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Extension and decomposition operators in products of strictly pseudoconvex sets

by Piotr Jakóbczak (Kraków)

Abstract. Let D be a strictly pseudoconvex domain in C^n and let M' be a closed analytic subvariety of C^n, which intersects ∂D transversally and has no singular points on ∂D; set $M = D \cap M'$. In this paper we prove the existence of linear and continuous extension operators from the spaces $A^k(M \times M)$, $H^{\infty,k}(M \times M)$ and $\Lambda_t(M \times M)$, to the corresponding function spaces on $D \times D$, provided that ∂D is sufficiently smooth. This result is then applied to the proof of decomposition theorems for functions in $A^k(M \times M)$, $H^{\infty,k}(M \times M)$ and $\Lambda_t(M \times M)$ which vanish on the diagonal; we examine the smoothness properties of the decomposition factors. We show also that if M' is a manifold, the extension operator can be defined by means of an integral formula, with a kernel defined recently by Henkin and Leiterer.

Notation and definitions. For every $z \in C^n$ and every $r > 0$, we denote by $B(z, r)$ the Euclidean ball in C^n with radius r centered at z.

If $\alpha, \beta \in Z_+^n$ are multiindices, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, we set $\alpha+\beta = (\alpha_1+\beta_1, \ldots, \alpha_n+\beta_n)$ and $|\alpha| = \alpha_1+\ldots+\alpha_n$.

For every $\alpha, \beta \in Z_+^n$ and $1 \leqslant j \leqslant n$, we set $D^j = \partial/\partial z_j$, $\bar{D}^j = \partial/\partial \bar{z}_j$, $D^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}$, etc.; subscript indicates that differentiation is performed with respect to a specified group of variables, e.g. $\bar{D}_\xi^\beta$.

We denote by $\mathcal{O}(D)$ the space of holomorphic functions in the domain $D \subset C^n$; more generally, if X is a complex subvariety of some C^n, $\mathcal{O}(X)$ will denote the space of holomorphic functions on X.

Let D be a domain in C^n with a minimal regular boundary (for definition see [15], Chapter VI, § 3.3). For every $k = 0, 1, \ldots$ set $A^k(D) = \mathcal{O}(D) \cap \mathscr{C}^k(\bar{D})$ where $\mathscr{C}^k(\bar{D})$ denotes, as usual, the space of all functions of class $\mathscr{C}^k$ in D such that their derivatives of order $\leqslant k$ extend continuously to $\bar{D}$, and let $H^{\infty,k}(D)$ be the space of all functions holomorphic in D such that their derivatives of order $\leqslant k-1$ extend continuously to $\bar{D}$, and the derivatives of order k are bounded in D. For $k = 0, 1, \ldots$, $A^k(D)$ and $H^{\infty,k}(D)$ are Banach algebras with the norm

$$\|f\|_{D,k} = \sum_{|\alpha| \leqslant k} \sup_D |D^\alpha f|.$$

We write $A(D) = A^0(D)$ and $H^\infty(D) = H^{\infty,0}(D)$.

If t is a positive real number which is not an integer and k is a non-negative integer such that $k < t < k+1$, let $\Lambda_t(D)$ denote the space of all functions $f \in A^k(D)$ such that for every $\alpha \in Z^n_+$ with $|\alpha| = k$, there exists a constant $c_\alpha > 0$ such that

$$|D^\alpha f(z) - D^\alpha f(z')| \leqslant c_\alpha |z-z'|^{t-k}, \quad z, z' \in D.$$

$\Lambda_t(D)$ is a Banach space with the norm

$$\|f\|_{D,t} = \|f\|_{D,k} + \sum_{|\alpha|=k} \sup_{z,z' \in D} |D^\alpha f(z) - D^\alpha f(z')|/|z-z'|^{t-k}.$$

If M' is a closed complex analytic subvariety of a domain Ω in C^n and M is a relatively compact subdomain of M' such that ∂M contains no singular points of M' and satisfies the minimal regularity condition with respect to local coordinates [15], Chapter VI, § 3.3, we can introduce the spaces $A^k(M)$, $H^{\infty,k}(M)$ and $\Lambda_t(M)$ (for description see [13]). The norms in these spaces will be denoted by $\| \ \|_{M,k}$ and $\| \ \|_{M,t}$, respectively.

A bounded domain $D \subset C^n$ is called a *domain with $\mathscr{C}^k$-boundary* if there exists a neighbourhood U of ∂D and a real-valued function $\varrho \in \mathscr{C}^k(U)$ such that

(i) $D = (D \setminus U) \cup \{z \in U: \varrho(z) < 0\}$,

(ii) $\operatorname{grad} \varrho(z) \neq 0$ for $z \in \partial D$.

The function ϱ is called a *defining function for D*. D is called *strictly pseudoconvex* (with $\mathscr{C}^k$-boundary, $k \geqslant 2$) if the function ϱ can be chosen in such a way that it is strictly plurisubharmonic in a neighbourhood of ∂D. Moreover, if the defining function ϱ can be chosen so that $\varrho \in \mathscr{C}^k(C^n)$ satisfies the condition $\lim_{z \to \infty} \varrho(z) = +\infty$ and has a real Hessian positive definite at every point of C^n, and $D = \{z \in C^n: \varrho(z) < 0\}$, then D is called a *strictly convex domain with $\mathscr{C}^k$-boundary* ([6], p. 529).

Let X be a complex subvariety of C^n and let M be an open connected and relatively compact set in X. We say [6], p. 546, that M is a *strictly pseudoconvex domain with $\mathscr{C}^p$-boundary on X* if ∂M contains no singular points of X and for every $\xi \in \partial M$ there exist a neighbourhood U of ξ in C^n and a function $\varrho \in \mathscr{C}^p(U)$ such that ϱ is strictly plurisubharmonic in U, $\operatorname{grad} \varrho(z) \neq 0$ for $z \in U$, and $U \cap M = \{z \in M': \varrho(z) < 0\}$.

If X is a complex submanifold of C^n, we denote by $T(X)$ and $T^*(X)$ the complex tangent and cotangent spaces to X, respectively. Similarly, given $z \in X$, $T_z(X)$ and $T_z^*(X)$ will denote the complex tangent and cotangent spaces to X at a point z.

For an arbitrary set E, we denote by $\Delta(X)$ the diagonal of X in the Cartesian product $E \times E$.

0. Introduction. Let M' be a closed analytic subvariety of some open subset Ω of C^n. Suppose that $M \subset\subset M'$ is a strictly pseudoconvex domain in M' with $\mathscr{C}^p$-boundary. Suppose also that $D \subset \Omega$ is a strictly pseudoconvex domain with $\mathscr{C}^p$-boundary, such that $M = D \cap M'$, and that M' intersects ∂D transversally.

Denote by $\tilde{A}(M \times M)$ one of the spaces $A^k(M \times M)$, $H^{\infty,k}(M \times M)$ or $\Lambda_t(M \times M)$. We shall prove the following theorem on the extension of holomorphic functions from $M \times M$ to $D \times D$:

THEOREM 1. *Let M, M', D and p be as above. Suppose that $p \geqslant k+5$ if $\tilde{A} = A^k$ or $H^{\infty,k}$, or $p \geqslant k+6$ in the case $\tilde{A} = \Lambda_t$, and $k < t < k+1$.*

Then there exists a linear and continuous extension operator

$$L\colon \tilde{A}(M \times M) \to \tilde{A}(D \times D).$$

The existence of the extension operator from $H^\infty(M)$ and $A(M)$ to $H^\infty(D)$ and $A(D)$, respectively, was proved by Henkin in [9] (under the assumption that ∂D is only of class $\mathscr{C}^2$), and the similar results for $A^\infty(M)$ were obtained in [5] and [1], and in [13] for other cases of $\tilde{A}(M)$. We prove Theorem 1 with the aid of the techniques similar to those used in [9] and [13], combined with some facts concerning holomorphic functions with values in Fréchet spaces, derived from [4].

As a corollary, we obtain the analogous results on the extension of functions in $\tilde{A}(M \times M)$ which vanish on the diagonal of $M \times M$. This corollary together with a certain construction due to Fornaess [6] and the results obtained in [12], enable us to prove the following decomposition theorem:

Denote by $\tilde{A}_0(M \times M)$ the space of all functions in $\tilde{A}(M \times M)$ which vanish on the diagonal. (Here, as above, $\tilde{A}$ denotes A^k, $H^{\infty,k}$ or Λ_t.)

THEOREM 2. *Suppose that M and p satisfy the assumptions of Theorem 1.*

Then for every $f \in \tilde{A}_0(M \times M)$ there exist functions $f_1(z, s), \ldots \ldots, f_n(z, s) \in \mathcal{O}(M \times M)$ which satisfy

$$f(z, s) = \sum_{i=1}^{n} (z_i - s_i) f_i(z, s), \qquad z, s \in M, \tag{0.1}$$

and such that:

(a) *If $\tilde{A} = A^k$, then $f_i(z, s)$ are of class $\mathscr{C}^k$ in $Q = (\bar{M} \times \bar{M}) \setminus \Delta(\partial M)$.*

(b) *If $\tilde{A} = H^{\infty,k}$, then for every compact subset K of Q the derivatives of $f_i(z, s)$ up to order $k-1$ are continuous on K and the derivatives of order k are bounded on K.*

(c) *If $\tilde{A} = \Lambda_t$, then for every compact $K \subset Q$ the derivatives of $f_i(z, s)$ up to order k are continuous on K and the derivatives of order k are $(t-k)$-Hölder continuous on K.*

Moreover, for each compact $K \subset Q$, the mapping

$$\tilde{A}_0(M \times M) \ni f \mapsto (f_1, \ldots, f_n) \in (\tilde{A}(K))^n$$

is linear and continuous.

In particular, for every fixed $s \in M$, the mapping

$$f \mapsto (f_1(\cdot, s), \ldots, f_n(\cdot, s))$$

is a linear continuous operator from $\tilde{A}_0(M \times M)$ to $(\tilde{A}(M))^n$.

We prove also that the functions $f_i(z, s)$ from Theorem 2 have the following properties:

PROPOSITION 3. *If $f \in A_0^k(M \times M)$, $H_0^{\gamma,k}(M \times M)$ or $(\Lambda_t)_0(M \times M)$, then for each $i = 1, \ldots, n$ the function $f_i(z, s)$ belongs to $A^{k-1}(M \times M)$, $H^{\gamma,k-1}(M \times M)$ or $\Lambda_{t-1}(M \times M)$, respectively.* (We assume here that $k \geq 1$ and $t > 1$.)

This proposition follows easily from the construction of the functions $f_i(z, s)$ in the proof of Theorem 2.

Finally, we shall show that under the assumption that M is a manifold and ∂M is sufficiently smooth, one can construct an extension operator given by an integral formula. We recall that if $D \subset C^n$ is a strictly pseudoconvex domain with $\mathscr{C}^p$-boundary and ϱ is a defining function for D, there exist [7], Lemma 2.4, [6], Theorem 16, a neighbourhood $\tilde{D}$ of $\bar{D}$ and a function $\Phi(\xi, z) \in \mathscr{C}^{p-1}(\tilde{D} \times \tilde{D})$ holomorphic in $z \in \tilde{D}$ such that

(i) $\Phi(\xi, \xi) = 0$ for all $\xi \in \tilde{D}$,

(ii) for every compact subset K of $\tilde{D}$ there exists a constant $\gamma = \gamma(K) > 0$ such that $\operatorname{Re} \Phi(\xi, z) \geq \varrho(\xi) - \varrho(z) + \gamma|\xi - z|^2$ for $(\xi, z) \in K \times K$,

(iii) there exists a neighbourhood U of ∂D such that the vectors $\operatorname{grad} \varrho(\xi)$ and $\operatorname{grad}_\xi \operatorname{Im} \Phi(\xi, z)|_{\xi = z}$ are linearly independent (over R) for every $\xi \in U$.

THEOREM 4. *Let M' be a complex submanifold of an open set $\Omega \subset C^n$,* $\dim M' = k$, *and let $D \subset\subset \Omega$ be strictly pseudoconvex with $\mathscr{C}^p$-boundary. Suppose that $M = M' \cap D$ is connected and that M' intersects ∂D transversally.*

There exist a function $\Phi(\xi, z)$ which satisfies properties (i)–(iii), *constructed with respect to the domain D, a neighbourhood V of $\bar{D}$, and a function $H(\xi, z) \in \mathscr{C}^{p-2}(\partial M \times V)$ holomorphic in $z \in V$ such that the formula*

$$L_1 f(z) = \int_{\partial M} f(\xi) \frac{H(\xi, z)}{\Phi(\xi, z)^k} \, d\sigma(\xi) \tag{0.2}$$

defines an extension operator from $H^\gamma(M)$ to $\mathcal{O}(D)$; moreover, L_1 is a linear and continuous extension operator from $\tilde{A}(M)$ to $\tilde{A}(D)$ (where $\tilde{A}$ denotes A^k, $H^{\gamma,k}$ or Λ_t), provided that p satisfies the assumptions of Theorem 1. (Here $d\sigma(\xi)$ denotes the volume measure on ∂M.)

We prove this theorem using the results of Henkin and Leiterer [10] on the existence of an integral formula for holomorphic functions in strictly pseudoconvex domains in complex manifolds, and the results of [12].

Iterating formula (0.2), we obtain an extension operator from $M \times M$ to $D \times D$:

THEOREM 4'. *Under the assumptions of Theorem* 4, *the formula*

$$L_2 f(z, s) = \int_{\partial M} \int_{\partial M} f(\xi, \eta) \frac{H(\xi, z)}{\Phi(\xi, z)^k} \frac{H(\eta, s)}{\Phi(\eta, s)^k} d\sigma(\xi) d\sigma(\eta)$$

defines a linear and continuous extension operator from $\tilde{A}(M \times M)$ *to* $\tilde{A}(D \times D)$.

This extension operator has a much more explicit form than the one constructed in Theorem 1.

1. Smoothness of some integral operators. We present here some results on the behaviour of integral operators with the kernel introduced by Henkin [7], acting on certain function spaces. They generalize the theorems obtained in [2] and [11].

Throughout this section we keep the following notation:

D is a strictly pseudoconvex domain in C^n and $\tilde{D}$ is a neighbourhood of $\bar{D}$. $\psi: \tilde{D} \to C^m$ is a biholomorphic mapping of $\tilde{D}$ onto a closed complex submanifold $\psi(\tilde{D})$ of C^m and $C \subset C^m$ is a strictly convex domain such that $\psi(D) \subset C$, $\psi(\tilde{D} \setminus \bar{D}) \subset C^m \setminus \bar{C}$, and $\psi(\tilde{D})$ intersects ∂C transversally. Moreover, if ϱ is a defining function for C, we set

$$f(\xi, z) = \sum_{i=1}^{m} \frac{\partial \varrho}{\partial \xi_i}(\xi)(\xi_i - z_i), \qquad \xi, z \in C^m.$$

Let $U \subset \psi(\tilde{D})$ and $\tilde{C} \subset C^m$ be neighbourhoods of $\psi(\bar{D})$ and $\bar{C}$ in $\psi(\tilde{D})$ and in C^m, respectively.

LEMMA 1.1. *Suppose that* ∂D *and* ∂C *are of class* $\mathscr{C}^p$, *where p satisfies the assumptions of Theorem* 1. *Let* $H(\xi, z)$ *be a function defined and of class* $\mathscr{C}^{p-2}$ *in* $U \times \tilde{C}$, *and holomorphic with respect to* $z \in C$. *For* $h \in \tilde{A}(\psi(D))$ *set*

$$Ph(z) = \int_{\psi(\partial D)} h(\xi) \frac{H(\xi, z) d\sigma(\xi)}{f(\xi, z)^n}, \qquad z \in C.$$

Then $Ph \in \tilde{A}(C)$ *and the operator*

$$\tilde{A}(\psi(D)) \ni h \to Ph \in \tilde{A}(C)$$

is linear and continuous.

This lemma, for domains with $\mathscr{C}^\infty$-boundaries and for $H(\xi, z) \in \mathscr{C}^\infty(U \times \tilde{C})$, is in [13]; the proof of it in the actual form is similar.

LEMMA 1.2. *Let ∂D, ∂C, p and $H(\xi, z)$ be as above. Suppose that $E \subset \mathbf{C}^k$ is a domain with $\mathscr{C}^1$-boundary. For $h \in \tilde{A}(\psi(D) \times E)$ set*

$$P_1 h(z, s) = \int_{\psi(\partial D)} h(\xi, s) \frac{H(\xi, z)\, d\sigma(\xi)}{f(\xi, z)^n}, \quad (z, s) \in C \times E.$$

Then P_1 is a linear and continuous operator from $\tilde{A}(\psi(D) \times E)$ to $\tilde{A}(C \times E)$.

Proof. Consider first the case $\tilde{A} = H^{\infty,k}$. Clearly $P_1 h \in \mathcal{O}(C \times E)$. Let $D_z^\alpha D_s^\beta$ be a differential operator with $|\alpha| + |\beta| \leqslant k$. Our task is to show that for every $h \in H^{\infty,k}(D \times E)$ and every $(z, s) \in C \times E$,

$$|D_z^\alpha D_s^\beta P_1 h(z, s)| \leqslant c\, \|h\|_{D \times E, k},$$

for some c independent of h, z, and s.

We have

$$D_z^\alpha D_s^\beta P_1 h(z, s) = D_z^\alpha \int_{\psi(\partial D)} D_s^\beta h(\xi, s) \frac{H(\xi, z)\, d\sigma(\xi)}{f(\xi, z)^n},$$

and for every fixed $s \in E$, $D_s^\beta f(\cdot, s) \in H^{\infty, k-|\beta|}(\psi(D))$. Therefore, by Lemma 1.1,

$$\begin{aligned} |D_z^\alpha D_s^\beta P_1 h(z, s)| &= |D^\alpha (P D_s^\beta h(\cdot, s))(z)| \leqslant c' \|D_s^\beta h(\cdot, s)\|_{\psi(D), k-|\beta|} \\ &\leqslant c\, \|h\|_{\psi(D) \times E, k}, \end{aligned}$$

where c' and c are independent of h, z and s.

Now let $\tilde{A} = A^k$. It is sufficient to prove that for every α, β with $|\alpha| + |\beta| = k$, the function $D_z^\alpha D_s^\beta P_1 h$ extends continuously to $\bar{C} \times \bar{E}$.

For every $h \in A^k(\psi(D) \times E)$ and every $s_0 \in \bar{E}$, $(D_s^\beta h)(\cdot, s_0) \in A^{k-|\beta|}(\psi(D))$. Hence, by Lemma 1.1, the function $P(D_s^\beta h)(\cdot, s_0)$ is in $A^{k-|\beta|}(C)$. For $z_0 \in \bar{C}$ set

$$P_{\alpha\beta} h(z_0, s_0) = D^\alpha (P(D_s^\beta h)(\cdot, s_0))(z_0).$$

Note that $P_{\alpha\beta} h(z, s) = D_z^\alpha D_s^\beta P_1 h(z, s)$ for $(z, s) \in C \times E$. Therefore it remains to show that $P_{\alpha\beta} h \in \mathscr{C}(\bar{C} \times \bar{E})$. Fix $(z_0, s_0) \in \bar{C} \times \bar{E}$ and let $\{(z_n, s_n)\}_{n=1}^\infty \subset \bar{C} \times \bar{E}$ be a sequence convergent to (z_0, s_0). Then

$$\begin{aligned} |P_{\alpha\beta} h(z_n, s_n) - P_{\alpha\beta} h(z_0, s_0)| &\leqslant |D^\alpha [P(D_s^\beta h)(\cdot, s_n) - P(D_s^\beta h)(\cdot, s_0)](z_n)| + \\ &\quad + |D^\alpha (P(D_s^\beta h)(\cdot, s_0))(z_n) - D^\alpha (P(D_s^\beta h)(\cdot, s_0))(z_0)|. \end{aligned}$$

The first term on the right is bounded by

$$c\, \|D_s^\beta h(\cdot, s_n) - D_s^\beta h(\cdot, s_0)\|_{\psi(D), k-|\beta|},$$

where c is independent of z_n, s_n, z_0, s_0 and h, and this expression is small if s_n is sufficiently close to s_0, since all derivatives of h of order $\leqslant k$ are uniformly continuous on $\psi(D)\times E$. The second term is small for large n, since $P(D_s^\beta h)(\cdot, s_0)\in A^{k-|\beta|}(C)$.

The proof of the lemma for the case $\tilde{A} = \Lambda_t$ is similar.

The following lemma is a special case of Lemma 1.2:

LEMMA 1.3. *If C, p, E and H are as above, then the operator P_2 defined by the formula*

$$\tilde{A}(C\times E)\ni h\mapsto P_2 h(z, s) = \int_{\partial C} h(\xi, s)\frac{H(\xi, z)\,d\sigma(\xi)}{f(\xi, z)^m}, \quad (z, s)\in C\times E,$$

is linear and continuous from $\tilde{A}(C\times E)$ into itself.

Now let D, C, ψ, p, H, U, $\tilde{C}$ and f be the same as in Lemma 1.1, and suppose we are given a second system $D_1 \subset C^{n_1}$, $C_1 \subset C^{m_1}$, ψ_1, H_1, U_1, $\tilde{C}_1$ and f_1 (with the same p), with analogous properties.

LEMMA 1.4. *The operator*

$$\tilde{A}(\psi(D)\times\psi_1(D_1))\ni h\mapsto P_3 h(z, s)$$
$$= \int_{\xi\in\psi(\partial D)}\int_{\eta\in\psi_1(\partial D_1)} h(\xi, \eta)\frac{H(\xi, z)H_1(\eta, s)\,d\sigma(\xi)\,d\sigma_1(\eta)}{f(\xi, z)^n f_1(\eta, s)^{n_1}}, \quad (z, s)\in C\times C_1,$$

is linear and continuous from $\tilde{A}(\psi(D)\times\psi_1(D_1))$ to $\tilde{A}(C\times C_1)$.

Proof. We have

$$P_3 h(z, s) = \int_{\psi(\partial D)} F_h(\xi, s)\frac{H(\xi, z)\,d\sigma(\xi)}{f(\xi, z)^n},$$

where

$$F_h(\xi, s) = \int_{\psi_1(\partial D_1)} h(\xi, \eta)\frac{H_1(\xi, \eta)\,d\sigma_1(\eta)}{f_1(\eta, s)^{n_1}}.$$

The conclusion follows by an iterated application of Lemma 1.2.

2. An extension theorem for holomorphic functions with values in Fréchet spaces. In this section we recall some facts from the theory of holomorphic functions with values in topological vector spaces. We start with lemmas on the identification of certain function spaces and then we recall a theorem of Bungart on the existence of an extension operator from $H^\infty(Y, E)$ to $\mathcal{O}(X, E)$, where Y is a subvariety of a Stein space X and E is a Fréchet space ([4], Corollary 18.2). Those results will be used in the proof of Theorem 1; therefore we restrict ourselves to the case of holomorphic functions with values in Fréchet spaces. (In fact, all function spaces, the ranges of holomorphic functions, which will be considered in Section 3, are actually Banach spaces.)

Let E be a Fréchet space. If D is an open subset of C^n, we denote by $\mathcal{O}(D, E)$ the space of holomorphic functions in D with values in E, $f\colon D \to E$, locally representable as convergent power series with coefficients in E. If X is an analytic space locally realizable as a subvariety of some open subset of C^n, then $f\colon X \to E$ is said to be *holomorphic* with values in E (we write $f \in \mathcal{O}(X, E)$) if for every $x \in X$ there exist a neigbourhood U of x in C^n and a function $F \in \mathcal{O}(U, E)$ such that $F|_{X \cap U} = f|_{X \cap U}$. We will denote by $H^\infty(X, E)$ the space of bounded holomorphic functions on X with values in E.

One can easily prove the following lemma:

LEMMA 2.1. *Let M' be a complex subvariety of an open set $\Omega_\alpha \subset C^{n_\alpha}$, and let $M_\alpha \subset\subset M'_\alpha$ be a domain with $\mathscr{C}^1$-boundary, $\alpha = 1, 2$. Suppose also that $N \subset\subset M_1$ is a domain with $\mathscr{C}^1$-boundary. Then the following mappings are linear and continuous*:

(a) $R_1\colon \tilde{A}(M_1 \times M_2) \to H^\infty(M_1, \tilde{A}(M_2))$, *where* $((R_1 f)(z))(s) = f(z, s)$;

(b) $R_2\colon \mathcal{O}(M_1, \tilde{A}(M_2)) \ni f \mapsto f|_N \in H^\infty(N, \tilde{A}(M_2))$;

(c) $R_3\colon \mathcal{O}(M_1, \tilde{A}(M_2)) \to \tilde{A}(N \times M_2)$, *given by the formula* $(R_3 f)(z, s) = (f(z))(s)$.

In [4] Bungart proved the following extension theorem:

THEOREM 2.2 ([4], Corollary 18.2). *Let X be a Stein space, Y a closed subvariety of X. Then there exists a continuous and linear extension operator* $P\colon H^\infty(Y, E) \to \mathcal{O}(X, E)$. (As usual, $\mathcal{O}(X, E)$ is endowed with the topology of uniform convergence on compact subsets of X, and $H^\infty(Y, E)$ has the topology of uniform convergence on Y.)

3. Extension operators in products of strictly pseudoconvex sets. The main result of this section is Theorem 1. As mentioned in the introduction, we adopt here the method of the proof of [9], Main Theorem, with convenient modifications. We first show that, under suitable assumptions on the smoothness of ∂D, there exists an extension operator

$$\tilde{L}\colon \tilde{A}(M \times M) \to \tilde{A}(D \times M). \tag{3.1}$$

In order not to repeat the details of the construction presented in [9], which forms a part of the proof of our theorem, we will only indicate the differences between the two proofs and give the necessary references to [9].

We construct the domains D_ν, $\tilde{D}_\nu$, M_ν and $\tilde{M}_\nu$ as in [9], p. 562–563; this can be done in such a way that the regularity of their boundaries is the same as that of ∂M, and $\bar{M}_\nu \setminus M$ and $\bar{\tilde{M}}_\nu \setminus M$ do not contain singular points of M'.

Define the operators R_ν^α, $\alpha = 0, 1$, acting on $f(z, s) \in \tilde{A}(M_{\nu-1} \times M)$, as in formula (4.9) of [9], with $f(z(\zeta, \zeta^*), s)$ in place of $f(z(\zeta, \zeta^*))$. In virtue of Lemma 1.3 the operator R_ν^0 is continuous and linear from $\tilde{A}(M_{\nu-1} \times M)$ to $\tilde{A}((M_\nu \cap S_{\zeta^*, 3\delta/4}) \times M)$, and R_ν^1 is continuous and linear as an operator from $\tilde{A}(M_{\nu-1} \times M)$ to $\tilde{A}((\tilde{M}_\nu \cap S_{\zeta^*, 3\delta/4}) \times M)$ and from $\tilde{A}(M_{\nu-1} \times M)$ to $\tilde{A}((M''_0 \cap$

$\cap M_1'')\times M)$. (We refer to [9], p. 562–563 for the description of all sets and quantities appearing above.)

Set $F=\tilde{A}((M_0''\cap M_1'')\times M)$. Consider the mapping $\delta\in\mathcal{O}(M_0''\cap M_1'', \mathcal{L}(F,\tilde{A}(M)))$ (where $\mathcal{L}(F,\tilde{A}(M))$ denotes the Banach space of all linear continuous operators from F to $\tilde{A}(M)$), defined by

$$((\delta(z))f)(s)=f(z,s),\quad f\in F,\ z\in M_0''\cap M_1''.$$

We conclude as in [9] that there exist functions $\delta^\alpha\in\mathcal{O}(M_\alpha'', \mathcal{L}(F,\tilde{A}(M)))$, $\alpha=0,1$, such that $\delta=\delta^0+\delta^1$ on $M_0''\cap M_1''$. Define the operators

$$T_\nu^\alpha\colon F\to\mathcal{O}(M_\alpha'',\tilde{A}(M)),\quad \alpha=0,1,$$

by the formula

$$(T_\nu^\alpha f)(z)=(\delta^\alpha(z))f,\quad f\in F,\ z\in M_\alpha''.$$

It follows that T_ν^α are linear and continuous, and that for every $f\in F$

$$f(z)=(T_\nu^0 f)(z)+(T_\nu^1 f)(z),\quad z\in M_0''\cap M_1''.$$

Proceeding now as in [9], p. 564, we conclude that the following lemma on the separation of singularities holds:

LEMMA 3.1. *For every $\varepsilon>0$ there exist a covering $\{\tilde{M}_i\}_{i=1}^N$ of ∂M by domains $\tilde{M}_i$ with $M\subset\tilde{M}_i$ and $\operatorname{diam}(\bar{M}\setminus\tilde{M}_i)<\varepsilon$ for every i, and the continuous and linear operators $L_i\colon \tilde{A}(M\times M)\to\tilde{A}(\tilde{M}_i\times M)$, $i=1,\ldots,N$, such that for every $f\in\tilde{A}(M\times M)$,*

$$f(z,s)=\sum_{i=1}^N (L_i f)(z,s),\quad z,s\in M.$$

In the second part of the proof of Theorem 1 we also keep the line of the proof of [9], Main Theorem, p. 564. The first modification is that the operator L defined by formula (4.20) of [9] is now a continuous and linear operator

$$L\colon \tilde{A}((\tilde{M}\cap G_{\zeta_0}^*)\times M)\to\tilde{A}(G_{\zeta_0}^*\times M).$$

This follows from Lemma 1.2. We rely now on Theorem 2.2 and Lemma 2.1 (a) to conclude that there exists a linear and continuous extension operator

$$B\colon \tilde{A}(\tilde{M}\cap D^*,\tilde{A}(M))\to\mathcal{O}(D^*,\tilde{A}(M))$$

(this is an analogue of the operator B defined in (4.21) of [9]).

The construction of the domains D'', G'', G_0'' and G_1'' is similar to that given in [9], p. 566. Let $F=H_{M''}^\infty(G'',\tilde{A}(M))$ be the space of functions holomorphic and bounded on G'', with values in $\tilde{A}(M)$, and vanishing on M''. Define a function $\delta\in\mathcal{O}_{M''}(G'',\mathcal{L}(F,\tilde{A}(M)))$, by $\delta(z)(f)=f(z)$, $f\in F$,

$z \in G''$. Exactly as in [9], p. 566, we conclude that there exist functions $\delta^\alpha \in \mathcal{O}_{M''}\big(G'', \mathcal{L}(F, \tilde{A}(M))\big)$, $\alpha = 0, 1$, such that $\delta = \delta^0 - \delta^1$ in G'', and the relevant operators

$$L_\alpha\colon F \to \mathcal{O}_{M''}(G'', \tilde{A}(M))$$

defined by the formula

$$(L_\alpha f)(z) = \delta^\alpha(z)(f),$$

are linear and continuous. (Here and before the subscript M'' indicates that we consider the space of functions which vanish on M''.) Proceeding as on p. 567 of [9] and using Lemmas 2.1 (b) and 2.1 (c), we obtain the desired operator $\tilde{L}$ from (3.1).

Interchanging the role of variables and repeating the process from the above construction, we obtain that there exists a linear and continuous extension operator

$$\tilde{\tilde{L}}\colon \tilde{A}(D \times M) \to \tilde{A}(D \times D).$$

Then $L = \tilde{\tilde{L}} \circ \tilde{L}$ is the operator whose existence was asserted in Theorem 1.

Consider now the space $\tilde{A}_0(M \times M)$. For any $f \in \tilde{A}_0(M \times M)$, set

$$L'f(z, s) = Lf(z, s) - Lf(s, s),$$

where $L\colon \tilde{A}(M \times M) \to \tilde{A}(D \times D)$ is the extension operator from Theorem 1. Then $L'f \in \tilde{A}_0(D \times D)$, and it is easy to show

PROPOSITION 3.2. *Under the hypothesis of Theorem* 1, L' *is a linear and continuous extension operator from* $\tilde{A}_0(M \times M)$ *to* $\tilde{A}_0(D \times D)$.

This proposition will be used in the next section in the proof of the decomposition theorem for functions from the space $\tilde{A}_0(M \times M)$.

Remark. It is clear from the proof of Theorem 1 that iterating the above procedure one can prove the existence of extension operators between the spaces $\tilde{A}(M_1 \times \ldots \times M_l)$ and $\tilde{A}(D_1 \times \ldots \times D_l)$, where M_i and D_i are related as M and D in Theorem 1.

4. The decomposition theorems in products of strictly pseudoconvex sets. We prove here Theorems 2 and 3. Many authors have considered Theorem 2 in a particularly important case when $f(z, s)$ has the form $g(z) - g(s)$, with $g \in \tilde{A}(M)$, and M is a strictly pseudoconvex domain in $\mathbf{C}^n$ with sufficiently smooth boundary; they proved the existence of decomposition operators for various spaces of holomorphic functions, not only of the form $\tilde{A}(M)$ (see e.g. [8], [14], [2], [3]).

To begin with the proof of Theorem 2, we remind an embedding theorem of Fornaess:

THEOREM 4.1 ([6], Theorem 9). *Let D be a strictly pseudoconvex domain in C^n with $\mathscr{C}^p$-boundary. Then there exist a neighbourhood $\tilde{D}$ of $\bar{D}$, a non-negative integer $m \geqslant n$, a holomorphic mapping ψ: $\tilde{D} \to C^m$ such that ψ maps $\tilde{D}$ biholomorphically onto a close subvariety $\psi(\tilde{D})$ of C^m, and a strictly convex domain $C \subset C^m$ with $\mathscr{C}^p$-boundary, such that $\psi(D) \subset C$, $\psi(\tilde{D} \setminus \bar{D}) \subset C^m \setminus \bar{C}$, and $\psi(\tilde{D})$ intersects ∂C transversally.*

In virtue of this theorem we may assume that D (and, of course, M) is embedded by ψ into some convex domain $C \subset C^m$. Since $\psi(M)$ intersects ∂C transversally, it follows from Proposition 3.2 that under the assumption on ∂M as in the hypotheses of Theorem 2, there exists a continuous and linear extension operator

$$L': \tilde{A}_0(\psi(M) \times \psi(M)) \to \tilde{A}_0(C \times C).$$

Suppose that Theorem 2 is proved in the case of strictly convex domains. In order to show that it holds in general, we proceed as in [12]. Namely, if $\psi = (\psi_1, \ldots, \psi_m)$, we can find, by Hefer's theorem, functions $\psi_{ij} \in \mathcal{O}(\tilde{D} \times \tilde{D})$ such that

$$\psi_i(w) - \psi_i(t) = \sum_{j=1}^{n} (w_j - t_j)\psi_{ij}(w, t), \qquad w, t \in D.$$

If $f \in \tilde{A}_0(M \times M)$, the function $Tf(z, s) = f(\psi^{-1}(z), \psi^{-1}(s))$, $z, s \in \psi(M)$, is in $\tilde{A}_0(\psi(M) \times \psi(M))$. Then

$$L' Tf(z, s) = \sum_{i=1}^{m} (z_i - s_i)\tilde{f}_i(z, s), \qquad z, s \in C,$$

for some $\tilde{f}_i \in \mathcal{O}(C \times C)$ satisfying the assertion of Theorem 2, according to the meaning of the symbol $\tilde{A}$. Eventually,

$$f(w, t) = L' Tf(\psi(w), \psi(t)) = \sum_{i=1}^{m} (\psi_i(w) - \psi_i(t))\tilde{f}_i(\psi(w), \psi(t))$$

$$= \sum_{j=1}^{n} (w_j - t_j) f_j(w, t)$$

with

$$f_j(w, t) = \sum_{i=1}^{m} \psi_{ij}(w, t)\tilde{f}_i(\psi(w), \psi(t)).$$

It is easy to see that $f_j(w, t)$ has the properties stated in Theorem 2.

This gives the desired decomposition provided that Theorem 2 holds for strictly convex domains.

Suppose now that $D = C$ is a strictly convex domain in C^n with $\mathscr{C}^2$-boundary. We will follow the construction of Lejbenzon [8] and will use the estimates from the proof of Proposition 10 in [12].

In the sequel, for functions $f(z, s)$ defined in the set $C \times C$, we use the notation $D_\xi^\alpha f$, $D_\eta^\beta f$, to denote differentiation with respect to the first and to the second group of variables, respectively.

Let $z, s \in C$ and let $f \in \tilde{A}_0(C \times C)$. Then

$$f(z, s) = f(z, s) - f(s, s) = \sum_{i=1}^{n} (z_i - s_i) \int_0^1 D_\xi^i f(s + \lambda(z-s), s) d\lambda.$$

Set

$$f_i(z, s) = \int_0^1 D_\xi^i f(s + \lambda(z-s), s) d\lambda = \int_0^{1/2} + \int_{1/2}^1 = F_i(z, s) + G_i(z, s).$$

It is clear that $f_i \in \mathscr{C}(C \times C)$, and it remains to prove the desired smoothness properties of f_i. We do this by showing the similar properties of F_i and G_i.

Since ∂C is of class $\mathscr{C}^2$, there exists a neighbourhood W of ∂C such that the orthogonal projection onto ∂C, π: $W \ni z \to \pi(z) \in \partial C$, is well defined and of class $\mathscr{C}^1$. For every $z \in \partial C$, choose vectors $v^{(r)}(z)$, $r = 1, \ldots, n-1$, spanning $T_z(\partial C)$, and set $v^{(r)}(z) = v^{(r)}(\pi(z))$, $z \in W$. Then for every $z_0 \in \partial C$ there exists a neighbourhood U of z_0 such that we can assume $v^{(r)}(z)$ to be of class $\mathscr{C}^1$ in U.

In order to prove the theorem, it is sufficient to show that for every $(z_0, s_0) \in (\bar{C} \times \bar{C}) \setminus \Delta(\partial C)$ there exist neighbourhoods U, V in $\bar{C}$ of z_0 and s_0, respectively, such that $(U \times V) \cap \Delta(\partial C) = \emptyset$ and that the mappings

$$\tilde{A}_0(C \times C) \ni f \to F_i|_{U \times V} \in \tilde{A}(U \times V), \tag{4.1}$$

$$\tilde{A}_0(C \times C) \ni f \to G_i|_{U \times V} \in \tilde{A}(U \times V) \tag{4.2}$$

are linear and continuous. We assume that $z_0, s_0 \in \partial C$ and $z_0 \neq s_0$; the other cases are easier.

Let z_0, s_0 be as above. It follows from the strict convexity of C that there exist neighbourhoods U and V of z_0 and s_0 in $\bar{C}$ and constants ε, γ, and $\delta > 0$ (depending only on C, U and V) such that:

(4.3) for every $(z, s) \in (U \times V) \cap (C \times C)$, for every $u \in T_z(\partial C)$ and $v \in T_s(\partial C)$, with $\|u\| \leqslant 1$ and $\|v\| \leqslant 1$ and for every $\lambda \in (1-\varepsilon, 1)$ (resp. for every $\lambda \in (0, \varepsilon)$), the analytic disc $\{s + \lambda(z-s) + xu\colon x \in \mathbf{C},\ |x| < \gamma\sqrt{1-\lambda}\}$ (resp. $\{s + \lambda(z-s) + xv\colon x \in \mathbf{C},\ |x| < \gamma\sqrt{\lambda}\}$) is contained in C;

(4.4) for every $(z, s) \in U \times V$, for every $\lambda \in [\varepsilon, 1-\varepsilon]$, the ball $B(s + \lambda(z-s), \delta)$ is contained in C.

Moreover, we may assume, taking smaller U and V if necessary, that $v^{(r)}(z)$ and $v^{(r)}(s)$ are of class $\mathscr{C}^1$ in U and V.

Now let $\tilde{A} = H^{\alpha,k}$. Consider the functions $F_i(z, s)$. Let $u^{(1)}, \ldots, u^{(n-1)}$ be an arbitrary sequence of $n-1$ vectors in C^n. We obtain a system of linear equations in $F_i(z, s)$:

$$\sum_{i=1}^{n} (z_i - s_i) F_i(z, s) = f(s + \tfrac{1}{2}(z-s), s) - f(s, s) = f(s + \tfrac{1}{2}(z-s), s),$$

$$\sum_{i=1}^{n} u_i^{(r)} F_i(z, s) = \int_0^{1/2} \frac{d}{dx} \big(f(s + \lambda(z-s) + xu^{(r)}, s)\big)_{x=0} d\lambda,$$

$r = 1, \ldots, n-1$, $z, s \in C$.

Acting on the both sides of the above system with the operator $D_z^\alpha D_s^\beta$, $|\alpha| + |\beta| \leqslant k$, we obtain another system,

$$\sum_{i=1}^{n} (z_i - s_i) D_z^\alpha D_s^\beta F_i(z, s) = A(z, s), \tag{4.5}$$

$$\sum_{i=1}^{n} u_i^{(r)} D_z^\alpha D_s^\beta F_i(z, s) \tag{4.5'}$$
$$= \sum_{\beta' + \beta'' = \beta} \int_0^{1/2} \frac{d}{dx} \big(D_\xi^\alpha D_\xi^{\beta'} D_\eta^{\beta''} f(s + \lambda(z-s) + xu^{(r)}, s)\big)_{x=0} P_{\alpha\beta'\beta''}(\lambda)\, d\lambda,$$

$r = 1, \ldots, n-1$, where $P_{\alpha\beta'\beta''}(\lambda)$ are polynomials in λ, depending only on α, β' and β'', and $A(z, s)$ is a function, which involves only derivatives of $F_i(z, s)$ of order $< |\alpha| + |\beta|$ and the derivatives of f of order $|\alpha| + |\beta|$.

Let $u^{(r)} = v^{(r)}(s)$. Again, taking smaller U and V, we may assume that the determinant $D(z, s)$ of system (4.5), (4.5)' is a $\mathscr{C}^1$-function on $U \times V$, which is bounded away from zero. Therefore, we can express $D_z^\alpha D_s^\beta F_i(z, s)$ by other functions occurring in (4.5), (4.5)', according to Cramer's rule. It is not difficult to see that the function $A(z, s)$ is bounded on $U \times V$ by $c\|f\|_{C \times C,k}$, where c does not depend on f. Therefore in order to prove (4.1) it remains to show that for every $(z, s) \in (U \times V) \cap (C \times C)$ the integrals on the right-hand sides of equations (4.5)' are bounded by $c_1 \|f\|_{C \times C,k}$ for some c_1 independent of f. This is easy to see for $|\alpha| + |\beta| < k$, so assume that $|\alpha| + |\beta| = k$.

It follows from (4.3), (4.4) and Cauchy's inequalities that

$$\left|\int_0^{1/2} \frac{d}{dx} \big(D_\xi^\alpha D_\xi^{\beta'} D_\eta^{\beta''} f(s + \lambda(z-s) + xv^{(r)}(s), s)\big)_{x=0} P_{\alpha\beta'\beta''}(\lambda)\, d\lambda\right| \leqslant \left|\int_0^{\varepsilon}\right| + \left|\int_\varepsilon^{1/2}\right|$$

$$\leqslant c_1 \|f\|_{C \times C,k} \int_0^{\varepsilon} \frac{d\lambda}{\sqrt{\lambda}} + c_2 \|f\|_{C \times C,k} \int_\varepsilon^{1/2} \frac{d\lambda}{\lambda} \leqslant c\|f\|_{C \times C,k},$$

where c_1, c_2 and c depend only on the choice of U and V, and on α, β' and β''. This ends the proof of (4.1) for the spaces $H_0^{\alpha,k}(C \times C)$.

The proof of the continuity of the mapping in (4.2) is similar.

Consider the space $A_0^k(C \times C)$. In virtue of the preceeding argument, it is sufficient to show that the derivatives of order k of the mappings $F_i|_{(U \times V) \cap (C \times C)}$ and $G_i|_{(U \times V) \cap (C \times C)}$ extend continuously to the whole of $U \times V$. The proof results by a minor modification of that of [8] or [12], and we recall it briefly.

First note that for every $h \in H^\infty(C \times C)$ and for every $(z, s) \in (U \times V) \cap \cap (C \times C)$ and $\lambda \in (0, \frac{1}{2}]$ we obtain from (4.3), (4.4) and Cauchy's inequalities the estimate

$$(4.6) \qquad \left|\frac{d}{dx}\left(h(s+\lambda(z-s)+vx^{(r)}(s), s)\right)_{x=0}\right| \leqslant c\,\frac{\|h\|_{C \times C}}{\sqrt{\lambda}},$$

c independent of h. Let $\tilde{z} \in U$ and $\tilde{s} \in V \cap \partial C$ be fixed. Expressing $D_z^\alpha D_s^\beta F_i(z, s)$ by Cramer's rule from system (4.5), (4.5)' (with $u^{(r)} = v^{(r)}(s)$) and using the Lebesgue dominated convergence theorem (which is legitimate in virtue of estimates (4.6)), we see that there exists $\lim\limits_{s \to \tilde{s}} D_z^\alpha D_s^\beta F_i(\tilde{z}, \tilde{s})$, where the limit is taken with respect to s from the interval $[\tilde{s}; \tilde{z}]$. We call this limit $D_z^\alpha D_s^\beta F_i(\tilde{z}, \tilde{s})$. Moreover, one can prove quite similarly that the function

$$V \cap \partial D \ni \tilde{s} \to D_z^\alpha D_s^\beta F_i(z, s)$$

is continuous on $V \cap \partial D$. Theorefore, for every fixed $\tilde{z} \in U$, $D_z^\alpha D_s^\beta F_i(\tilde{z}, s)$ extends continuously to $V \cap \partial C$. Another standard argument, which will not be presented here, shows that $D_z^\alpha D_s^\beta F_i(z, s)$ is continuous as a function of $(z, s) \in U \times V$.

In the proof of the continuity of $G_i|_{U \times V}$ on $U \times V$ we use the same method.

To end the proof of Theorem 2, it remains to show that if $f \in (A_t)_0 \times \times (C \times C)$, $k < t < k+1$ and $|\alpha| + |\beta| = k$, then $D_z^\alpha D_s^\beta F_i(z, s)$ satisfies the Hölder condition with the exponent $t-k$ on $(U \times V) \cap (C \times C)$, and that the same is true for $G_i(z, s)$.

We consider again system (4.5), (4.5)' with $u^{(r)} = v^{(r)}(s)$, and express $D_z^\alpha D_s^\beta F_i(z, s)$ by Cramer's rule. As noted above, the determinant $D(z, s)$ of this system can be chosen to be $\mathscr{C}^1$ and bounded away from 0 on $U \times V$, and it is not difficult to prove that $A(z, s)$ are $(t-k)$-Hölder continuous on $(U \times V) \cap (C \times C)$. To apply the standard argument, it is sufficient to verify that the functions defined by the integrals in the right-hand side of (4.5)' are $(t-k)$-Hölder continuous. Fix β', β'' with $\beta' + \beta'' = \beta$, and denote

$$H(z,s) = \int_0^{1/2} \frac{d}{dx}\left(D_\xi^\alpha D_\xi^{\beta'} D_\eta^{\beta''} f\left(s+\lambda(z-s)+xv^{(r)}(s), s\right)\right)_{x=0} P_{\alpha\beta'\beta''}(\lambda)\,d\lambda.$$

By (4.3), (4.4) and Cauchy's inequalities we obtain the estimate

$$|H(z,s)-H(z',s')| \leqslant c_1 \|f\|_{D\times D,t} \sup\{|(s+\lambda(z-s)+xv^{(r)}(s), s)-$$

$$-(s'+\lambda(z'-s')+xv^{(r)}(s'), s')|^{t-k}:\ \lambda\in(0,\varepsilon],\ x\in B(0,\gamma\sqrt{\lambda})\}\cdot\int_0^{\varepsilon}\frac{d\lambda}{\lambda}+$$

$$+c_2\|f\|_{C\times C,t}\sup\{|(s+\lambda(z-s)+xv^{(r)}(s), s)-(s'+\lambda(z'-s')+xv^{(r)}(s'), s')|^{t-k}:$$

$$\lambda\in[\varepsilon,\tfrac{1}{2}],\ x\in B(0,\delta)\}\int_{\varepsilon}^{1/2}\frac{d\lambda}{\lambda} \leqslant c\|f\|_{C\times C,t}|(z,s)-(z',s')|^{t-k},$$

(z, s), $(z', s')\in(U\times V)\cap(C\times C)$, where c is independent of z, z', s, s' and f. Since β' and β'' and the points z, z', s, s' were chosen arbitrarily, we obtain the required estimate.

We can prove the corresponding statement for $G_i(z, s)$ in the similar way.

This ends the proof of Theorem 2.

Proposition 3 is an easy consequence of the proof of Theorem 2.

Note. Theorem 2 can also be proved directly, by applying the techniques of [2] (see also [11]). In the proof one can apply a more complicated version of Lemma 1.3, involving the dependence of the integrated function f on the variables ξ, z and s (we shall not present the details here). However, we do not know how to prove Proposition 3 by this method.

5. The explicit integral extension operators in strictly pseudoconvex domains in complex manifolds. Let M', Ω, D and M be as in the introduction, and suppose now that M' is a complex submanifold of Ω and $\dim M' = k$. We shall show that in this case there exists an extension operator from $\tilde{A}(M)$ to $\tilde{A}(D)$, given by an explicit integral formula; this is the contents of Theorem 4.

We recall briefly the construction given in [10], which leads to the integral representation formula in strictly pseudoconvex domains on complex manifolds [10], Theorem 3.2.1; this is the main ingredient of our proof.

In virtue of [10], Lemma 2.1.1, there exists a holomorphic mapping s: $M'\times M'\to T(M')$ such that for every $(\xi, z)\in M'\times M'$ we have $s(\xi, z)\in T_z(M')$ and $s(z, z) = 0$ for $z\in M'$. Denote by $\mathscr{F}_s$ the analytic subsheaf of the sheaf ${}_{M'\times M'}\mathcal{O}$ of germs of holomorphic functions on $M'\times M'$, generated by the components of s with respect to local holomorphic coordinates. Then there is a function $\varphi \in \mathcal{O}(M \times M)$ such that $\varphi(z, z) = 1$ for all $z \in M$, and the restriction of φ to $(M \times M) \setminus \Delta(M)$ belongs to $\mathcal{F}_s|_{(M \times M)\setminus \Delta(M)}$ ([10], Lemma 2.1.2). Let $\Phi(\xi, z)$ be a function satisfying properties (i)–(iii) of section 0 constructed with respect to the domain D.

LEMMA 5.1 ([10], Corollary 3.1.3). *Suppose that ∂D is of class of $\mathcal{C}^p$. There exist neighbourhoods $\tilde{M}$ and U (in M') of the sets $\bar{M}$ and ∂M, respectively, and a $\mathcal{C}^p$-mapping s^*: $U \times \tilde{M} \to T^*(M')$ such that for every $(\xi, z) \in U \times \tilde{M}$, $s^*(\xi, z) \in T_z^*(M')$, s^* depends holomorphically on $z \in \tilde{M}$, and*

$$\varphi(\xi, z)\, \Phi(\xi, z) = \langle s^*(\xi, z), s(\xi, z)\rangle, \quad (\xi, z) \in U \times \tilde{M}.$$

(If $s_i(\xi, z)$, $s_i^*(\xi, z)$, $i = 1, \ldots, k$, denote the components of s and s^* with respect to local holomorphic coordinates, then we set $\langle s^*(\xi, z), s(\xi, z)\rangle = \sum_{i=1}^{k} s_i^*(\xi, z)\, s_i(\xi, z)$.)

If $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_k)$ are collections of complex $\mathcal{C}^1$-functions on a complex manifold X, set

$$\omega(u) = du_1 \wedge \ldots \wedge du_k, \qquad \omega'(v) = \sum_{j=1}^{k} (-1)^{j-1} v_j \wedge \bigwedge_{l \neq j} dv_l.$$

THEOREM 5.2 ([10], Theorem 3.2.1). *There exists a non-negative integer ν such that for every $f \in A(M)$,*

$$(5.1) \qquad f(z) = \frac{(k-1)!}{(2\pi i)^k} \int_{\partial M} \frac{f(\xi)}{\Phi(\xi, z)^k}\, \omega_\xi'\big(\varphi^\nu(\xi, z) s^*(\xi, z)\big) \wedge \omega_\xi\big(s(\xi, z)\big), \quad z \in M.$$

Here the forms ω' and ω act on $u(\xi, z) = \big(s_1(\xi, z), \ldots, s_k(\xi, z)\big)$ and $v(\xi, z) = \big(\varphi^\nu(\xi, z) s_1^*(\xi, z), \ldots, \varphi^\nu(\xi, z) s_k^*(\xi, z)\big)$ (where s_i, s_i^* are as above), regarded as functions of ξ. The fact that the differential form on the right-hand side of (5.1) is independent of the choice of the local holomorphic coordinates follows from Lemma on p. 2 of [10].

Since $\Omega(\xi, z) = \omega_\xi'\big(\varphi^\nu(\xi, z) s^*(\xi, z)\big) \wedge \omega_\xi\big(s(\xi, z)\big)$ is a $(k, k-1)$-form with respect to ξ, there exists a function $h(\xi, z) \in \mathcal{C}^{p-2}(U \times \tilde{M})$, holomorphic in $z \in \tilde{M}$, such that formula (5.1) can be rewritten in the form

$$(5.2) \qquad f(z) = \frac{(k-1)!}{(2\pi i)^k} \int_{\partial M} f(\xi)\, \frac{h(\xi, z)}{\Phi(\xi, z)^k}\, d\sigma(\xi), \quad z \in M,$$

$d\sigma(\xi)$ being the volume form on ∂M. This formula is valid also for $f \in H^\infty(M)$; in this case $f(\xi)$ denotes the boundary values of f on ∂M, which exist a.e. with respect to the volume measure on ∂M.

To prove Theorem 4 we use the same techniques as in the proof of [13], Theorem 3. Choose $\tilde{D}$, m, ψ and C with respect to the domain D, according to Theorem 4.1. Let ϱ be a defining function for C. Set

$$f(\zeta, w) = \sum_{i=1}^{m} \varrho_i(\zeta)(\zeta_i - w_i), \quad \text{where } \varrho_i(\zeta) = \frac{\partial \varrho}{\partial \zeta_i}(\zeta),$$

and let

$$\Phi(\xi, z) = \sum_{i=1}^{m} \varrho_i(\psi(\xi))(\psi_i(\xi) - \psi_i(z)).$$

Then $f(\zeta, w)$ and $\Phi(\xi, z)$ satisfy properties (i)–(iii) of section 0 with respect to the domains C and D, respectively ([6], p. 554). By (5.2), there exists a function $h(\zeta, w) \in \mathscr{C}^{p-2}(\psi(U) \times \psi(\tilde{M}))$ holomorphic in $w \in \psi(\tilde{M})$, such that for every $f \in H^{\infty}(\psi(M))$,

$$f(w) = \int_{\psi(\partial M)} f(\zeta) \frac{h(\zeta, w)}{f(\zeta, w)^k} d\tilde{\sigma}(\zeta), \qquad w \in \psi(M),$$

where $d\tilde{\sigma}(\zeta)$ denotes the volume form on $\psi(\partial M)$. Let $\tilde{H}(\zeta, w)$ be the extension of $h(\zeta, w)$ to a $\mathscr{C}^{p-2}$-function defined in $\psi(U) \times C'$ and holomorphic in $w \in C'$, where C' is some neighbourhood of $\bar{C}$. For $f \in H^{\infty}(\psi(M))$ and $w \in C$, define

$$\tilde{L}f(w) = \int_{\psi(\partial M)} f(\zeta) \frac{\tilde{H}(\zeta, w)}{f(\zeta, w)^k} d\tilde{\sigma}(\zeta).$$

We claim that $\tilde{L}$ is a continuous extension operator from $\tilde{A}(\psi(M))$ to $\tilde{A}(C)$, provided that p satisfies the assumptions of Theorem 4. This can be verified similarly to [13], where it is shown that the operator $\tilde{P}$, occurring in the proof of [13], Theorem 3, has the corresponding regularity properties. We apply here Lemma 1.1 instead of [13], Lemma 5. For $z \in D$ and $f \in \tilde{A}(M)$ we have

$$\begin{aligned}\tilde{L}(f \circ \psi^{-1})(\psi(z)) &= \int_{\partial M} f(\xi) \frac{\tilde{H}(\psi(\xi), \psi(z))}{\Phi(\xi, z)^k} (\psi^* d\tilde{\sigma})(\xi) \\ &= \int_{\partial M} f(\xi) \frac{H(\xi, z)}{\Phi(\xi, z)^k} d\sigma(\xi)\end{aligned}$$

for some function $H(\xi, z) \in \mathscr{C}^{p-2}(U \times \tilde{D})$ holomorphic in $z \in \tilde{D}$, where $\tilde{D}$ is a neighbourhood of $\bar{D}$. We set

$$L_1 f(z) = \int_{\partial M} f(\xi) \frac{H(\xi, z)}{\Phi(\xi, z)^k} d\sigma(\xi), \quad f \in H^\infty(M),\ z \in D.$$

It follows from the above considerations that L_1 has the properties asserted in Theorem 4.

Theorem 4′ can be proved in a similar way; in the proof we use Lemmas 1.2 and 1.4 instead of Lemma 1.1.

It is seen that the above proof can be easily adapted as in the case of the extension operators from Section 3 to produce an integral extension operator from $\tilde{A}(M_1 \times \ldots \times M_l)$ to the space $\tilde{A}(D_1 \times \ldots \times D_l)$, where each M_i is a complex manifold related to the strictly pseudoconvex domain D_i in the same way as M and D from Theorem 4. One can also obtain extension operators given by integral formulas, provided that each M_i is an analytic variety described as the zero set of a global holomorphic function in D_i; for further details we refer to [13].

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