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REGENERATIVE RENEWAL PROCESSES

Abstract. The paper deals with the description of a few point processes defined on the basis of the renewal process. First, we introduce the regenerative renewal process as a spline process consisting of consecutive editions of some renewal process initiated at the shock moments, which also form a renewal process. Then a superposition of the delayed renewal processes is introduced and its special version under the assumption that the superposed delayed processes are terminated. In the paper the expected value and the second moment of the point processes are investigated and also their asymptotic expansions are found.

1. Introduction. The definitions of the point processes considered in the paper are based on some renewal process called, for terminological simplicity, the *shock process*. This subterfuge, which has terminological character merely, is analogous to the one used in Esary et al. [2] in the description of the compound Poisson process with the difference that here the shock involves no random variable but some renewal processes initiated at the shock moment. If we assume that every shock breaks the evolution of the actual renewal process and, independently of the past, initiates a new edition of the renewal process, it is easy to see that we remain in the domain of regenerative stochastic processes in Smith's sense (see [5] and [6]). If the shock does not break the evolution of the previously initiated renewal processes but initiates a new renewal process which is added to the previous one, then we deal with a superposition of delayed renewal processes. We get a special version of this superposition with the assumption that the superposed processes are terminated.

In the paper we consider the expected value of the point process and its variance. We find the equations for them, the solutions of the equations and their asymptotical estimation. In Section 2 we recall familiar facts from renewal theory. We do this because the cited papers and the famous monographs contain small formal differences or mistakes which may have disagreeable consequences in later considerations.

The notation introduced for the basis shock process is obligatory in the paper. Other notation may have a local character.

2. Shock process. Let us assume that the shock process is an ordinary renewal process generated by the sequence X_1, X_2, \dots of independent random variables, nonnegative and equidistributed. The common probability distribution function of these random variables is denoted by F , and it is sometimes represented by a random variable X ; then we write

$$F(x) = \Pr(X < x).$$

Let us put

$$\bar{F} = 1 - F, \quad \mu_i = EX^i, \quad i = 1, 2, \dots, \quad \sigma^2 = \mu_2 - \mu_1^2.$$

Let $S_0 = 0$, $S_n = S_{n-1} + X_n$, $n = 1, 2, \dots$

The *shock process* $N(t)$, $t \geq 0$, is defined as the number of shocks in the interval $(0, t)$, e.g.,

$$N(t) = \sum_{n=1}^{\infty} 1_{S_n < t},$$

where $1_{S_n < t}$ is the indicator of the random event $S_n < t$.

Let us write

$$H(t) = EN(t) \quad \text{and} \quad M(t) = EN^2(t).$$

It is obvious that the functions are equal to zero for a negative argument. Hence we restrict our considerations to a nonnegative argument, and this is not to be recalled.

PROPOSITION 1. *We have*

$$(1) \quad N(t) \stackrel{d}{=} 1_{X < t} (1 + N'(t - X)),$$

where $\stackrel{d}{=}$ denotes the equality in distributions, the random variable X and the random process N' are mutually independent, X has the distribution function F , N' is the probability copy of the process N ,

$$(2) \quad H(t) = F(t) + H * F(t),$$

$$(3) \quad M(t) = 2H(t) - F(t) + M * F(t),$$

where $*$ is the convolution symbol.

PROPOSITION 2. *The solutions of equations (2) and (3) are of the form*

$$H(t) = \sum_{n=1}^{\infty} F^{*n}(t),$$

$$(4) \quad M(t) = 2H * H(t) + H(t).$$

PROPOSITION 3. We have

$$(5) \quad H(t) = \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 + o(1),$$

$$(6) \quad M(t) = \frac{t^2}{\mu_1^2} + \left(\frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1} \right) t + \frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{3\mu_2}{2\mu_1^2} + 1 + o(1),$$

$$D^2 N(t) = \frac{\sigma^2 t}{\mu_1^3} + \frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} + o(1),$$

where $o(1) \rightarrow 0$ under $t \rightarrow \infty$, provided that the used moments are finite.

PROPOSITION 4. If $\mu_3 < \infty$, then

$$\int_0^t \left(H(u) - \frac{u}{\mu_1} \right) du = \left(\frac{\mu_2}{2\mu_1^2} - 1 \right) t + \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2} + o(1).$$

For the proof see [1], p. 57.

PROPOSITION 5. If H_0 is the renewal function of some renewal process generated by a probability distribution with the moments λ_i , $i = 1, 2, \dots$, such that

$$H_0(t) = \frac{t}{\lambda_1} + \frac{\lambda_2}{2\lambda_1^2} - 1 + o(1),$$

then

$$\left(H_0(t) - \frac{t}{\lambda_1} \right)^* \left(H(t) - \frac{t}{\mu_1} \right) = \left(\frac{\lambda_2}{2\lambda_1^2} - 1 \right) \left(\frac{\mu_2}{2\mu_1^2} - 1 \right) + o(1),$$

provided that the used moments are finite.

Proof. Let $U(t) = H_0(t) - t/\lambda_1$ and $V(t) = H(t) - t/\mu_1$. Using in the appropriate place the mean-value theorem of integration we get

$$\begin{aligned} U^* V(t) &= \left(\int_0^{t/2} + \int_{t/2}^t \right) U(t-x) dV(x) \\ &= U(\xi) V(t/2) + U(t/2) V(t/2) - U(t/2) V(\zeta), \end{aligned}$$

where $t/2 < \xi < t$ and $t/2 < \zeta < t$. Hence

$$U^* V(t) = U(\infty) V(\infty) + o(1).$$

Let us suppose that the sequence X_1, X_2, \dots , which generates the renewal process, is the sequence of independent nonnegative random variables, the random variable X_1 has the probability distribution function K and the random variables X_2, X_3, \dots are equidistributed and have the distribution

function F . Then the renewal process N_K , defined as the number of renewals in the interval $(0, t)$, is the modified renewal process. Now, let us put

$$H_K(t) = EN_K(t) \quad \text{and} \quad M_K(t) = EN_K^2(t).$$

We denote the moments in the probability distribution function K by $\mu_i(K)$, $i = 1, 2, \dots$

It is easy to see that

$$N_K(t) \stackrel{d}{=} 1_{X_1 < t}(1 + N'(t - X_1)),$$

where the random variable X_1 and the random process N' are mutually independent, X_1 has the probability distribution function K , and N' is a probabilistic copy of the process N . This implies the formulae

$$(7) \quad H_K(t) = K(t) + H * K(t),$$

$$(8) \quad M_K(t) = K(t) + 2H * K(t) + M * K(t) = K(t) + 3H * K(t) + 2H * H * K(t).$$

PROPOSITION 6. *We have*

$$(9) \quad H_K(t) = \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - \frac{\mu_1(K)}{\mu_1} + o(1),$$

$$(10) \quad M_K(t) = \frac{t^2}{\mu_1^2} + t \left(\frac{2\mu_2}{\mu_1^3} - \frac{1}{\mu_1} - \frac{2\mu_1(K)}{\mu_1^2} \right) + \frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} + \frac{\mu_2(K)}{\mu_1^2} - \frac{2\mu_2\mu_1(K)}{\mu_1^3} + \frac{\mu_1(K)}{\mu_1} + o(1),$$

$$(11) \quad D^2 N_K(t) = \frac{t\sigma^2}{\mu_1^3} + \frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} + \frac{\mu_2(K)}{\mu_1^2} - \frac{\mu_2\mu_1(K)}{\mu_1^3} + \frac{\mu_1(K)}{\mu_1} - \frac{\mu_1^2(K)}{\mu_1^2} + o(1),$$

provided that the used moments are finite.

Proof. The key theorem of renewal theory applied to the function $\bar{K} = 1 - K$ and the renewal function H implies

$$H_K(t) = K(t) + H(t) - \bar{K} * H(t) = \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - \frac{\mu_1(K)}{\mu_1} + o(1).$$

Before analyzing the function M_K we calculate

$$\bar{K} * H(t) = \frac{1}{\mu_1} \int_0^\infty \bar{K}(u) du + o(1) = \mu_1(K)/\mu_1 + o(1).$$

Now, from (7) and (8) under $K(t) = 1 + o(1)$ we get

$$\bar{K}^* t^2 = t^2 - K^* t^2 = t^2 - \int_0^t (t-u)^2 dK(u) = 2\mu_1(K)t - \mu_2(K) + o(1).$$

Let us consider the function $U(t) = M(t) - t^2/\mu_1^2$. It follows from (6) that U satisfies the condition

$$\lim_{t \rightarrow \infty} \frac{1}{h} (U(t+h) - U(t)) = \frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1}.$$

It is a condition of the same type as the assertion of the Blackwell theorem in renewal theory. Recalling the familiar relation between the Blackwell theorem and the key theorem of the renewal theory ([6], p. 296) it may analogously be proved that

$$\bar{K}^* U(t) = \left(\frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1} \right) \int_0^\infty \bar{K}(u) du + o(1) = \left(\frac{2\mu_2}{\mu_1^2} - 3 \right) \frac{\mu_1(K)}{\mu_1} + o(1).$$

Now, from the first equation in (8) we have

$$\begin{aligned} M_K(t) &= K(t) + 2H^* K(t) + M^* K(t) \\ &= K(t) + 2(H_K(t) - K(t)) + M(t) - \bar{K}^* \frac{t^2}{\mu_1^2} - \bar{K}^* \left(M(t) - \frac{t^2}{\mu_1^2} \right) \\ &= \frac{t^2}{\mu_1^2} + t \left(\frac{2\mu_2}{\mu_1^3} - \frac{1}{\mu_1} - \frac{2\mu_1(K)}{\mu_1^2} \right) \\ &\quad + \left(\frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} + \frac{\mu_2(K)}{\mu_1^2} - \frac{2\mu_2\mu_1(K)}{\mu_1^3} + \frac{\mu_1(K)}{\mu_1} \right). \end{aligned}$$

Now we may calculate the variance $D^2 N_K(t)$. We subtract side by side

$$H_K^2(t) = \frac{t^2}{\mu_1^2} + \frac{2t}{\mu_1} \left(\frac{\mu_2}{2\mu_1^2} - \frac{\mu_1(K)}{\mu_1} \right) + \frac{\mu_2^2}{4\mu_1^4} - \frac{\mu_2\mu_1(K)}{\mu_1^3} - \frac{\mu_1^2(K)}{\mu_1^2} + o(1)$$

and we get (11).

3. Regenerative renewal processes. Let us introduce the notation on which we base the construction of the regenerative renewal process. We have the shock process defined in Section 2. At the shock moment we start the renewal process which lasts endless, but (in this section) we are interested in it to the moment of the nearest shock. The renewal processes which start in consecutive shocks are mutually independent and independent of the shock process.

Let us define the point process $\check{N}(t)$, $t \geq 0$, in the following way:

$$\check{N}(t) = \begin{cases} N_0(t) & \text{if } 0 \leq t < X_1, \\ N_0(X_1) + N_1(X_2) + \dots + N_{k-1}(X_k) + N_k(t - S_k) & \text{if } S_k \leq t < S_{k+1}, k = 1, 2, \dots, \end{cases}$$

where the random processes N_0, N_1, \dots are mutually independent copies of

some process and are independent of the random variables X_1, X_2, \dots which generate the shock process.

For the process N_0 we introduce the following notation: F_0 denotes the probability distribution function of intersignal time and

$$H_0(t) = EN_0(t), \quad M_0(t) = EN_0^2(t).$$

The moments in F_0 are denoted by

$$\lambda_i = \int_0^\infty x^i dF_0(x), \quad i = 1, 2, \dots$$

Note that we construct the process \check{N} in the following way: at the moment $t = 0$ we start the renewal process N_0 and this develops to the moment X_1 of the first shock. At the moment S_n of the n -th shock the evolution of the process N_{n-1} is broken and the evolution of the delayed process $N_n(t - S_n)$ starts, where N_n is the copy of the renewal process N_0 . Let us write

$$\check{H}(t) = E\check{N}(t) \quad \text{and} \quad \check{M}(t) = E\check{N}^2(t).$$

THEOREM 1. *We have*

$$(12) \quad \check{N}(t) \stackrel{d}{=} 1_{X \geq t} N_0(t) + 1_{X < t} (N_0(X) + \check{N}'(t - X)),$$

where the random variable X and the point processes N_0, \check{N}' are mutually independent, X has the probability distribution function F , and \check{N}' is the copy of the renewal process \check{N} ,

$$(13) \quad \check{H}(t) = F_1(t) + \check{H} * F(t),$$

$$(14) \quad \check{M}(t) = F_3(t) + 2F_2 * \check{H}(t) + \check{M} * F(t),$$

where

$$F_1(t) = \int_0^t \bar{F}(x) dH_0(x), \quad F_2(t) = \int_0^t H_0(x) dF(x),$$

$$F_3(t) = \int_0^t \bar{F}(x) dM_0(x).$$

Proof. Formula (12) is the analogue of (1). The expectation of (12) gives

$$\check{H}(t) = \bar{F}(t)H_0(t) + \int_0^t (H_0(x) + \check{H}(t-x))dF(x),$$

which implies (13). Squaring (12) and taking the expectation result in

$$\check{M}(t) = M_0(t)\bar{F}(t) + \int_0^t (M_0(x) + 2H_0(x)\check{H}(t-x) + \check{M}(t-x))dF(x).$$

That implies (14) because

$$\begin{aligned} M_0(t)\bar{F}(t) + \int_0^t (M_0(x) + 2H_0(x)\check{H}(t-x))dF(x) \\ = \int_0^t \bar{F}(x)dM_0(x) + 2 \int_0^t H_0(x)\check{H}(t-x)dF(x) = F_3(t) + 2F_2^* \check{H}(t). \end{aligned}$$

THEOREM 2. The solutions of (13) and (14) are

$$(15) \quad \check{H}(t) = \kappa_1 H_{K_1}(t),$$

$$(16) \quad \check{M}(t) = \kappa_2 H_{K_3}(t) + \kappa_1^2 (H_{K_1^* K_2}(t) + M_{K_1^* K_2}(t)),$$

where

$$\kappa_1 = EN_0(X), \quad \kappa_2 = EN_0^2(X),$$

$$K_1 = F_1/\kappa_1, \quad K_2 = F_2/\kappa_1, \quad K_3 = F_3/\kappa_2,$$

and H_{K_1} , H_{K_3} , $H_{K_1^* K_2}$ are the modified renewal functions, $M_{K_1^* K_2}$ is the modified function of the second moment, provided that λ_2 and μ_2 are finite.

Proof. Note that the functions K_1 , K_2 , K_3 have the following properties: $K_i(0) = 0$, K_i are nondecreasing,

$$\kappa_1 = \int_0^\infty \bar{F}(x)dH_0(x) = \int_0^\infty H_0(x)dF(x) = EN_0(X),$$

$$\kappa_2 = \int_0^\infty \bar{F}(x)dM_0(x) = EN_0^2(X),$$

and the moments are finite if λ_2 and μ_2 are finite. Hence the functions K_1 , K_2 , K_3 are proper distribution functions and modified by the probability distribution functions K_1 , K_3 or $K_1^* K_2$ the renewal processes are properly defined. From (13) we get

$$(17) \quad \check{H}(t) = F_1(t) + F_1^* H(t),$$

hence

$$\check{H}(t) = \kappa_1 (K_1(t) + K_1^* H(t)) = \kappa_1 H_{K_1}(t).$$

From equation (14) we get

$$\check{M}(t) = F_3(t) + 2F_2^* \check{H}(t) + (F_3(t) + 2F_2^* \check{H}(t))^* H(t),$$

and from (17) and (8) we obtain

$$\begin{aligned}\check{M}(t) &= F_3(t) + 2F_2^* \check{H}(t) + F_3^* H(t) + 2F_2^* \check{H}^* H(t) \\ &= F_3(t) + 2F_1^* F_2(t) + 2F_1^* F_2^* H(t) + F_3^* H(t) + 2F_1^* F_2^* H(t) + 2F_1^* F_2^* H^* H(t) \\ &= \kappa_2 H_{K_3}(t) + \kappa_1^2 (H_{K_1^* K_2}(t) + M_{K_1^* K_2}(t)).\end{aligned}$$

THEOREM 3. *We have*

$$(18) \quad \check{H}(t) = \kappa_1 H(t) - \frac{1}{\mu_1} \text{cov}(N_0(X), X) + \mathbb{E} N_0(\gamma) + o(1),$$

$$\begin{aligned}(19) \quad \mathbb{D}^2 \check{N}(t) &= \frac{t}{\mu_1} \left(\mathbb{D}^2 N_0(X) - \frac{2\kappa_1}{\mu_1} \text{cov}(N_0(X), X) + \left(\frac{\kappa_1}{\mu_1} \right)^2 \sigma^2 \right) \\ &\quad + \mathbb{D}^2 N_0(\gamma) + \mathbb{D}^2 N_0(X) \left(\frac{\mu_2}{2\mu_1^2} - 1 \right) + \frac{2\kappa_1}{\mu_1^2} \text{cov}(N_0(X), X^2) \\ &\quad - \frac{2\kappa_1}{\mu_1} \text{cov}(N_0(\gamma), \gamma) - \frac{1}{\mu_1} \text{cov}(N_0^2(X), X) + \frac{1}{\mu_1^2} \text{cov}^2(N_0(X), X) \\ &\quad - \frac{\kappa_1}{\mu_1} \left(\frac{3\mu_2}{\mu_1^2} - 2 \right) \text{cov}(N_0(X), X) + \kappa_1^2 \left(\frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} \right) + o(1),\end{aligned}$$

where γ is a random variable with the probability distribution function

$$\Pr(\gamma < x) = \frac{1}{\mu_1} \int_0^x \bar{F}(u) du,$$

under the condition that the used moments are finite.

Before proving the theorem let us calculate some of the moments

$$\mu_j(K_i) = \int_0^\infty x^j dK_i(x), \quad i = 1, 2, 3, j = 1, 2.$$

LEMMA 1. *We have*

$$\begin{aligned}\kappa_1 \mu_1(K_1) &= \int_0^\infty x \bar{F}(x) dH_0(x) = \int_0^\infty H_0(x) (x dF(x) - \bar{F}(x) dx) \\ &= \mathbb{E} N_0(X) X - \mu_1 \mathbb{E} N_0(\gamma),\end{aligned}$$

$$\kappa_1 \mu_1(K_2) = \int_0^\infty x H_0(x) dF(x) = \mathbb{E} N_0(X) X,$$

$$\begin{aligned}\kappa_2 \mu_1(K_3) &= \int_0^\infty x \bar{F}(x) dM_0(x) = \int_0^\infty M_0(x) (x dF(x) - \bar{F}(x) dx) \\ &= \mathbb{E} N_0^2(X) X - \mu_1 \mathbb{E} N_0^2(\gamma),\end{aligned}$$

$$\kappa_1 \mu_2(K_1) = \int_0^\infty x^2 \bar{F}(x) dH_0(x) = \int_0^\infty H_0(x) (x^2 dF(x) - 2x\bar{F}(x)dx)$$

$$= EN_0(X)X^2 - 2\mu_1 EN_0(\gamma)\gamma,$$

$$\kappa_1 \mu_2(K_2) = \int_0^\infty x^2 H_0(x) dF(x) = EN_0(X)X^2,$$

$$\mu_1(K_1^* K_2) = \mu_1(K_1) + \mu_1(K_2),$$

$$\mu_2(K_1^* K_2) = \mu_2(K_1) + 2\mu_1(K_1)\mu_1(K_2) + \mu_2(K_2),$$

provided that the used moments are finite.

Proof of Theorem 3. From (15) and (9) we get

$$\check{H}(t) = \kappa_1 \left(\frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - \frac{\mu_1(K_1)}{\mu_1} \right) + o(1);$$

hence and from Lemma 1 we get (18). From (16), using (9) and (10), we obtain

$$\begin{aligned} \check{M}(t) = & \frac{t^2 \kappa_1^2}{\mu_1^2} + t \left(\kappa_2 + \kappa_1^2 \left(\frac{2\mu_2}{\mu_1^3} - \frac{2(\mu_1(K_1) + \mu_1(K_2))}{\mu_1^2} \right) \right) \\ & + \kappa_2 \left(\frac{\mu_2}{2\mu_1^2} - \frac{\mu_1(K_3)}{\mu_1} \right) + \kappa_1^2 \left(\frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} + \frac{\mu_2(K_1^* K_2)}{\mu_1^2} \right. \\ & \left. - \frac{2\mu_2(\mu_1(K_1) + \mu_1(K_2))}{\mu_1^3} \right) + o(1). \end{aligned}$$

Subtracting side by side

$$\check{H}^2(t) = \kappa_1^2 \left(\frac{t^2}{\mu_1^2} + \frac{t}{\mu_1} \left(\frac{\mu_2}{\mu_1^2} - \frac{2\mu_1(K_1)}{\mu_1} \right) + \frac{\mu_2^2}{4\mu_1^4} - \frac{\mu_2\mu_1(K_1)}{\mu_1^3} + \frac{\mu_1^2(K_1)}{\mu_1^2} \right) + o(1),$$

we get

$$\begin{aligned} D^2 \check{N}(t) = & \frac{t}{\mu_1} \left(\kappa_2 + \kappa_1^2 \left(-\frac{\mu_2}{\mu_1^2} + \frac{2\mu_1(K_1)}{\mu_1} + \frac{2\mu_2}{\mu_1^2} - \frac{2(\mu_1(K_1) + \mu_1(K_2))}{\mu_1} \right) \right) \\ & + \kappa_2 \left(\frac{\mu_2}{2\mu_1^2} - \frac{\mu_1(K_3)}{\mu_1} \right) + \kappa_1^2 \left(\frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} + \frac{\mu_2(K_1^* K_2)}{\mu_1^2} \right. \\ & \left. - \frac{\mu_2\mu_1(K_1)}{\mu_1^3} - \frac{2\mu_2\mu_1(K_2)}{\mu_1^3} - \frac{\mu_1^2(K_1)}{\mu_1^2} \right) + o(1). \end{aligned}$$

Using Lemma 1, after suitable transformations we get (19).

The process $\check{N}(t)$, $t \geq 0$, is the cumulative type process in the Smith sense. This follows from the fact that the sequence of the random variables

$$Y_n = \check{N}(S_{n+1}) - \check{N}(S_n) = N_n(X_{n+1}), \quad n = 0, 1, \dots,$$

consists of a sequence of equidistributed random variables. From the general theory of cumulative processes we obtain the estimation of the mean and the variance of the process and its asymptotical normality. In this way the first terms of the asymptotical expansion in formulae (18) and (19) may be found. The knowledge of the evolution of the regenerative renewal process between the regenerative moments (shocks) enables a continuation of the expansion.

In formula (19) we have the covariance of the random variables X and $N_0(X)$. It is interesting that the random variables X and $X - N_0(X)$ are mutually independent if and only if X is exponentially distributed (see [4]). Any expressions from (19) have interpretations in terms of the residual renewal process. Define the point process

$$\Gamma(t) = \check{N}(S_{N(t)+1}) - \check{N}(t).$$

Introduce the residual time and the distribution function of the residual time:

$$\gamma_1(t) = t - S_{N(t)}, \quad \gamma(t) = S_{N(t)+1} - t.$$

It is obvious that

$$\lim_{t \rightarrow \infty} \Pr(\gamma(t) \geq u, \gamma_1(t) \geq v) = \frac{1}{\mu_1} \int_{u+v}^{\infty} \bar{F}(x) dx, \quad u \geq 0, v \geq 0,$$

$$\lim_{t \rightarrow \infty} \Pr(\gamma(t) + \gamma_1(t) < w) = \frac{1}{\mu_1} \int_0^w x dF(x), \quad w \geq 0.$$

The probability distribution functions on the right-hand side of these relations are represented in the sequel by some random variables γ and γ_1 .

THEOREM 4. *If $\mu_2 < \infty$ and $\lambda_1 < \infty$, then*

$$\lim_{t \rightarrow \infty} (\mu_1 \kappa_1(H(t) + 1) - \check{H}(t)) = \lim_{t \rightarrow \infty} E\Gamma(t) = \frac{1}{\mu_1} EN_0(X)X - EN_0(\gamma).$$

Proof. From the Wald equality after suitable transformations we get

$$E\check{N}(S_{N(t)+1}) = \mu_1 EN_0(X)(H(t) + 1),$$

$$E\Gamma(t) = E(\check{N}(S_{N(t)+1}) - \check{N}(t)) = E(N_0(\gamma(t) + \gamma_1(t)) - N_0(\gamma_1(t)))$$

$$= EN_0(\gamma + \gamma_1) - EN_0(\gamma_1) + o(1) = \frac{1}{\mu_1} EN_0(X)X - EN_0(\gamma) + o(1).$$

4. Superpositions of the delayed renewal processes. Assume the conditions for the construction of the regenerative renewal process. Let us consider the random process

$$\check{N}(t) = \sum_{n=0}^{\infty} 1_{S_n < t} N_n(t - S_n), \quad t \geq 0,$$

where N_0, N_1, \dots are mutually independent copies of some renewal process, independent of the sequence X_1, X_2, \dots .

Note that we construct the process \check{N} in the following way. At the moment $t = 0$ we start the renewal process N_0 and it is performed eternally. At the moment of the n -th shock we start the renewal process N_n which is a copy of the process N_0 and it also performs eternally. The process \check{N} is the superposition of all delayed renewal processes initiated at the shock moments.

For simplicity of the notation the new process, defined in this section, is denoted by \check{N} just as in the previous section. Also as previously we denote the characteristics of the process \check{N} and, naturally, as previously we denote the terms which define the process \check{N} .

THEOREM 5. *We have*

$$(20) \quad \check{N}(t) \stackrel{d}{=} N_0(t) + 1_{X < t} \check{N}'(t - X),$$

where the random variable X and the random processes N_0, \check{N}' are mutually independent, X has the probability distribution function F , and \check{N}' is the probabilistic copy of the process \check{N} ,

$$(21) \quad \check{H}(t) = H_0(t) + \check{H} * F(t),$$

$$(22) \quad \check{M}(t) = F_0(t) + \check{M} * F(t),$$

where

$$(23) \quad F_0(t) = M_0(t) + 2H_0(t)(\check{H}(t) - H_0(t)),$$

and the solution of equations (21) and (22) is of the form

$$(24) \quad \check{H}(t) = H_0(t) + H_0^* H(t),$$

$$(25) \quad \check{M}(t) = F_0(t) + F_0^* H(t).$$

Proof. Formula (20) is the analogue of (1) and (12). Also formulae (21)–(25) have their analogues in Propositions 1 and 2 and in Theorem 1.

THEOREM 6. *We have*

$$(26) \quad \check{H}(t) = \frac{t^2}{2\lambda_1\mu_1} + \frac{t}{2\lambda_1\mu_1} \left(\frac{\mu_2}{\mu_1} + \frac{\lambda_2}{\lambda_1} - 2\lambda_1 \right) + o(t),$$

$$(27) \quad D^2 \check{N}(t) = \frac{t^3 \sigma^2}{3\lambda_1^2 \mu_1^3} + o(t^3),$$

provided that the used moments are finite.

Proof. In virtue of Propositions 4 and 5 we have

$$\begin{aligned}
 H_0^* H(t) &= \left(H_0(t) - \frac{t}{\lambda_1}\right)^* \left(H(t) - \frac{t}{\mu_1}\right) + \left(H_0(t) - \frac{t}{\lambda_1}\right)^* \frac{t}{\mu_1} \\
 &\quad + \left(H(t) - \frac{t}{\mu_1}\right)^* \frac{t}{\lambda_1} + \left(\frac{t}{\mu_1}\right)^* \frac{t}{\lambda_1} \\
 &= \left(\frac{\lambda_2}{2\lambda_1^2} - 1\right) \left(\frac{\mu_2}{2\mu_1^2} - 1\right) + \frac{1}{\mu_1} \left(\frac{\lambda_2}{2\lambda_1^2} - 1\right) t + \frac{1}{\lambda_1} \left(\frac{\mu_2}{2\mu_1^2} - 1\right) t + \frac{t^2}{2\lambda_1\mu_1} + o(t) \\
 &= \frac{t^2}{2\lambda_1\mu_1} + t \left(\frac{1}{\lambda_1} \left(\frac{\mu_2}{2\mu_1^2} - 1\right) + \frac{1}{\mu_1} \left(\frac{\lambda_2}{2\lambda_1^2} - 1\right) \right) + o(t).
 \end{aligned}$$

Adding to this $H_0(t) = t/\lambda_1 + o(t)$ we get (26). Using (5) and (6) we calculate

$$\begin{aligned}
 F_6(t) &= M_0(t) + 2H_0(t)(\check{H}(t) - H_0(t)) = \frac{t^2}{\lambda_1^2} + \left(\frac{2\lambda_2}{\lambda_1^3} - \frac{3}{\lambda_1}\right)t \\
 &\quad + 2\left(\frac{t}{\lambda_1} + \frac{\lambda_2}{2\lambda_1^2} - 1\right) \left(\frac{t^2}{2\lambda_1\mu_1} + t \left(\frac{1}{\lambda_1} \left(\frac{\mu_2}{2\mu_1^2} - 1\right) + \frac{1}{\mu_1} \left(\frac{\lambda_2}{2\lambda_1^2} - 1\right) \right) \right) + o(t) \\
 &\stackrel{\text{df}}{=} At^3 + Bt^2 + o(t^2).
 \end{aligned}$$

Hence for

$$H(t) = \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 + o(1) \stackrel{\text{df}}{=} a't + b' + o(1)$$

we have

$$F_6^* H(t) = \frac{Aa'}{4}t^4 + \left(Ab' + \frac{Ba'}{3}\right)t^3 + o(t^3).$$

Adding to this $F_6(t)$ we get

$$\check{M}(t) = \frac{Aa'}{4}t^4 + \left(Ab' + \frac{Ba'}{3} + A\right)t^3 + o(t^3).$$

Now, by subtraction side by side $\check{H}^2(t) = a^2t^4 + 2abt^3 + o(t^3)$ we get (27), because

$$Aa' = 4a^2, \quad Ab' + \frac{Ba'}{3} + A - 2ab = \frac{\sigma^2}{3\lambda_1^2\mu_1^3}.$$

5. Defective case. The cumulative process of the renewal processes considered at the beginning of Section 4 is rather disagreeable for uses of parabolical drift. The phenomenon vanishes under the assumption that N_0 is a terminated renewal process.

The theory of terminating renewal processes (see [3], p. 374) has no essentially new theoretical elements but it has interesting practical uses. Recall that the renewal process N_0 is *terminating* if the probability distribution function F_0 is defective, e.g., $F_0(\infty) = p < 1$. Let $F_0(x) = pG(x)$, where G is a proper probability distribution function. Then the renewal function takes the form

$$H_0(t) = \sum_{n=1}^{\infty} p^n G^{*n}(t).$$

The function H_0 has the following properties: $H_0(0) = 0$, H_0 is nondecreasing, $H_0(\infty) = r$, $r = p/q$, $q = 1 - p$. Hence

$$R(t) = \frac{1}{r} H_0(t)$$

is the probability distribution function of a random variable.

The function $M_0(t) = EN_0^2(t)$ in virtue of (4) has the following properties:

$$M_0(t) = 2H_0^* H_0(t) + H_0(t) = 2r^2 R^* R(t) + rR(t);$$

hence $M_0(t)/(2r^2 + r)$ is the probability distribution function of a random variable.

The general solutions which describe the cumulative renewal processes are significant in the case of terminated renewal processes. New asymptotical expansions may be obtained because the asymptotical order of these functions is new.

THEOREM 7. *If the renewal process N_0 is terminating, then*

$$(28) \quad \check{H}(t) = rH_R(t),$$

$$\check{M}(t) = F_6(t) + F_6^* H(t),$$

where H_R is the modified renewal function generated by the probability distribution functions R and F , and F_6 is defined by (23).

Proof. From (24) we have

$$\check{H}(t) = H_0(t) + H_0^* H(t) = r(R(t) + R^* H(t)) = rH_R(t).$$

THEOREM 8. *We have*

$$(29) \quad \check{H}(t) = \frac{r}{\mu_1} \left(t + \frac{\mu_2}{2\mu_1} - \frac{\lambda_1}{pq} \right) + o(1),$$

$$D^2 \check{N}(t) = \frac{t}{\mu_1} \left(\sigma^2 \left(\frac{r}{\mu_1} \right)^2 + \frac{r}{q} \right) + o(t).$$

Proof. Let us calculate

$$\mu_1(R) = \int_0^\infty x dR(x) = \frac{1}{r} \int_0^\infty x dH_0(x) = \lambda_1/pq,$$

where

$$\lambda_1 = \int_0^\infty x dF_0(x).$$

From (28) we get (29). Now, starting from (23) we get the estimation

$$F_6(t) = r + 2r^2 \left(\frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - \frac{\lambda_1}{\mu_1 pq} \right) + o(1).$$

Then in virtue of Proposition 4 we obtain

$$\begin{aligned} F_6^* H(t) &= \left(\frac{2r^2 t}{\mu_1} \right)^* \left(\frac{t}{\mu_1} \right) + \left(F_6(t) - \frac{2r^2 t}{\mu_1} \right)^* H(t) \\ &\quad + F_6^* \left(H(t) - \frac{t}{\mu_1} \right) - \left(F_6(t) - \frac{2r^2 t}{\mu_1} \right)^* \left(H(t) - \frac{t}{\mu_1} \right) \\ &= \frac{r^2 t^2}{\mu_1^2} + \frac{1}{\mu_1} \int_0^\infty \left(F_6(t) - \frac{2r^2 t}{\mu_1} \right) dt + \frac{2r^2}{\mu_1} \int_0^\infty \left(H(t) - \frac{t}{\mu_1} \right) dt + o(t) \\ &= \frac{r^2 t^2}{\mu_1^2} + \frac{2r^2 t}{\mu_1} \left(\frac{\mu_2}{\mu_1^2} - \frac{\lambda_1}{\mu_1 pq} + \frac{1}{2r} - 1 \right) + o(t). \end{aligned}$$

Adding to this $F_6(t) = 2r^2 t/\mu_1 + o(t)$ we get

$$\check{M}(t) = \frac{r^2 t^2}{\mu_1^2} + \frac{2r^2 t}{\mu_1} \left(\frac{\mu_2}{\mu_1^2} - \frac{\lambda_1}{\mu_1 pq} + \frac{1}{2r} \right) + o(t).$$

Subtracting side by side

$$\check{H}^2(t) = \frac{r^2}{\mu_1^2} \left(t^2 + 2t \left(\frac{\mu_2}{2\mu_1} - \frac{\lambda_1}{pq} \right) \right) + o(t)$$

we obtain

$$D^2 \check{N}(t) = \frac{r^2 t}{\mu_1^3} \left(\mu_2 + \frac{\mu_1^2}{r} \right) = \frac{t}{\mu_1} \left(\sigma^2 \left(\frac{r}{\mu_1} \right)^2 + \frac{r}{q} \right) + o(t).$$

Let $N_0(\infty)$ denote the number of points in a terminated renewal process. It is obvious that this is a proper random variable, geometrically distributed:

$$\Pr(N_0(\infty) = k) = p^k q, \quad k = 0, 1, \dots$$

Also $EN_0(\infty) = r$, $D^2 N_0(\infty) = r/q$, and hence in Theorem 8 we may write

$$\check{H}(t) = EN_0(\infty)H(t) + o(1),$$

$$D^2 \check{N}(t) = \frac{t}{\mu_1} \left(D^2 N_0(\infty) + \left(\frac{EN_0(\infty)}{\mu_1} \right)^2 \sigma^2 \right) + o(t).$$

This proves that Theorems 3 and 8 are associated.

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