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New generating functions for the G-function

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1. In this paper an attempt has been made to obtain some new generating functions involving the hypergeometric function and Meijers G-function. The results established are of a general character and include as particular cases certain previous results. The results obtained are believed to be new.

In our investigation we employ the generalized function of two variables, defined by Sharma [9] as follows:

$$(1)\quad S\left[x,y\left|\begin{bmatrix} m_1 & 0 \\ p_1 & q_1\end{bmatrix}_{b_{q_1}}^{a_{p_1}}\right|\begin{pmatrix} n_2 & m_2 \\ p_2 & q_2\end{pmatrix}_{d_{q_2}}^{c_{p_2}}\left|\begin{pmatrix} n_3 & m_3 \\ p_3 & q_3\end{pmatrix}_{f_{q_3}}^{e_{p_3}}\right.\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{c_1}\int_{c_2}\frac{\prod_{j=1}^{m_1}\Gamma(a_j+s+t)\prod_{j=1}^{m_2}\Gamma(1-c_j+s)\prod_{j=1}^{n_2}\Gamma(d_j-s)}{\prod_{j=m_1+1}^{p_1}\Gamma(1-a_j-s-t)\prod_{j=1}^{q_1}\Gamma(b_j+s+t)\prod_{j=m_2+1}^{p_2}\Gamma(c_j-s)}\times$$

$$\times\frac{\prod_{j=1}^{m_3}\Gamma(1-e_j+t)\prod_{j=1}^{n_3}\Gamma(f_j-t)}{\prod_{j=n_2+1}^{q_2}\Gamma(1-d_j+s)\prod_{j=m_3+1}^{p_3}\Gamma(e_j-t)\prod_{j=n_3+1}^{q_3}\Gamma(1-f_j+t)}x^s y^t\,ds\,dt,$$

where c_1 and c_2 are two suitable contours and positive integers $p_1, p_2, p_3, q_1, q_2, q_3, m_1, m_2, m_3, n_2$ and n_3 satisfy the following inequalities: $q_2 \geqslant 1$, $q_3 \geqslant 1$, $p_1 \geqslant 0$, $q_1 \geqslant 0$, $0 \leqslant m_1 \leqslant p_1$, $0 \leqslant m_2 \leqslant p_2$, $0 \leqslant n_2 \leqslant q_2$, $0 \leqslant m_3 \leqslant p_3$, $0 \leqslant n_3 \leqslant q_3$, $p_1+p_2 \leqslant q_1+q_2$, $p_1+p_3 \leqslant q_1+q_3$. The values $x = 0$, $y = 0$ are excluded.

2. We first prove the following formula:

$$(2)\quad \sum_{r=0}^{\infty}\frac{1}{r!}\,{}_{u+1}F_v[-r, a_u;\ \beta_v;\ x]\,G_{p+1,q}^{m,n+1}\left[y\left|\begin{matrix}1-\lambda-r, a_p\\ b_q\end{matrix}\right.\right]t^r$$

$$= S\left[\frac{y}{1-t}, \frac{xt}{1-t}\middle|\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}^{\lambda}\middle|\begin{pmatrix}m & n\\ p & q\end{pmatrix}_{b_q}^{a_p}\middle|\begin{pmatrix}1 & u\\ u & v+1\end{pmatrix}_{0,-1-\beta_v}^{-1-a_u}\right]\times$$

$$\times(1-t)^{-\lambda}\frac{\prod_{j=1}^{v}\Gamma(\beta_j)}{\prod_{j=1}^{u}\Gamma(a_j)},$$

valid for $|t|<1$, $p+q<2(m+n)+1$, $|\arg y|<(m+n-\frac{1}{2}p-\frac{1}{2}\varrho+\frac{1}{2})\pi$, $x>0$, $R(\beta_j)>0$, $j=1,2,\ldots,v$; $R(a_j)>0$, $j=1,2,\ldots,u$.

Proof. In the left-hand side of (2), we express the G-function ([2], p. 204, equation (1)) as a Mellin–Barnes type integral and interchanging the order of integration and summation, which is justified in view of the absolute convergence of the integral and the series involved, we get

$$(3)\quad \frac{1}{2\pi i}\int_{c_1}\frac{\prod_{j=1}^{m}\Gamma(b_j-s_1)\prod_{j=1}^{n}\Gamma(1-a_j+s_1)\Gamma(\lambda+s_1)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s_1)\prod_{j=n+1}^{p}\Gamma(a_j-s_1)}y^{s_1}\times$$

$$\times\left\{\sum_{r=0}^{\infty}\frac{(\lambda+s_1)_r}{r!}\,{}_{u+1}F_v[-r, a_u;\ \beta_v;\ x]t^r\right\}ds_1.$$

Applying the formula Chaundy ([3], p. 267, equation (22))

$$(4)\quad (1-t)^{-\lambda}{}_{p+1}F_q\left[\lambda, a_p;\ \beta_q;\ -\frac{xt}{1-t}\right]=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{p+1}F_q[-n, a_q;\ \beta_q;\ x]t^n$$

to (3), we get

$$(5)\quad \frac{1}{2\pi i}\int_{c_1}\frac{\prod_{j=1}^{m}\Gamma(b_j-s_1)\prod_{j=1}^{n}\Gamma(1-a_j+s_1)\Gamma(\lambda+s_1)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s_1)\prod_{j=n+1}^{p}\Gamma(a_j-s_1)}y^{s_1}(1-t)^{-(\lambda+s_1)}\times$$

$$\times{}_{u+1}F_v\left[\lambda+s_1, a_u;\ \beta_v;\ -\frac{xt}{1-t}\right]ds_1.$$

If we now write (5) with use of the notation (1), we get just (2). This completes the proof of (2).

Particular cases. We discuss below some particular cases of (2).

(a) Taking $x\to 0$ and the formula due to Sharma [8]

$$(6)\quad \mathop{Lt}_{y\to 0} S\left[x, y\middle|\begin{bmatrix}m_1 & 0\\ p_1 & q_1\end{bmatrix}_{1-b_{q_1}}^{1-a_{p_1}}\middle|\begin{pmatrix}n_2 & m_2\\ p_2 & q_2\end{pmatrix}_{d_{q_2}}^{c_{p_2}}\middle|\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_0\right]$$

$$= G_{p_1+p_2,q_1+q_2}^{n_2,m_1+m_2}\left[x\middle|\begin{matrix}a_1,\ldots,a_{m_1}, c_{p_2}, a_{m_1+1},\ldots,a_{p_1}\\ d_{q_2}, b_{q_1}\end{matrix}\right],$$

in (2), we get a known result due to Meijer ([5], p. 487, equation (47)).

(b) Taking $x = 1$, $u = 1$, $v = 1$ in (2), we get

$$(7)\quad \sum_{r=0}^{\infty} \frac{(\beta-\alpha)_r}{(\beta)_r r!} G_{p+1,q}^{m,n+1}\left[y\left|\begin{matrix}1-\lambda-r,\ a_p\\ b_q\end{matrix}\right.\right] t^r$$

$$= (1-t)^{-\lambda}\frac{\Gamma(\beta)}{\Gamma(\alpha)} S\left[\frac{y}{1-t}, \frac{t}{1-t}\left\|\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}^{\lambda}\right\|\begin{pmatrix}m & n\\ p & q\end{pmatrix}_{b_q}^{a_p}\left|\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}_{0,-1-\beta}^{-1-\alpha}\right|\right].$$

(c) Taking $n = p$, $m = 1$, and using the result of Erdelyi ([2], p. 215, equation (1)) in (2), we get

$$(8)\quad \sum_{r=0}^{\infty} \frac{(\lambda)_r}{r!}\, {}_{m+1}F_n[-r, \alpha_m;\ \beta_n;\ x]\, {}_{p+1}F_q[\lambda+r, a_p;\ b_q;\ y]t^r$$

$$= (1-t)^{-\lambda} F\left[\begin{matrix}\lambda;\ a_p;\ \alpha_m;\ \dfrac{-y}{1-t}, \dfrac{-xt}{1-t}\\ -;\ b_q;\ \beta_n\end{matrix}\right],$$

where the notation due to Chaundy [1] has been used to represent the hypergeometric function of higher order and of two variables.

In particular, all results obtained by Manocha [6] for generalized Rice polynomials can easily be derived from (8) with use of the definition of generalized Rice polynomial, given by Khandekar [4].

Next we use the formula due to Rainville ([7], p. 200, equation (1)) in (8); we thus obtain an interesting formula for Laguerre polynomials:

$$(9)\quad \sum_{r=0}^{\infty} \frac{m+r!}{(1+\alpha)_r} L_r^{\alpha}(x) L_{m+r}^{\beta}(y) t^r$$

$$= (1+\lambda)_m (1-t)^{-(1+\lambda+m)} e^y \psi_2\left[1+\lambda+m;\ 1+\lambda, 1+\alpha;\ \frac{y}{1-t}, \frac{-xt}{1-t}\right].$$

For ψ_2 see Erdelyi ([2], p. 225, equation (24)).

If we take $\alpha = \beta$, $m = 0$ and use the relation (Sharma [8], p. 62, equation (47))

$$(10)\quad e^{-(x+y)} I_v(2\sqrt{xy}) = \frac{(xy)^{\frac{1}{2}v}}{\Gamma(1+v)} \psi_2(v+1;\ v+1, v+1;\ -x, -y),$$

then (9) reduces to a formula of Rainville ([7], p. 211, equation (120)).

3. The second formula to be proved is this:

$$(11)\quad \sum_{r=0}^{\infty} \frac{2^r}{(2v)_r} C_r^v(x) G_{p+2,q}^{m,n+2}\left[y\left|\begin{matrix}1-\lambda-\frac{1}{2}r,\ \frac{1}{2}-\lambda-\frac{1}{2}r,\ a_p\\ b_q\end{matrix}\right.\right] t^r$$

$$= \frac{\Gamma(v+\frac{1}{2})}{(1-xt)^{2\lambda}} S\left[\frac{y}{(1-xt)^2}, \frac{t^2(1-x^2)}{(1-xt)^2}\left\|\begin{bmatrix}2 & 0\\ 2 & 0\end{bmatrix}^{\lambda,\frac{1}{2}+\lambda}\right\|\begin{pmatrix}m & n\\ p & q\end{pmatrix}_{b_q}^{a_p}\left|\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}_{0,\frac{1}{2}-v}\right|\right],$$

valid for $p+q<2(m+n+1)$, $|\arg y|<(m+n-\frac{1}{2}p-\frac{1}{2}q+1)\pi$, $-1<x<1$, $R(v)>-\frac{1}{2}$.

Proof. (11) is derived in the same way as (2) with use of the formula (Rainville [7], p. 279, equation (8))

$$(12)\qquad (1-xt)^{-\gamma}{}_2F_1\left[\begin{matrix}\frac{1}{2}\gamma,\ \frac{1}{2}\gamma+\frac{1}{2};\\ v+\frac{1}{2}\end{matrix}\ \frac{t^2(x^2-1)}{(1-xt)^2}\right]=\sum_{n=0}^{\infty}\frac{(\gamma)_n C_n^v(x)t^n}{(2v)_n},$$

instead of (4).

In particular, if we use the formula of [7], p. 280, equation (20), in (11), we get after a little simplification the following generating function:

$$(13)\qquad \sum_{r=0}^{\infty}\frac{\overline{m+r}!}{(2v)_r}C_r^v(x)C_{r+m}^a(y)t^r=(2a)_m(y-xt)^{-2a-m}\times$$
$$\times F_4\left[a+\tfrac{1}{2}m,\ a+\tfrac{1}{2}m+\tfrac{1}{2};\ a+\tfrac{1}{2},\ v+\tfrac{1}{2};\ \frac{y^2-1}{(y-xt)^2},\ \frac{(x^2-1)t^2}{(y-xt)^2}\right].$$

Since the G-function is a generalization of a great many of special functions occurring in applied mathematics, (2) and (11) can yield results involving Bessel, Legendre, Whittaker function and related functions.

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