| &= \sup\{|f(x)|: f\epsilon E^*,\ \|f\|\leqslant 1\}\\ &\leqslant \sup\Big\{\sum_{i=1}^{\infty} |f_i(x)f(x_i)|; f\epsilon E^*,\ \|f\|\leqslant 1\Big\} \leqslant |x|.\end{aligned}$$

For the reverse in equality let $f\epsilon E^*$, $\|f\|\leqslant 1$. Choose ε_i, $|\varepsilon_i| = 1$ such that

$$\sum_{i=1}^{\infty} |f_i(x)f(x_i)| = \Big|f\Big(\sum_{i=1}^{\infty} \varepsilon_i f_i(x)x_i\Big)\Big|.$$

Then we have

$$\sum_{i=1}^{\infty} |f_i(x)f(x_i)| \leqslant \Big\|\sum_{i=1}^{\infty} \varepsilon_i f_i(x)x_i\Big\| \leqslant 4K\|x\|.$$

Also $\|f_i(x)x_i\| \leqslant 2K\|x\|$ and so $|x| \leqslant 6K\|x\|$.

To see that $|\cdot|$ has the desired properties, let $\sigma\epsilon\Sigma$ and $y = \sum_{i\epsilon\sigma} b_i x_i \epsilon I_{\sigma}$. Then, for $x = \sum_{i=1}^{\infty} f_i(x)x_i \epsilon E$,

$$|x-y| = \sup_{\|f\|\leqslant 1}\Big[\sum_{i\epsilon\sigma}\big|\big(f_i(x)-b_i\big)f(x_i)\big| + \sum_{i\epsilon\omega\setminus\sigma} |f_i(x)f(x_i)| + \qquad (*)$$
$$+\sum_{i\epsilon\sigma}\big\|\big(f_i(x)-b_i\big)x_i\big\|2^{-i} + \sum_{i\epsilon\omega\setminus\sigma}\|f_i(x)x_i\|2^{-i}$$

and it is clear that $\inf\{\|x-y\|: y\epsilon L_\sigma\}$ is attained when $y = S_\sigma(x) = \sum_{i\epsilon\sigma} f_i(x)x_i$.

Also, if $i \in \sigma$ is such that $b_i \neq f_i(x)$, then the third term on the right of (*) is not zero and if follows that, for $y = \sum_{i \in \sigma} b_i x_i$, $|x-y| > |x - S_\sigma(x)|$, i.e.

$$B_\sigma(x) = \{S_\sigma(x)\} \text{ for each } x \in E \text{ and } \sigma \in \Sigma.$$

For many applications the following version of (5.6) is useful. The proof follows from (5.6) and (2.7).

(5.8) THEOREM. *Let T denote the class of all permutations of ω into ω. A norm, $\|\cdot\|$, on a Banach space E is an*

(i) *NT-norm relative to (x_n) if and only if $(x_{\tau(i)})$ is strictly co-monotone for each $\tau \in T$; and*

(ii) *NK-norm relative to (x_n) if and only if $(x_{\tau(i)})$ is strictly monotone for each $\tau \in T$.*

We end this section with the observation that (4.2), (4.4) and (4.5) are valid also for the directed set Σ. In fact, let p be as in § 3 and let $\mu_\sigma(p) = \inf\{\|p-y\|: y \in L_\sigma\}$. Then

(5.9) THEOREM. *If (x_n) is a fundamental sequence in E, $x_n \neq 0$, then (x_n) is an unconditional basis for E if and only if there is a constant $C \geqslant 1$ such that*

$$\|p - S_\sigma(p)\| \leqslant C\mu_\sigma(p) \text{ for all } p \in P \text{ and } \sigma \in \Sigma.$$

The proof is analogous to that of (4.2).

(5.10) THEOREM. *If (x_i, f_i) is a biorthogonal sequence with (x_i) fundamental in E, then (x_n) is an unconditional basis for E if and only if $\lim_{\sigma \in \Sigma} \|S_\sigma\| \mu_\sigma(x) = 0$ for each $x \in E$.*

Finally, if (x_i, f_i) is as in (5.10) let $|||x||| = \sup_{\sigma \in \Sigma} \|S_\sigma(x)\|$. Then

(5.11) THEOREM. *The biorthogonal sequence (x_i, f_i) is an unconditional basis for E if and only if for each $x \in E$ there is a $z_\sigma \in B_\sigma(x)$ such that $(|||z_\sigma|||)_{\sigma \in \Sigma}$ is bounded.*

The proofs of (5.10) and (5.11) follow as in (4.4) and (4.5) with the observation that the net Σ contains a co-final sequence, applying the result of [11].

6. On bases in $C[0,1]$. In view of the preceding sections it is natural to ask is there a Banach space E containing no strictly (co-)orthogonal or strictly (co-)monotone basis?

Somewhat surprisingly the answer is yes, and the example is $C[0,1]$.

Of course, as is well known (see e.g. [7]) $C[0,1]$ has no unconditional basis. Thus, by (5.2), $C[0,1]$ has no (strictly) orthogonal or (strictly) co-orthogonal fundamental sequence.

To show that $C[0,1]$ has no co-monotone fundamental sequence requires much more work. This result has been given by Vaniček [18]. To the author's knowledge this work appears in print only in Czech.

A detailed exposition of this result appears in the Master's thesis of J. R. Holub [5]. We sketch the proof here.

(6.1) THEOREM V. *No fundametal sequence for $C[0,1)$ is co-monotone.*

Sketch of proof. If (x_i) were such a sequence then of course (x_i) would be a co-monotone basis for $C[0,1]$. Since, without loss of generality, we may assume $\|x_i\| = 1$ for each i, we see that there is a $K > 0$ such that $\sup_i \|f_i\| = K < +\infty$, where $f_i(x_j) = \delta_{ij}$. Fix $n \geqslant 2$. Since each x_i is uniformly continuous, there is a $\delta' > 0$ such that

$$|x_i(t_1) - x_i(t_2)| < \min\left[[12K\cdot n]^{-1}, 12^{-1}\right] \quad \text{for } i = 1, 2, \ldots, n,$$

whenever $|t_1 - t_2| < \delta'$. Let I be a closed interval of length $\delta = \delta'/2$ with the property that

$$\sup_{t\in I} |x_1(t)| = \sup_{t\in[0,1]} |x_1(t)|.$$

Let c denote the midpoint of I and define z_0, z_1, z_2 by:

$$z_1(t) = \begin{cases} 0 & \text{for } t\in[0, c-\delta/2] \cup [c, 1], \\ 4/\delta(t-c+\delta/2) & \text{for } t\in[c-\delta/2, c-\delta/4], \\ -4/\delta(t-c) & \text{for } t\in[c-\delta/4, c], \end{cases}$$

$$z_2 = \begin{cases} z_1(t-\delta/2) & \text{for } t\in[c, c+\delta/2], \\ 0 & \text{for all other } t, \end{cases}$$

$$z_0 = z_1 + z_2.$$

Clearly, $z_j(t) \in C[0,1]$, $j = 0, 1, 2$, $\|z_j\| = 1$ and for all $t_1, t_2 \in I$

$$|S_n z_j(t_1) - S_n z_j(t_2)| \leqslant \sum_{i=1}^{n} |f_i(z_j)|\, |x_i(t_1) - x_i(t_2)| < 1/12, \quad j = 0, 1, 2.$$

By a long, but rather straightforward argument the assumption of co-monotonicity yields

$$(*) \qquad S_n \delta_j(t) \in \left(\frac{1}{3}, \frac{2}{3}\right) \quad \text{for all } t\in I \text{ and } j = 0, 1, 2.$$

Since S_n is a linear operator on $C[0,1]$, $S_n(z_0) = S_n(z_1) + S_n(z_2)$ and this is clearly incompatible with $(*)$.

Of course, $C[0,1]$ has numerous monotone basis (e.g. the usual Schauder basis for $C[0,1]$ consisting of the indefinite integrals of the Haar system), so we turn to the problem of the existence of a *strictly* monotone basis for $C[0,1]$.

But, Phelps [10] has shown the following:

(6.2) THEOREM P. *Let X be a compact Hausdorff space. If $C(X)$ contains a Čebyšev subspace Y with* $\operatorname{codim} Y \geqslant 2$, *then X is totally disconnected.*

Using (2.7) we obtain an immediate corollary.

(6.3) COROLLARY. *The space $C[0,1]$ has no strictly monotone basis.*

Now we are left with one final question: Is there a Banach space E with a basis but having no monotone basis?

This question has recently been settled by Gurariĭ [3], [4]. As a special case of his results we state

(6.4) THEOREM G. *The closed linear span $[t^{n^2}]$ of the sequence $\{t^{n^2}\}$ in $C[0,1]$ has a basis but admits no monotone basis.*

Details of Gurariĭ's results appeared in Russian in [3] and with some indication of proof in the English translation [4]. Full details appeared in the above-mentioned work of Holub. Holub also gives some monotonicity criteria for fundamental sequences in $C[0,1]$.

The following problem appears still to be open (see e.g. [15]):

(6.5) PROBLEM. Does every infinite dimensional Banach space admit a monotone basic sequence? (**P 693**).

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