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OPTIMAL CONTROL OF DYNAMICAL PROCESSES IN TWO-PHASE SYSTEMS OF SOLID–LIQUID TYPE

IRENA PAWŁOW*

Systems Research Institute, Polish Academy of Sciences,
Warsaw, Poland

Introduction

Free boundaries arise in modelling of various nonlinear evolutionary processes as a result of mathematical idealization. Macroscopically observed, the layers separating regions of strongly different physical properties (phases) admit a well-justified approximation by some hypersurfaces (perhaps moving in a way that is a priori unknown), as usual referred to as free (or moving) boundaries.

The presence of such components contributes to structural nonlinearity of the corresponding models. In many real processes, the way an appropriate free boundary moves heavily affects the whole evolution. Then, it would be of a practical value to find out how to apply external variables so that to provide a desired process development. But, against the needs, free boundaries as a rule are carriers of discontinuous nonlinearities, hence they contribute to an irregular evolution of the complete system (cf. [R1], [L1], [P3]).

Therefore, an analysis of such systems requires the use of their weak formulations. This, in turn, leads to an implicit treatment of the free boundaries whose recovery becomes first possible a posteriori since they are interpreted as some level sets of the resulting weak solutions (cf. [E1], [P1], [P3]). Their regularity is hardly known; what one expects in physically realistic situations are the dendritic or cellular growth and formation of singular spatial patterns (cf. [L1]). Any control of such developments turns out to be a nontrivial task, far from being elementary and straightforward.

This category of effects is characteristic of phase transitions in solid–liquid systems like freezing of liquids or melting of solids. Growth of crystals, their purification (recrystallization), solidification of metallic alloys, fabrication of

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semiconductors, electrochemical processing are all examples of such processes. Control is of technological importance in all of them.

The process activation may be performed either by means of conditions imposed on the fixed boundary (or its part) or by means of a distributed source terms in the governing equations, considered as the external control variables. The corresponding control problems may be formulated so that to minimize an objective functional over a given class of admissible controls or to stabilize the process dynamics within a given closed-loop feedback structure. The former refers to simple technological and economical criteria, the latter reflect an attempt to stabilize the process development in a certain neighbourhood of a desired trajectory (cf. [H1], [P3]).

In this paper, a class of standard optimal control problems for general two-phase Stefan-like processes transformed to variational inequalities (weak formulations) is considered. We develop a theoretical analysis of the problems under consideration. The processes of Stefan type are treated as mathematical models of simple phase transitions of the solid–liquid type (cf. [E1], [R1]).

In particular, we have in mind the situations that arise in artificial freezing of deep geologic formations (cf. [N2], [N3]).

By exploiting a family of regularization procedures, we derive a constructive characterization of the suitable optimal solutions in the form of explicit necessary optimality conditions which give rise to practically efficient computational schemes (cf. [P3]).

Although an extensive literature is devoted to various aspects of optimal control in nonlinear parabolic systems (cf. the monographs [A1], [B1], [B4], [L2], [L3], in particular), almost all results exposed there apply to regular structures and smooth developments, exclusively. Discontinuous nonlinearities that are attributes of systems including free boundaries like those governed by two-phase Stefan-like problems are nonadmissible, as a rule. To our knowledge, the only monographic sources in this respect are [B1], [P3], [S1]).

A comprehensive numerical analysis of the relating approximation aspects was developed in [P3]. The present paper is based on the author's lectures at the Banach Center during the Spring semester '87.

1. Formulation of the control problems

Let Ω be an open bounded domain in R^N, $N \geq 2$, with boundary Γ assumed regular enough. For $0 < T < \infty$, we shall denote $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$.

To be more specific, we shall consider the following boundary control problem:

(1.1) Minimize

$$J(\vartheta, u) = \tfrac{1}{2}\|\vartheta - \vartheta_d\|^2_{L^2(Q)} + \tfrac{1}{2}\alpha\|u\|^2_{\mathcal{U}},$$

over the set of states $\vartheta \in L^2(Q)$ and admissible controls $u \in \mathcal{U}$, subject to the state equations (corresponding to the two-phase Stefan problem, cf. [R1])

$$\begin{cases} w' - \Delta\vartheta = \lambda, \quad w \in \gamma_0(\vartheta) \quad \text{in } Q, & (1.2a) \\ \partial_\nu \vartheta + p\vartheta = u \quad \text{on } \Sigma, & (1.2b) \\ \vartheta(0) = \vartheta_0, \quad w(0) = w_0 \in \gamma_0(\vartheta_0) \quad \text{in } \Omega. & (1.2c) \end{cases}$$

Here $w' = \partial w/\partial t$, ∂_ν denotes the outward normal derivative on Γ; $\vartheta_d = \vartheta_d(x, t)$, $\lambda = \lambda(x, t)$, $\vartheta_0 = \vartheta_0(x)$, $w_0 = w_0(x)$, $p = p(x) \geqslant 0$ are given functions, α is a positive constant; γ_0 is a monotone graph (multivalued, in general) in $\boldsymbol{R}\times\boldsymbol{R}$. The control space $\mathcal{U}$ is defined as either $\mathcal{U} = \mathcal{U}^0 \equiv L^2(\Sigma)$ or $\mathcal{U} = \mathcal{U}^1 \equiv H^1(0, T; L^2(\Gamma))$, alternatively, equipped with the standard norms

$$\|u\|_{\mathcal{U}^0} = \|u\|_{L^2(\Sigma)}, \qquad \|u\|_{\mathcal{U}^1} = \|u(0)\|_{L^2(\Gamma)} + \|u'\|_{L^2(\Sigma)}.$$

System (1.2) is often referred to as the enthalpy fixed-domain formulation of two-phase Stefan problem (cf. [E1], [P1]), with the enthalpy graph γ_0,

$$\gamma_0(r) = \tilde{\gamma}_0(r) + L\,\mathrm{sign}^+(r), \qquad r \in \boldsymbol{R}, \tag{1.3}$$

where $\tilde{\gamma}_0(r) = \int_0^r \varrho(\xi)\,d\xi$, $\varrho(r) = c(r)/k(r) \geqslant 0$, $L \geqslant 0$,

$$\mathrm{sign}^+(r) = \begin{cases} 0 & \text{if } r < 0, \\ [0, 1] & \text{if } r = 0, \\ 1 & \text{if } r > 0. \end{cases}$$

In the sequel, we shall interpret the problems under consideration as related to heat conduction processes with phase transitions. In this case, ϑ will represent temperature, λ distributed heat sources, L latent heat of phase transition, c, k heat capacity and heat conductivity, respectively. Both thermal parameters c and k are assumed to be smooth up to finite jumps at $\vartheta = 0$ which represents the critical point of phase transition. Further, k is assumed positively bounded from below while c is only nonnegative, possibly vanishing at some values of ϑ (both parameters are temperature-dependent).

Hence, the governing equation (1.2a) is of mixed parabolic-elliptic type. In the boundary condition (1.2b), p represents the heat exchange through Γ and $u = u(x, t)$ is the boundary control. Problem (1.2), (1.3) can be also treated as a mathematical model of two-phase flows with saturation in porous media (without gravity) and various processes in electrochemical technology (cf. [E1], [H2]).

In heat conduction processes, the cost functional (1.1) is chosen so that to express an interest in approaching a given temperature profile $\vartheta_d \in L^2(Q)$ with using boundary controls $u \in \mathcal{U}$ that have possibly small norm; the coefficient α is an arbitrary weight which expresses the energy cost of control. The choice of a concrete control space is motivated by the specific application: controls from $\mathcal{U}^0$ represent an action of L^2-boundary heat flux or environmental temperature, directly. In turn, taking the control space $\mathcal{U}^1$ means that the environment temperature is specified by heat power of external heat sources (control variable u is then specified as the solution of an additional ordinary differential equation).

We now specify the notion of a solution to the problem (1.2), (1.3) by introducing an appropriate variational inequality formulation in terms of the freezing index

$$y(x, t) := \int_0^t \vartheta(x, s)\, ds.$$

Formally expressed with respect to y, system (1.2) takes the form

$$\begin{cases} \gamma_0(y') - \Delta y \ni f_0 & \text{in } Q, & (1.4a) \\ \partial_\nu y + py = g & \text{on } \Sigma, & (1.4b) \\ y(0) = 0 & \text{in } \Omega, & (1.4c) \end{cases}$$

with

$$f_0(x, t) = \tilde{f}(x, t) + w_0(x), \quad \tilde{f}(x, t) = \int_0^t \lambda(x, s)\, ds, \quad g(x, t) = \int_0^t u(x, s)\, ds.$$

System (1.4) admits the following variational inequality form (cf. [P1], [P3]):

(VI_t). (instant-in-time-form). Determine a function $y \in W^{1,\infty}(0, T; V)$ such that

$$\begin{cases} F_1(y(t), z; \tilde{\gamma}_0, \Psi_0, f_0(t), g(t)) \\ \qquad \equiv (\tilde{\gamma}_0(y'(t)) - f_0(t), z - y'(t)) + a(y(t), z - y'(t)) \\ \qquad\quad - (g(t), z - y'(t))_\Gamma + \Psi_0(z) - \Psi_0(y'(t)) \geq 0, \\ \qquad\qquad \text{for all } z \in V, \text{ a.a. } t \in [0, T], & (1.5a) \\ y(0) = 0 \quad \text{in } \Omega, & (1.5b) \end{cases}$$

with: $V = H^1(\Omega)$, $H = L^2(\Omega)$, $\|\cdot\|_V$, $\|\cdot\|_H$ the corresponding standard norms; $(\cdot,\cdot)$, $(\cdot,\cdot)_\Gamma$ standard scalar products in H and $L^2(\Gamma)$, respectively;

$$a(y, z) = (\nabla y, \nabla z) + (py, z)_\Gamma,$$

$$\Psi_0(z) = L \int_\Omega \psi_0(z(x))\, dx, \quad \psi_0(z) = \max\{0, z\}.$$

Besides, we introduce the following formulation integrated in time, corresponding to (VI_t):

(VI). Determine $y \in H^1(0, T; V)$ such that

$$\begin{cases} \int_0^T F_1\big(y(t), z(t); \tilde{\gamma}_0, \Psi_0, f_0(t), g(t)\big)\,dt \geqslant 0, & \text{for all } z \in L^2(0, T; V), \quad (1.6a)\\ y(0) = 0 \quad \text{in } \Omega. & (1.6b)\end{cases}$$

Clearly, any function y satisfying (VI_t) fulfils (VI), as well. Conversely, one can show that if y is a solution of (VI) and $y \in W^{1,\infty}(0, T; V)$, then it solves (VI_t), too. In this sense, variational inequality formulations (VI_t) and (VI) are equivalent.

We are now ready to introduce the following

DEFINITION. By *weak solution* of the Stefan problem (1.2), (1.3) we mean a function y (which represents the freezing index of the system) that satisfies variational inequality (VI).

Since $y' = \vartheta$ a.e. in Q, the function y' (or y) can be treated as the state variable of the system, corresponding to the control u. Hence, the control problem under study can be given the formulation

(CP). Minimize $J(y', u)$ over $y' \in L^2(Q)$ and $u \in \mathscr{U}$, subject to $y = y(u)$ being the solution of (VI) that corresponds to the control u.

2. Basic structural properties of variational inequality (VI)

2.1. Underlying hypotheses. Throughout the paper we shall assume that the following hypotheses on the mathematical structure of the problems considered are fulfilled:

(A1) ϱ admits representation $\varrho(r) = \tilde{\varrho}(r) + \tilde{\tilde{\varrho}}\,\mathrm{sign}^+(r)$, where $\tilde{\varrho} \in C^1(\boldsymbol{R})$ and $\tilde{\tilde{\varrho}}$ is a finite constant, $0 \leqslant \underline{\varrho} \leqslant \varrho(r) \leqslant \bar{\varrho} < +\infty$ for $r \in \boldsymbol{R}$.

Everywhere, two cases are to be distinguished: (i) parabolic if $\underline{\varrho} > 0$, and (ii) degenerate (parabolic-elliptic) if $\underline{\varrho} = 0$.

$(\mathrm{A2})^0$ $\lambda \in L^2(Q)$;

$(\mathrm{A2})^1$ $\lambda \in H^1(0, T; H)$;

(A3) $\vartheta_0 \in V \cap L^\infty(\Omega)$, $w_0 \in H$, $w_0 = (\gamma_0)^0(\vartheta_0)$, where $(\gamma_0)^0$ is the minimum-norm section of the graph γ_0; $L \geqslant 0$ is the parameter which characterizes discontinuous behaviour on the free boundary;

(A4) $p \in L^\infty(\Gamma)$, $p \geqslant 0$ a.e. on Γ, the set $\{x \in \Gamma \mid p(x) > 0\}$ has positive Lebesgue measure in Γ;

$(\mathrm{A5})^0$ $u \in \mathscr{U}^0$;

$(\mathrm{A5})^1$ $u \in \mathscr{U}^1$.

By assumption (A1), the mapping $\tilde{\gamma}_0$: $H \to H$ is Lipschitz continuous with Lipschitz constant $\bar{\bar{\varrho}}$, and monotone (strictly monotone if $\bar{\varrho} > 0$),

(2.1) $$(\tilde{\gamma}_0(y)-\tilde{\gamma}_0(z), y-z) \geqslant \bar{\varrho}\, \| y-z\|_H^2, \quad \text{for all } y, z\in H.$$

The functional Ψ_0: $V \to \boldsymbol{R}$ is bounded, convex and lower semicontinuous (l.s.c.); the bilinear form $a(\cdot, \cdot)$: $V\times V \to \boldsymbol{R}$ is symmetric, continuous and V-elliptic.

2.2. Regularization. We introduce auxiliary regularized problems which approximate (VI). The procedure comprises parabolic regularization of the problem (in the degenerate case) and its smoothing. The former is based on approximating γ_0 by strictly monotone graphs γ_μ, $\mu\in(0, 1]$, in the latter γ_μ is approximated by a family of smooth function $\gamma_{\mu\varepsilon}$, $\varepsilon\in(0, 1]$.

2.2.1. Parabolic regularization. To tackle the parabolic and degenerate cases simultaneously, we modify (1.4) by replacing γ_0 with

(2.2) $$\gamma_\mu(r) = \tilde{\gamma}_\mu(r)+L\,\mathrm{sign}^+(r), \quad r\in \boldsymbol{R},\ \mu\in[0, 1],$$

where

$$\tilde{\gamma}_\mu(r) = \int_0^r \varrho_\mu(s)\,ds, \quad \varrho_\mu(r) = \varrho(r)+\mu.$$

Clearly, at $\mu = 0$, $\gamma_\mu = \gamma_0$. We also have

$$0 \leqslant \bar{\varrho}+\mu = \bar{\varrho}_\mu \leqslant \varrho_\mu(r) \leqslant \bar{\bar{\varrho}}_\mu = \bar{\bar{\varrho}}+\mu < +\infty, \quad r\in \boldsymbol{R}.$$

Correspondingly, we substitute f_0 in (1.4) with

(2.3) $$f_\mu(x, t) = \tilde{f}(x, t)+w_\mu(x), \quad w_\mu(x) = w_0(x)+\mu\vartheta_0(x),$$

to come eventually to the variational inequality

$(\mathrm{VI}_t)^\mu$, $\mu\in[0, 1]$. Determine $y_\mu\in W^{1,\infty}(0, T; V)$ such that

$$\begin{cases} F_1(y_\mu(t), z;\ \tilde{\gamma}_\mu, \Psi_0, f_\mu(t), g(t)) \geqslant 0, & (2.4a)\\ \qquad\qquad\qquad\qquad \text{for all } z\in V,\ \text{a.a. } t\in[0, T], & \\ y_\mu(0) = 0 \quad \text{in } \Omega. & (2.4b)\end{cases}$$

Clearly, $(\mathrm{VI}_t)^\mu$ at $\mu = 0$ coincides with (VI_t). Further, observe that $\tilde{\gamma}_\mu$: $H \to H$ is Lipschitz continuous with Lipschitz constant $\bar{\bar{\varrho}}_\mu$, and satisfies (2.1) with $\bar{\varrho}_\mu$. If $\bar{\varrho}_\mu > 0$, then $(\mathrm{VI}_t)^\mu$ is of parabolic type; if $\bar{\varrho}_\mu = 0$, the variational inequality is degenerate.

It turns out useful in some situations to note that variational inequality $(\mathrm{VI}_t)^\mu$ can be formulated in an alternative way.

LEMMA 2.1. *Variational inequality* (2.4a) *admits the equivalent form*

$$F_2\big(y_\mu(t), z;\ F^\mu, f_\mu(t), g(t)\big) \equiv a\big(y_\mu(t), z-y'_\mu(t)\big)-\big(f_\mu(t), z-y'_\mu(t)\big)-\big(g(t), z-y'_\mu(t)\big)_\Gamma + F^\mu(z)-F^\mu\big(y'_\mu(t)\big) \geqslant 0 \quad \textit{for all } z\in V,\ \textit{a.a. } t\in[0, T], \tag{2.5}$$

where

$$F^\mu(z) = B^\mu(z)+\Psi_0(z), \qquad B^\mu(z) = \int_\Omega \beta^\mu\big(z(x)\big)\,dx, \qquad \beta^\mu(r) = \int_0^r \tilde{\gamma}_\mu(s)\,ds.$$

Proof. The Gateaux differential DB^μ of functional B^μ: $H\to \mathbf{R}$ admits the characterization

$$\big(DB^\mu(y), z\big) = \big(\tilde{\gamma}_\mu(y), z\big), \qquad \text{for all } y, z\in H, \tag{2.6}$$

hence, due to the monotonicity (strict monotonicity if $\bar{\varrho}_\mu > 0$) of $\tilde{\gamma}_\mu$, B^μ is convex (strictly convex) and

$$\big(\tilde{\gamma}_\mu(y), z-y\big) \leqslant B^\mu(z)-B^\mu(y), \qquad \text{for all } y, z\in H.$$

This implies that (2.5) follows from (2.4a). To prove the converse, let us take

$$z = y'_\mu(t)+\delta\big(\bar{z}-y'_\mu(t)\big), \quad \delta\in(0, 1),\ \bar{z}\in V, \text{ in (2.5).}$$

Then, by virtue of the convexity of Ψ_0, we arrive at the following variational inequality

$$a\big(y_\mu(t), \bar{z}-y'_\mu(t)\big)-\big(f_\mu(t), \bar{z}-y'_\mu(t)\big)-\big(g(t), \bar{z}-y'_\mu(t)\big)_\Gamma + \frac{1}{\delta}\Big(B^\mu\big(y'_\mu(t)+\delta\big(\bar{z}-y'_\mu(t)\big)\big)-B^\mu\big(y'_\mu(t)\big)\Big) + \Psi_0(\bar{z})-\Psi_0\big(y'_\mu(t)\big) \geqslant 0, \qquad \text{for all } \bar{z}\in V.$$

Let us pass to the limit with $\delta\to 0$ in the above inequality. Then, by virtue of the representation (2.6) we get (2.4a). ∎

Remark 2.1. For any $\mu\in[0, 1]$, the functional F^μ: $V\to \mathbf{R}$ is convex (strictly convex if $\bar{\varrho}_\mu > 0$), weakly l.s.c. on V, and lower bounded,

$$F^\mu(z) \geqslant \tfrac{1}{2}\bar{\varrho}_\mu \|z\|_H^2, \qquad \text{for all } z\in H.$$

Moreover,

$$\lim_{\mu\to 0}\int_0^T F^\mu\big(z(t)\big)\,dt = \int_0^T F\big(z(t)\big)\,dt, \qquad \text{for all } z\in L^2(0, T;\ V); \tag{2.7a}$$

if $z_\mu\to z$ weakly in $L^2(0, T;\ V)$, then

$$\liminf_{\mu\to 0}\int_0^T F^\mu\big(z_\mu(t)\big)\,dt \geqslant \int_0^T F\big(z(t)\big)\,dt. \tag{2.7b}$$

Remark 2.2. The integrated-in-time formulation corresponding to $(\mathrm{VI}_t)^\mu$ is

$(\mathrm{VI})^\mu$, $\mu\in[0, 1]$. Determine $y_\mu\in H^1(0, T; V)$ such that

$$\begin{cases} \int\limits_0^T F_1\big(y_\mu(t), z(t);\ \tilde{\gamma}_\mu, \Psi_0, f_\mu(t), g(t)\big)\,dt \geqslant 0, & \text{(2.8a)}\\ \qquad\qquad\qquad \text{for all } z\in L^2(0, T; V), & \\ y_\mu(0) = 0 \quad \text{in } \Omega. & \text{(2.8b)} \end{cases}$$

By virtue of Lemma 2.1, (2.8a) is equivalent to

$$\int\limits_0^T F_2\big(y_\mu(t), z(t);\ F^\mu, f_\mu(t), g(t)\big)\,dt \geqslant 0, \quad \text{for all } z\in L^2(0, T; V). \tag{2.8c}$$

2.2.2. *Smooth approximation.* With the purpose to approximate the graph γ_μ, we define the following single-valued functions

$$\gamma_{\mu\varepsilon}(r) = \tilde{\gamma}_{\mu\varepsilon}(r) + L\chi_\varepsilon(r), \qquad r\in \boldsymbol{R},\ \varepsilon\in(0, 1], \tag{2.9}$$

with

$$\tilde{\gamma}_{\mu\varepsilon}(r) = \int\limits_0^r \varrho_{\mu\varepsilon}(s)\,ds, \qquad \varrho_{\mu\varepsilon}(r) = \tilde{\varrho}(r) + \bar{\bar{\varrho}}\chi_\varepsilon(r) + \mu.$$

Here, $\chi_\varepsilon(\cdot)$ is a C^2-approximation of the Heaviside graph $\mathrm{sign}^+(\cdot)$ (e.g., polynomial as in [P1]). Then, $\gamma_{\mu\varepsilon}\in C^2(\boldsymbol{R})$ and $\gamma_{\mu\varepsilon}$ approximate graph γ_μ in the sense of uniform convergence on compact subsets of $\boldsymbol{R}\setminus\{0\}$. Moreover,

$$D\gamma_{\mu\varepsilon}(r) = D\gamma_\mu(r), \qquad \text{for } r\in(-\infty, 0]\cup[\varepsilon, +\infty),$$

$$\bar{\varrho}_\mu \leqslant D\gamma_{\mu\varepsilon}(r) \leqslant \frac{C}{\varepsilon}, \qquad \bar{\varrho}_\mu \leqslant D\tilde{\gamma}_{\mu\varepsilon}(r) \leqslant \bar{\bar{\varrho}}_\mu,$$

$$|D^2\gamma_{\mu\varepsilon}(r)| \leqslant \frac{C}{\varepsilon^2}, \qquad \text{for all } r\in \boldsymbol{R}, \tag{2.10}$$

where $C \neq C(\mu, \varepsilon)$ is a constant. By virtue of (A3), the approximations we have already introduced induce the following compatible smooth approximations to f_μ,

$$f_{\mu\varepsilon}(x, t) = \bar{f}(x, t) + w_{\mu\varepsilon}(x), \qquad w_{\mu\varepsilon}(x) = \gamma_{\mu\varepsilon}(\vartheta_0(x)). \tag{2.11a}$$

Indeed, for any $\mu \geqslant 0$

$$w_{\mu\varepsilon} \to w_\mu \qquad \text{a.e. in } \Omega \text{ as } \varepsilon\to 0, \tag{2.11b}$$

because $\|w_{\mu\varepsilon}\|_{L^\infty(\Omega)} \leqslant C$ with a constant C independent of μ, ε.

The corresponding smooth approximation of system (1.4) take the form

$$\begin{cases} \gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}) - \Delta y_{\mu\varepsilon} = f_{\mu\varepsilon} \quad \text{in } Q,\\ (1.4\text{b}), \quad (1.4\text{c}), \end{cases} \tag{2.12}$$

which gives rise to the following approximating variational inequality

$(\mathrm{VI}_t)^\mu_\varepsilon$, $\mu \in [0, 1]$, $\varepsilon \in (0, 1]$. Determine $y_{\mu\varepsilon} \in W^{1,\infty}(0, T; V)$ such that

$$\begin{cases} F_1\big(y_{\mu\varepsilon}(t), z; \tilde{\gamma}_{\mu\varepsilon}, \Psi_\varepsilon, f_{\mu\varepsilon}(t), g(t)\big) \geq 0, & \text{(2.13a)} \\ \qquad\qquad \text{for all } z \in V, \text{ a.a. } t \in [0, T], & \\ y_{\mu\varepsilon}(0) = 0 \quad \text{in } \Omega, & \text{(2.13b)} \end{cases}$$

where

$$\Psi_\varepsilon(z) = L \int_\Omega \psi_\varepsilon(z(x))\,dx, \qquad \psi_\varepsilon(r) = \int_0^r \chi_\varepsilon(s)\,ds;$$

$\psi_\varepsilon \in C^3(\boldsymbol{R})$ and uniformly approximate ψ_0:

$$|\psi_0(r) - \psi_\varepsilon(r)| \leq \tfrac{1}{2}\varepsilon, \qquad \text{for all } r \in \boldsymbol{R}. \tag{2.13c}$$

Correspondingly, for any $\varepsilon > 0$, the functionals $\Psi_\varepsilon \colon V \to \boldsymbol{R}$ are bounded, convex, l.s.c. and Gateaux differentiable; besides, convergences analogous to (2.7) are true as $\varepsilon \to 0$.

Remark 2.3. (2.13a) can be equivalently formulated in the form (2.5), with F^μ, f_μ substituted with F^μ_ε and $f_{\mu\varepsilon}$, respectively, where

$$F^\mu_\varepsilon(z) = B^\mu_\varepsilon(z) + \Psi_\varepsilon(z), \qquad B^\mu_\varepsilon(z) = \int_\Omega \beta^\mu_\varepsilon(z(x))\,dx, \qquad \beta^\mu_\varepsilon(r) = \int_0^r \tilde{\gamma}_{\mu\varepsilon}(s)\,ds.$$

The functional $F^\mu_\varepsilon \colon V \to \boldsymbol{R}$ preserves properties of F^μ. Furthermore, convergences analogous of (2.7) take place, both as $\varepsilon \to 0$ (for any fixed $\mu \in [0, 1]$) and as μ, $\varepsilon \to 0$ simultaneously.

Remark 2.4. $(\mathrm{VI})^\mu_\varepsilon$ at $\varepsilon = 0$ is to be identified with $(\mathrm{VI})^\mu$. In the sequel, we shall skip indices whenever equal to zero.

2.3. Existence of solutions to variational inequality (VI). Because we are primarily interested in studying control problem (CP) formulated for (VI) as a process model, we shall consider (VI) with various classes of admissible controls and with degeneration of the governing equations from parabolic to parabolic-elliptic, taken into account.

The proof of existence of solutions to (VI) is accomplished along the following lines. First we consider the problem in the parabolic case $\bar{\varrho}_\mu > 0$ and with controls $u \in \mathscr{U}^1$ more regular in time (see Theorem 2.1). By a priori estimates established in this case, because $\mathscr{U}^1$ is densely embedded in $\mathscr{U}^0$, the existence of solutions that correspond to $u \in \mathscr{U}^0$ is deduced (see Theorem 2.2). At the same time, by the established estimates we can pass in $(\mathrm{VI})^\mu$ to the limit as $\mu \to 0$, and hence to conclude the existence of solutions to (VI) in the degenerate case with $\bar{\varrho} = 0$, too (see Theorems 2.3 and 2.4).

2.3.1. Parabolic case

THEOREM 2.1 ($\bar{\varrho}_\mu > 0$, $u \in \mathscr{U}^1$). *Let* (A1), (A2)^0, (A3), (A4) *be satisfied. Then there exists at least one solution* $y_\mu \in W^{1,\infty}(0, T; V) \cap H^2(0, T; H)$ *of* $(\mathrm{VI}_t)^\mu$, $\mu \in [0, 1]$, *such that* $y'_\mu(0) = \vartheta_0$ *in* Ω *and the following a priori bounds are true*:

(2.14a) $$\| y_\mu \|_{H^1(0,T;V)} + \bar{\varrho}_\mu^{1/2} \| y'_\mu \|_{L^\infty(0,T;H)} \leqslant C_0,$$

(2.14b) $$\| y'_\mu \|_{L^\infty(0,T;V)} + \bar{\varrho}_\mu^{-1/2} \| y''_\mu \|_{L^2(Q)} \leqslant C_1, \qquad \text{if } \bar{\varrho} > 0,$$

as well as, provided (A2)^1 *is satisfied,*

(2.14c) $$\| y'_\mu \|_{L^\infty(0,T;V)} + \mu^{1/2} \| y''_\mu \|_{L^2(Q)} \leqslant C_2, \qquad \text{if } \bar{\varrho} = 0,$$

with positive constants C_i *dependent on the following data*:

(2.15) $$\begin{aligned} C_0 &= C_0(\|\lambda\|_{L^2(Q)}, \|\vartheta_0\|_H, \|u\|_{\mathscr{U}^0}), \\ C_1 &= C_1(\|\lambda\|_{L^2(Q)}, \|\vartheta_0\|_V, \|u\|_{\mathscr{U}^1}), \\ C_2 &= C_2(\|\lambda\|_{H^1(0,T;H)}, \|\vartheta_0\|_V, \|u\|_{\mathscr{U}^1}). \end{aligned}$$

Proof (a sketch; see [P1], [P3] for details). A Galerkin approximation to the regularized variational inequality $(\mathrm{VI})^\mu_\varepsilon$, $\varepsilon \in (0, 1]$, is to be constructed. Let $\{v_1, \ldots, v_m\}$ be a system of linearly independent elements in V, such that $\mathrm{cl}(\bigcup_{m\in N} V_m) = V$ where $V_m = \mathrm{span}\{v_1, \ldots, v_m\}$. Elements v_1, v_2 are to be selected so as to provide $0, \vartheta_0 \in \mathrm{span}\{v_1, v_2\}$. Let us introduce the following approximating semidiscrete Galerkin problems:

(2.16) $$\begin{cases} \text{Determine } y_m = y_{\mu\varepsilon m} \in W^{1,\infty}(0, T; V_m) \ (m \geqslant 2), \text{ such that} \\ \big(\gamma_{\mu\varepsilon}(y'_m(t)) - f_{\mu\varepsilon}(t), z_m\big) + a(y_m(t); z_m) - (g(t), z_m)_\Gamma = 0, \\ \qquad\qquad\qquad\qquad\qquad \text{for all } z_m \in V_m, \text{ a.a. } t \in [0, T], \\ y_m(0) = 0 \quad \text{in } \Omega. \end{cases}$$

System (2.16) of nonlinear ordinary differential equations has a solution y_m on the interval $[0, T]$ that satisfies bounds (2.14) uniformly with respect to m, ε. This enables us to pass in (2.16) to the limit as $m \to \infty$ (at ε fixed) and to show that the resulting limit function $y_{\mu\varepsilon}$ fulfils $(\mathrm{VI}_t)^\mu_\varepsilon$. Next, due to the analogous uniform bounds for $y_{\mu\varepsilon}$, we pass to the limit as $\varepsilon \to 0$ in $(\mathrm{VI}_t)^\mu_\varepsilon$, and conclude that the appropriate limit y_μ is a solution to $(\mathrm{VI}_t)^\mu$. Estimates (2.14) on $y_{\mu\varepsilon}$ imply analogous bounds on y_μ. ∎

THEOREM 2.2 ($\bar{\varrho}_\mu > 0$, $u \in \mathscr{U}^0$). *Let* (A1), (A2)^0, (A3), (A4) *be satisfied. Then there exists at least one solution* $y_\mu \in H^1(0, T; V) \cap W^{1,\infty}(0, T; H)$ *of* $(\mathrm{VI})^\mu$, $\mu \in [0, 1]$, *which satisfies the bound* (2.14a). *This solution may be constructed as a limit of solutions* $y_{\mu n}$ *to problems* $(\mathrm{VI})^{\mu,n}$ *which correspond to* $u_n \in \mathscr{U}^1$:

(2.17a) *if* $u_n \to u$ *strongly in* $\mathcal{U}^0$ *as* $n \to \infty$, *then*

(2.17b) $y_{\mu n} \to y_\mu$ *weakly in* $H^1(0, T; V)$,
weakly- in* $W^{1,\infty}(0, T; H)$,

(2.17c) $y'_{\mu n} \to y'_\mu$ *strongly in* $L^2(Q)$,

where y_μ *is a solution of* (VI)$^\mu$ *corresponding to* $u \in \mathcal{U}^0$.

Proof. Since $\mathcal{U}^1$ is dense in $\mathcal{U}^0$, for any $u \in \mathcal{U}^0$ there exists a sequence $\{u_n\} \subset \mathcal{U}^1$ which satisfies (2.17a). Let $\{y_{\mu n}\}$ be the sequence of solutions to (VI)$^{\mu,n}$ with u_n. By (2.14a) and due to the boundedness of $\{u_n\}$ in $\mathcal{U}^0$, $\{y_{\mu n}\}$ is uniformly bounded in $H^1(0, T; V) \cap W^{1,\infty}(0, T; H)$. For a subsequence, this implies (2.17b).

To conclude (2.17c), it is enough to show that $\{y'_{\mu n}\}$ is a Cauchy sequence in $L^2(Q)$. To this end, let us consider problems (VI$_t$)$^{\mu,n}$ and (VI$_t$)$^{\mu,m}$, corresponding to u_n and u_m, respectively. Take $z = y'_{\mu n}(t)$ in the inequality (2.4a) with u_m, and $z = y'_{\mu m}(t)$ in the same inequality with u_n. By combining both inequalities integrated over time interval $[0, t]$ with $0 < t \leqslant T$, due to the strict monotonicity of $\tilde{\gamma}_\mu$ we get

$$(2.18)\qquad \bar{\varrho}_\mu \int_0^t \| y'_{\mu n}(\tau) - y'_{\mu m}(\tau)\|_H^2 \, d\tau + \tfrac{1}{2} a\big(y_{\mu n}(t) - y_{\mu m}(t),\, y_{\mu n}(t) - y_{\mu m}(t)\big)$$
$$\leqslant \int_0^t \Big(\int_0^\tau (u_n(s) - u_m(s))\, ds,\, y'_{\mu n}(\tau) - y'_{\mu m}(\tau)\Big)_\Gamma d\tau.$$

Hence, by the uniform boundedness of $y'_{\mu n}|_\Sigma$ in $L^2(\Sigma)$, we see that $\{y'_{\mu n}\}$ is a Cauchy sequence in $L^2(Q)$, indeed. To complete the proof, we still need to show that the limit y_μ fulfils (VI)$^\mu$ with u. For this, let us take the upper limit as $n \to \infty$ in (2.8c) including u_n. By virtue of (2.17) and due to the weak l.s.c. of F^μ, y_μ actually satisfies (VI)$^\mu$ including u. Clearly, the bounds (2.14a) on $y_{\mu n}$ imply the analogous inequalities for y_μ. ∎

2.3.2. Degenerate case. For proving the existence of solutions also in the degenerate case, we shall make use of the parabolically regularized approximations (VI)$^\mu_\varepsilon$, $\varepsilon \in (0, 1]$, of variational inequality (VI). We shall consider the problems both with controls u from $\mathcal{U}^1$ and $\mathcal{U}^0$.

THEOREM 2.3 ($\bar{\varrho} = 0$, $u \in \mathcal{U}^1$). *Let* (A1), (A2)^0, (A3), (A4) *hold. Then there exists at least one solution* $y \in H^1(0, T; V)$ *to* (VI), *such that the bound*

$$(2.19a)\qquad \|y\|_{H^1(0,T;V)} \leqslant C_0$$

is satisfied with the same constant C_0 *as in* (2.15). *This solution may be constructed by taking the limit of solutions* y_μ *to* (VI)$^\mu$ *as* $\mu \to 0$:

(2.20a) $y_\mu \to y$ *weakly in* $H^1(0, T; V)$,

(2.20b) $y'_\mu \to y'$ *strongly in* $L^2(Q)$,

(2.20c) $\mu y'_\mu \to 0$ *strongly in* $L^\infty(0, T; H)$.

Moreover, if (A2)^1 *is satisfied, then* $y \in W^{1,\infty}(0, T; V)$ *and, in addition to* (2.19a),

$$\|y\|_{W^{1,\infty}(0,T;V)} \leqslant C_2, \tag{2.19b}$$

with the same constant C_2 *as in* (2.15).

Proof. By the estimate (2.14a) on $\{y_\mu\}$, convergences (2.20a,c) follow for a subsequence. To show (2.20b), let us consider problems (VI)$^\mu$ and (VI)$^\lambda$ with μ, $\lambda \in (0, 1]$. Take $z = y'_\lambda$ in (2.8a), and $z = y'_\mu$ in the same inequality corresponding to λ. By combining both inequalities, due to the monotonicity of $\tilde{\gamma}_0$, we get

$$\int_0^T \big(\mu(y'_\mu(t)-\vartheta_0)-\lambda(y'_\lambda(t)-\vartheta_0),\ y'_\mu(t)-y'_\lambda(t)\big)\,dt$$
$$+\tfrac{1}{2}a\big(y_\mu(T)-y_\lambda(T),\ y_\mu(T)-y_\lambda(T)\big) \leqslant 0.$$

Hence, in particular,

$$(\mu(y'_\mu-\vartheta_0)-\lambda(y'_\lambda-\vartheta_0),\ y'_\mu-y'_\lambda)_{L^2(Q)} \leqslant 0 \quad \text{for all } \mu,\ \lambda \in (0, 1]. \tag{2.21}$$

Due to the Crandall–Pazy lemma [C1], the boundedness of $\{y'_\mu\}$ in $L^2(Q)$ together with (2.21) imply that $\|y'_\mu\|_{L^2(Q)}$ is nondecreasing in μ, and y'_μ converge to y' strongly in $L^2(Q)$ as $\mu \to 0$.

In order to prove that y satisfies (VI), take the upper limit as $\mu \to 0$ in inequality (2.8c). On account of (2.20a,c) and by virtue of the property (2.7), true for functional F^μ, we conclude that y fulfils inequality (2.8c) with F^μ, f_μ substituted by F and f_0, respectively. Hence, because $y(0) = 0$ in Ω, y actually is a solution of (VI). Estimate (2.19a) follows directly from the analogous estimates on y_μ.

To complete the proof, observe that if (A2)^1 holds, then due to bounds (2.14c) on $\{y_\mu\}$, y satisfies (2.19b). ∎

THEOREM 2.4 ($\bar{\varrho} = 0$, $u \in \mathscr{U}^0$). *Let* (A1), (A2)^0, (A3), (A4) *hold. Then there exists at least one solution* $y \in H^1(0, T; V)$ *of* (VI) *for which the bound* (2.19a) *holds and convergences* (2.20a–c) *are true.*

The proof of this theorem proceeds in the same way as that of Theorem 2.3.

Remark 2.5. In the degenerate case ($\bar{\varrho} = 0$), a solution y of (VI) with $u \in \mathscr{U}^0$ can also be constructed as the limit of solutions y_n to (VI)^n ($\bar{\varrho} = 0$) corresponding to $u_n \in \mathscr{U}^1$:

(2.22a) if $u_n \to u$ strongly in $\mathscr{U}^0$ as $n \to \infty$,

(2.22b) then $y_n \to y$ weakly in $H^1(0, T; V)$,

where y is a solution of (VI) corresponding to $u \in \mathscr{U}^0$ (see the proof of Theorem 2.2).

2.4. Uniqueness of solutions to (VI) and their stability. We recall here results on the continuous dependence of solutions to (VI) upon perturbations of data, first established in [P1, P3]. These results will directly imply the uniqueness of solutions to (VI) in all the situations which have been considered in Theorems 2.1 – 2.4.

THEOREM 2.5 (parabolic case, $\bar{\varrho} > 0$). *Let* y, y_* *be solutions of* (VI) *which correspond to the data* $\varrho(\cdot)$, λ, p, u, L, w_0 *and* $\varrho_*(\cdot)$, λ_*, p_*, u_*, L_*, w_{0*}, *respectively. Then there exists a positive constant* C *independent of the data, such that*

$$(2.23)\qquad \|\delta y\|_{L^\infty(0,T;V)} + \|\delta y'\|_{L^2(Q)} \leqslant C\Big\{\Big\|\int_0^{y'_*} \delta\varrho(s)\,ds\Big\|_{L^2(Q)} + \|\delta\lambda\|_{L^2(Q)} + \|y'_*\,\delta p\|_{L^2(\Sigma)} + \|\delta u\|_{L^2(\Sigma)} + |\delta L| + \|\delta w_0\|_H\Big\},$$

where $\delta y = y - y_*$, $\delta\lambda = \lambda - \lambda_*$, $\delta\varrho(r) = \varrho(r) - \varrho_*(r)$, *etc.*

In the degenerate case, the stability result is restricted to perturbations of λ, p, u, w_0.

THEOREM 2.6 (degenerate case, $\bar{\varrho} = 0$). *Let* y, y_* *be solutions of* (VI) *corresponding to* $\varrho(\cdot)$, λ, p, u, L, w_0 *and* $\varrho(\cdot)$, λ_*, p_*, u_*, L, w_{0*} *as the corresponding data, respectively. Then there exists a positive constant* C *independent of the data, such that*

$$(2.24)\qquad \|\delta y\|_{L^\infty(0,T;V)} \leqslant C\{\|\delta\lambda\|_{L^2(Q)} + \|y'_*\,\delta p\|_{L^2(\Sigma)} + \|\delta u\|_{L^2(\Sigma)} + \|\delta w_0\|_H\}.$$

Proof of Theorems 2.5, 2.6 (an outline). At first we assume that $u \in \mathcal{U}^1$, i.e. y, y_* satisfy (VI_t). Take $z = y'_*(t)$ in inequality (1.5a) and $z = y'(t)$ in the appropriate inequality corresponding to the perturbed data, add both inequalities by sides and next integrate over $[0, t]$ for $t \in (0, T]$. By integrating by parts the appropriate terms and applying Young's and Gronwall's inequalities, due to the monotonicity of $\gamma_0(\cdot)$ and V-ellipticity of $a(\cdot,\cdot)$, we immediately conclude the stability estimates (2.23) and (2.24).

To complete the proof, let us note that on account of the convergences (2.17) and (2.22), estimates (2.23) and (2.24) remain also valid for $u \in \mathcal{U}^0$. Hence, the assertions of Theorems 2.5, 2.6 apply to solutions y, y_* of (VI). ∎

COROLLARY 2.1. *In the parabolic case* ($\bar{\varrho} > 0$):

(i) *the mapping* $\{\lambda, u, w_0, L\} \to \{y, y'\}$ *is Lipschitz continuous from* $L^2(Q) \times \mathcal{U}^0 \times H \times \mathbf{R}$ *into* $L^\infty(0, T; V) \times L^2(Q)$;

(ii) *in Theorems* 2.1 *and* 2.2, *solutions* y *are uniquely defined, i.e., if* y *and* y_* *are two solutions that correspond to the same data, then* $y = y_*$ *a.e. in* Q.

In the degenerate case ($\bar{\varrho} = 0$):

(iii) *the mapping* $\{\lambda, u, w_0\} \to y$ *is Lipschitz continuous from* $L^2(Q) \times \mathcal{U}^0 \times H$ *into* $L^\infty(0, T; V)$;

(iv) *in Theorems* 2.3 *and* 2.4, *the solutions* y *are unique.*

2.5. Estimates on the regularization error. In [P1], [P3] we have established estimates on the error due to regularization. Here we recall these results because of their usefulness in the further exposition.

The error resulting from the parabolic regularization is estimated by the following

THEOREM 2.7. *Let y and y_μ be the solutions of* $(\mathrm{VI})^0$ *and* $(\mathrm{VI})^\mu$, $\mu \in (0, 1]$, *respectively. Then*

$$\| y - y_\mu \|_{L^\infty(0,T;V)} + \bar{\varrho}_\mu^{1/2} \| y' - y'_\mu \|_{L^2(Q)} \leqslant C_0 \mu^{1/2}, \tag{2.25}$$

with a constant C_0 defined as in (2.15).

The error due to smoothing is characterized by

THEOREM 2.8. *Let y_μ and $y_{\mu\varepsilon}$ be the solutions of* $(\mathrm{VI})^\mu$ *and* $(\mathrm{VI})^\mu_\varepsilon$, $\mu \in [0, 1]$, $\varepsilon \in (0, 1]$, *respectively. Suppose also that*

$$\operatorname{mes}\{x \in \Omega \mid 0 < \vartheta_0(x) < \varepsilon\} \leqslant C_\vartheta \varepsilon, \tag{A6}$$

with a constant C_ϑ independent of ε. Then

$$\| y_\mu - y_{\mu\varepsilon} \|_{L^\infty(0,T;V)} + \bar{\varrho}^{1/2} \| y'_\mu - y'_{\mu\varepsilon} \|_{L^2(Q)} \leqslant C \varepsilon^{1/2}, \tag{2.26}$$

where C is a positive constant independent of μ, ε. More precisely, if $\bar{\varrho} > 0$ then $C = \bar{C} \bar{\varrho}^{-1/2}$ with a constant $\bar{C}$ which depends only upon C_ϑ, L, $\bar{\varrho}$, $|\tilde{\varrho}|$, T and $\operatorname{mes}\Omega$; *if $\bar{\varrho} = 0$, then $C = C_0$ is defined as in* (2.15).

Estimates (2.25), (2.26) immediately follow upon adding by sides the corresponding variational inequalities with appropriately chosen test functions, by arguments similar to those used in Theorems 2.5 and 2.6. In particular, the bound (2.19a) on the solutions of $(\mathrm{VI})^\mu$ and $(\mathrm{VI})^\mu_\varepsilon$, as well as the estimate (2.13c) and assumption (A6) are essentially used. For the detailed proofs, we refer to [P3].

3. Structural properties of control problem (CP)

3.1. State observation mapping. Let $\Xi\colon \mathscr{U} \to L^2(Q)$ denote the state observation mapping defined by $\Xi(u) = y'$, where y is the solution of (VI) corresponding to the control u. Then, control problem (CP) admits the equivalent formulation in the control space $\mathscr{U}$:

(CP). $\inf_{u \in \mathscr{U}} \{ I(u) = J(\Xi(u), u) \}$.

Observation mapping Ξ has the following properties.

THEOREM 3.1. *Assume that* (A1), $(\mathrm{A2})^0$, (A3), (A4) *and, alternatively,* $(\mathrm{A5})^0$ *or* $(\mathrm{A5})^1$ *hold. Then*:

(I) *in the degenerate case* ($\bar{\varrho} = 0$):

(i) Ξ *is continuous from $\mathscr{U}$ (weak) into $L^2(0, T; V)$ (weak)*;

(II) *in the parabolic case* ($\bar{\varrho} > 0$): (i) *holds, moreover,*

(ii) Ξ *is compact from* $\mathscr{U}^0$ *into* $L^2(Q)$;

(iii) Ξ *is Lipschitz continuous from* $\mathscr{U}^0$ *into* $L^2(Q)$.

Proof. (I) Consider a sequence $\{u_n\} \subset \mathscr{U}$, such that $u_n \to u$ weakly in $\mathscr{U}$. Let $\{y_n\}$ be the sequence of solutions to (VI) which correspond to controls u_n. Since the sequence $\{u_n\}$ is bounded in $\mathscr{U}$, by virtue of (2.19a) (see Theorem 2.4) $\{y_n\}$ is uniformly bounded in $H^1(0, T; V)$. Therefore, for a subsequence,

$$y_n \to y \text{ weakly in } H^1(0, T; V). \tag{3.1}$$

To show that y is the solution of (VI) corresponding to u, we pass to the limit as $n \to \infty$ in (VI) including u_n. Observe that, in view of (3.1), $y(0) = 0$ in Ω; besides, as $n \to \infty$,

$$\int_0^T \Big(\int_0^t u_n(s)\,ds,\ y_n'(t)\Big)_\Gamma dt = -\int_0^T (u_n(t),\ y_n(t))_\Gamma\, dt + \Big(\int_0^T u_n(s)\,ds,\ y_n(T)\Big)_\Gamma \to \int_0^T \Big(\int_0^t u(s)\,ds,\ y'(t)\Big)_\Gamma dt. \tag{3.2}$$

Let us take the upper limit as $n \to \infty$ in (2.8c) at $\mu = 0$, with u_n as the appropriate control. By (3.2) and due to the weak l.s.c. of F, we can conclude that y satisfies (2.8c) with u_n to be substituted with u. Thus, y is the solution of (VI) that corresponds to control u. Therefore, (i) has been proved.

(II) In the parabolic case, assertion (ii) follows arguments similar to those used in the proof of Theorem 2.2 (see, (2.18)). Indeed, we have

$$\begin{aligned} \bar{\varrho}\, \| y_n' - y_m' \|^2_{L^2(Q)} + \tfrac{1}{2} a\big(y_n(T) - y_m(T),\ y_n(T) - y_m(T)\big) \\ \leqslant -\int_0^T \big(u_n(t) - u_m(t),\ y_n(t) - y_m(t)\big)_\Gamma\, dt \\ + \Big(\int_0^T \big(u_n(s) - u_m(s)\big)\,ds,\ y_n(T) - y_m(T)\Big)_\Gamma, \quad \text{for all } n, m \in N. \end{aligned} \tag{3.3}$$

Hence, due to (3.1), $\{y_n'\}$ is a Cauchy sequence in $L^2(Q)$. Thus, $y_n' \to y'$ strongly in $L^2(Q)$. At last, assertion (iii) is a direct consequence of the stability result in Theorem 2.5. The proof is complete. ∎

3.2. Existence of optimal solutions. As an immediate conclusion from the continuity of Ξ we have

THEOREM 3.2. *Control problem* (CP) *has a nonempty set of optimal solutions.*

Proof. Let $\{u_n\} \subset \mathcal{U}$ be a minimizing sequence for the functional I, i.e.

$$\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} J(\Xi(u_n), u_n) = \hat{I} \equiv \inf_{u\in\mathcal{U}} I(u). \tag{3.4}$$

Hence, by the radial unboundedness of I, the sequence $\{u_n\}$ is uniformly bounded in $\mathcal{U}$. Thus, for a subsequence, $u_n \to \hat{u}$ weakly in $\mathcal{U}$ as $n \to \infty$. Due to Theorem 3.1, (i), by the weak l.s.c. of the norms in J, we immediately get

$$J(\Xi(\hat{u}), \hat{u}) \leqslant \liminf_{n\to\infty} J(\Xi(u_n), u_n) = \hat{I},$$

and hence conclude that $\hat{u}$ is an optimal control for (CP). ∎

3.3. Role of control spaces $\mathcal{U}^0$ and $\mathcal{U}^1$. When comparing properties of solutions to the problems with control spaces $\mathcal{U}^0$ and $\mathcal{U}^1$, one can recognize a regularizing role of controls being elements of $\mathcal{U}^1$ against those from $\mathcal{U}^0$.

THEOREM 3.3. *Consider the family of control problems*

(CP)$_\nu$, $\nu \geqslant 0$. $\inf\{I_\nu(u) = \|\Xi(u)-\vartheta_d\|^2_{L^2(Q)} + \|u\|^2_{\mathcal{U}^0} + \nu\|u'\|^2_{\mathcal{U}^0}\}$, *over* $u\in\mathcal{U}^1$ *if* $\nu > 0$, *and over* $u\in\mathcal{U}^0$ *if* $\nu = 0$:

Assume that (A1), (A2)^0, (A3), (A4) *hold and problem* (VI) *is parabolic* ($\bar{\varrho} > 0$). *Then there exists a sequence* $\{\hat{u}_\nu\} \subset \mathcal{U}^1$ *of optimal solutions to* (CP)$_\nu$, *such that as* $\nu \to 0$,

$$\hat{u}_\nu \to \hat{u} \text{ strongly in } \mathcal{U}^0, \tag{3.5}$$

$$\Xi(\hat{u}_\nu) \to \Xi(\hat{u}) \text{ weakly in } L^2(0, T; V), \quad \text{strongly in } L^2(Q), \tag{3.6}$$

$$\hat{I}_\nu \to \hat{I}_0, \tag{3.7}$$

where $\hat{u}\in\mathcal{U}^0$ *is an optimal control for* (CP)$_0$; $\hat{I}_\nu = I_\nu(\hat{u}_\nu)$, $\hat{I}_0 = \hat{I}_0(\hat{u})$.

Proof. Let us observe that, for every $\nu > 0$,

$$\hat{I}_\nu = \inf_{u\in\mathcal{U}^1} I_\nu(u) \geqslant \inf_{u\in\mathcal{U}^0} I_0(u) = \hat{I}_0. \tag{3.8}$$

We now show that for every $\sigma > 0$ there is $\nu(\sigma)$ such that

$$\hat{I}_\nu \leqslant \hat{I}_0 + \sigma, \quad \text{for } \nu \leqslant \nu(\sigma). \tag{3.9}$$

For any pair $u_1, u_2 \in \mathcal{U}^0$, due to the Lipschitz continuity and boundedness of Ξ (cf. (2.23), (2.14a)),

$$|I_0(u_1) - I_0(u_2)| \leqslant C\|u_1 - u_2\|_{\mathcal{U}^0}, \qquad C \equiv C(\|u_1\|_{\mathcal{U}^0}, \|u_2\|_{\mathcal{U}^0}). \tag{3.10}$$

By the density of $\mathcal{U}^1$ in $\mathcal{U}^0$, (3.10) implies the existence of $w\in\mathcal{U}^1$ such that

$$|I_0(w) - I_0(\hat{u}_0)| \leqslant C\|w - \hat{u}_0\|_{\mathcal{U}^0} \leqslant \tfrac{1}{2}\sigma,$$

where $\hat{u}_0$ is any optimal control for problem (CP)$_0$; hence

$$\hat{I}_\nu \leqslant I_\nu(w) \leqslant I_0(\hat{u}_0) + \tfrac{1}{2}\sigma + \nu\|w'\|^2_{\mathcal{U}^0}.$$

Thus, after adjusting $\nu = \nu(\sigma)$ so that $\nu(\sigma)\|w'\|^2_{\mathscr{U}^0} \leqslant \frac{1}{2}\sigma$, we get (3.9) and therefore

(3.11) $$\limsup_{\nu\to 0} \hat{I}_\nu \leqslant \hat{I}_0.$$

(3.11) together with (3.8) imply (3.7). By (3.11), for $\nu \leqslant \nu(\sigma)$,

$$\|\hat{u}_\nu\|^2_{\mathscr{U}^0} + \nu\,\|\hat{u}'_\nu\|^2_{\mathscr{U}^0} \leqslant \hat{I}_0,$$

thus, for a subsequence,

(3.12a) $$\hat{u}_\nu \to \hat{u} \text{ weakly in } \mathscr{U}^0,$$

(3.12b) $$\nu\hat{u}'_\nu \to 0 \text{ strongly in } \mathscr{U}^0.$$

To show that $\hat{u}$ is an optimal control for $(\mathrm{CP})_0$, observe that by Theorem 3.1 (i), (ii),

$$\Xi(\hat{u}_\nu) \to \Xi(\hat{u}) \text{ weakly in } L^2(0, T; V), \quad \text{strongly in } L^2(Q).$$

Hence, $\liminf_{\nu\to 0} I_\nu(\hat{u}_\nu) \geqslant I_0(\hat{u})$. At the same time, by virtue of (3.11)

$$I_0(\hat{u}) \leqslant \liminf_{\nu\to 0} I_\nu(\hat{u}_\nu) \leqslant \hat{I}_0,$$

implying that $\hat{u}$ actually is optimal for $(\mathrm{CP})_0$. Thus, assertion (3.5) with the weak convergence as well as (3.6) have been shown. To complete the proof, we still have to show the strong convergence in (3.5). To this end, let us note that there is $\eta \geqslant 0$ such that for a subsequence $\{\hat{u}_{\nu'}\}$,

(3.13) $$\|\hat{u}_{\nu'}\|^2_{\mathscr{U}^0} \to \eta \qquad \text{as } \nu' \to 0.$$

Due to Theorem 3.1 (i), by virtue of (3.12) and (3.13), we have

(3.14) $$\liminf_{\nu'\to 0} I_{\nu'}(\hat{u}_{\nu'}) \geqslant \|\Xi(\hat{u}) - \vartheta_d\|^2_{L^2(Q)} + \eta.$$

(3.14) together with (3.11) yield the inequality $\eta \leqslant \|\hat{u}\|^2_{\mathscr{U}^0}$. At the same time, by the weak l.s.c. of the norm,

$$\|\hat{u}\|^2_{\mathscr{U}^0} \leqslant \liminf_{\nu'\to 0} \|\hat{u}_{\nu'}\|^2_{\mathscr{U}^0} = \eta.$$

Hence,

$$\lim_{\nu'\to 0} \|\hat{u}_{\nu'}\|_{\mathscr{U}^0} = \|\hat{u}\|_{\mathscr{U}^0}.$$

This implies (3.5) for the subsequence $\{\hat{u}_{\nu'}\}$. It still remains to show that (3.5) holds for the entire sequence $\{\hat{u}_\nu\}$ which satisfies (3.12a). To this purpose, it is enough to show that $\lim_{\nu\to 0}\|\hat{u}_\nu\|_{\mathscr{U}^0} = \|\hat{u}\|_{\mathscr{U}^0}$. Suppose the converse, i.e., for a subsequence $\{\hat{u}_{\nu''}\}$,

(3.15) $$\lim_{\nu''\to 0} \|\hat{u}_{\nu''}\|^2_{\mathscr{U}^0} = \bar{\eta} \neq \|\hat{u}\|^2_{\mathscr{U}^0}.$$

By repeating the arguments we have used for $\{\hat{u}_{\nu'}\}$, we can conclude that $\bar{\eta} = \|\hat{u}\|^2_{\mathscr{U}^0}$, though. This contradicts (3.15), hence the proof is complete. ∎

4. Characterization of optimal solutions

For numerical reasons, it is of interest to develop efficient gradient-type algorithms for solving the control problems under consideration. In this connection, differentiability of the state observation mapping and cost functional become of primary importance.

The control problems we study exhibit structural nonsmoothness due to the lack of a sufficient regularity of the solution to variational inequality (VI). In order to ensure differentiability of the state observation mapping, with a direct characterization of the differential, we apply the regularization procedures to (VI) as exposed in Section 2.2. Consequently, after constructing discretizations to the regularized control problem, gradient-type minimization techniques can be applied for solving the problem numerically.

An alternative approach consists in a direct discretization of the control problems, and afterwards in employing techniques of nonsmooth optimization that recently have been intensively developed in literature.

Here we shall use the regularization approach to constructing optimal controls and to performing a theoretical analysis of the proposed numerical methods (cf. [P3]), at the same time.

4.1. Regularized control problem. For μ, $\varepsilon \in [0, 1]$, let $\Xi^\mu_\varepsilon\colon \mathscr{U} \to L^2(Q)$ be defined by $\Xi^\mu_\varepsilon(u) = y'_{\mu\varepsilon}$, where $y_{\mu\varepsilon}$ is the unique solution of $(\mathrm{VI})^\mu_\varepsilon$. Recall that the regularized variational inequality $(\mathrm{VI})^\mu_\varepsilon$ comprehends both $\bar{\varrho}_\mu > 0$ and $\varepsilon > 0$. Then, the regularized counterpart of control problem (CP) assumes the form

$(\mathrm{CP})^\mu_\varepsilon$, $\mu \in [0, 1]$, $\varepsilon \in (0, 1]$. $\inf_{u \in \mathscr{U}} \{I^\mu_\varepsilon(u) = J(\Xi^\mu_\varepsilon(u), u)\}$.

As up to now, for any of the parameters μ, ε that vanish, we skip index "0" in all related notations.

By the same arguments as in the proof of Theorem 3.1 it follows that, provided $\bar{\varrho}_\mu > 0$,

(4.1) Ξ^μ_ε is continuous from $\mathscr{U}$ (weak) into $L^2(0, T; V)$ (weak),
compact from $\mathscr{U}^0$ into $L^2(Q)$,
Lipschitz continuous from $\mathscr{U}^0$ into $L^2(Q)$,
with Lipschitz constant independent of ε.

Clearly, problem $(\mathrm{CP})^\mu_\varepsilon$ has nonempty set of optimal solutions.

The regularized state observation mapping Ξ^μ_ε is differentiable in the following sense.

THEOREM 4.1. *Assume that $\bar{\varrho}_\mu$, $\varepsilon > 0$. Then Ξ^μ_ε is Gateaux differentiable in $\mathscr{U}^0$. Its Gateaux differential $D\Xi^\mu_\varepsilon$ is characterized by*

$$D\Xi^\mu_\varepsilon(u)v = \xi'_{\mu\varepsilon}, \quad \textit{for all } u, v \in \mathscr{U}^0, \tag{4.2}$$

where $\xi_{\mu\varepsilon}$ is the unique solution of the problem

$$\begin{cases} \left(D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}(t))\,\xi'_{\mu\varepsilon}(t),\, z\right)+a\left(\xi_{\mu\varepsilon}(t),\, z\right) = \left(\int_0^t v(s)\,ds,\, z\right)_\Gamma, & (4.3a)\\ \qquad\qquad\qquad \textit{for all } z\in V,\ \text{a.a. } t\in[0,\, T], & \\ \xi_{\mu\varepsilon}(0) = 0 \quad \textit{in } \Omega, & (4.3b) \end{cases}$$

with $y'_{\mu\varepsilon} = \Xi^{\mu}_{\varepsilon}(u)$, and $D\gamma_{\mu\varepsilon}(\cdot)$ being the Gateaux differential of $\gamma_{\mu\varepsilon}$: $H\to H$.

Proof. For $\lambda>0$, denote

$$y'_{\lambda\mu\varepsilon} = \Xi^{\mu}_{\varepsilon}(u+\lambda v), \qquad \xi_{\lambda\mu\varepsilon} = (y_{\lambda\mu\varepsilon}-y_{\mu\varepsilon})/\lambda, \qquad \eta_{\lambda\mu\varepsilon} = \left(\gamma_{\mu\varepsilon}(y'_{\lambda\mu\varepsilon})-\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon})\right)/\lambda.$$

Observe that $\xi_{\lambda\mu\varepsilon}$ satisfies the system

$$\begin{cases} \left(\eta_{\lambda\mu\varepsilon}(t),\, z\right)+a\left(\xi_{\lambda\mu\varepsilon}(t),\, z\right) = \left(\int_0^t v(s)\,ds,\, z\right)_\Gamma & (4.4a)\\ \qquad\qquad\qquad \text{for all } z\in V,\ \text{a.a. } t\in[0,\, T], & \\ \xi_{\lambda\mu\varepsilon}(0) = 0 \quad \text{in } \Omega. & (4.4b) \end{cases}$$

The main point now consists in establishing estimates on $\xi_{\lambda\mu\varepsilon}$ that permit to pass in (4.4) to the limit as $\lambda\to 0$. For this, let us set $z = \xi'_{\lambda\mu\varepsilon}(t)$ in (4.4a) and integrate it over $[0,\, t]$, with $t\in(0,\, T]$. Upon integrating the right-hand side of the resulting inequality by parts and then applying Young's and Gronwall's inequalities, by strict monotonicity of $\gamma_{\mu\varepsilon}$, we get

$$\|\xi_{\lambda\mu\varepsilon}\|_{L^\infty(0,T;V)}+\bar{\varrho}_\mu^{1/2}\,\|\xi'_{\lambda\mu\varepsilon}\|_{L^2(Q)} \leqslant C'\,\|v\|_{\mathscr{U}^0} \leqslant C, \tag{4.5a}$$

with constant C independent of λ, μ, ε. Moreover, due to (2.10),

$$\|\eta_{\lambda\mu\varepsilon}\|_{L^2(Q)} \leqslant \frac{C}{\varepsilon}\,\|\xi'_{\lambda\mu\varepsilon}\|_{L^2(Q)} \leqslant \frac{C}{\varepsilon\bar{\varrho}_\mu^{1/2}}, \tag{4.5b}$$

with constant C independent of λ, μ, ε. By (4.5), as $\lambda\to 0$,

$$\begin{aligned} &\xi_{\lambda\mu\varepsilon}\to\xi_{\mu\varepsilon} \quad \text{weakly-* in } L^\infty(0,\, T;\, V), \text{ weakly in } H^1(0,\, T;\, H),\\ &\eta_{\lambda\mu\varepsilon}\to\eta_{\mu\varepsilon} \quad \text{weakly in } L^2(Q). \end{aligned} \tag{4.6}$$

Hence, after passing in (4.4) to the limit as $\lambda\to 0$, we can see that equality (4.3a) is satisfied, with $\eta_{\mu\varepsilon}$ substituting $D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon})\,\xi'_{\mu\varepsilon}$. Besides, due to (4.4b), (4.3b) holds, too. Let us observe that relation (4.2) follows directly by definition of the Gateaux differential. Hence, in order to complete the proof, it only remains to show that

$$\eta_{\mu\varepsilon} = D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon})\,\xi'_{\mu\varepsilon}. \tag{4.7}$$

By the mean-value theorem,

$$\eta_{\lambda\mu\varepsilon} = D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon})\,\xi'_{\lambda\mu\varepsilon}+\tfrac{1}{2}\lambda D^2\gamma_{\mu\varepsilon}(\bar{y}'_{\mu\varepsilon})\,(\xi'_{\lambda\mu\varepsilon})^2, \tag{4.8}$$

where $\bar{y}_{\mu\varepsilon} = (1-\beta)\, y_{\mu\varepsilon} + \beta y_{\lambda\mu\varepsilon}$, with some $\beta \in [0, 1]$. At the same time, by virtue of (2.10) and (4.5) we have

$$\|\lambda D^2 \gamma_{\mu\varepsilon}(\bar{y}'_{\mu\varepsilon})(\xi'_{\lambda\mu\varepsilon})^2\|_{L^1(Q)} \leqslant \frac{C|\lambda|}{\varepsilon^2} \|\xi'_{\lambda\mu\varepsilon}\|^2_{L^2(Q)} \leqslant \frac{C|\lambda|}{\varepsilon^2 \bar{\varrho}_\mu}, \tag{4.9}$$

where C is a positive constant independent of λ, μ, ε. Hence, as $\lambda \to 0$,

$$\lambda D^2 \gamma_{\mu\varepsilon}(\bar{y}'_{\mu\varepsilon})(\xi'_{\lambda\mu\varepsilon})^2 \to 0 \quad \text{strongly in } L^1(Q). \tag{4.10}$$

According to (4.8) and (4.10), by (4.6) we conclude that (4.7) holds. This completes the proof. ■

We remark that problem (4.3) has unique solution $\xi_{\mu\varepsilon} \in L^\infty(0, T; V) \cap H^1(0, T; H)$ that satisfies estimate (4.5a).

To study the convergence of regularized control problems, the estimates introduced in Section 2.5 that characterize an influence of the regularization applied to variational inequality (VI), are of importance. In particular, by virtue of (2.26) (with $\bar{\varrho} > 0$), for any $u \in \mathscr{U}^0$ we have

$$\bar{\varrho}\, \|\Xi(u) - \Xi_\varepsilon(u)\|_{L^2(Q)} \leqslant \bar{C}\varepsilon^{1/2}, \tag{4.11}$$

where $\bar{C}$ is a constant independent of ε and u.

4.2. Necessary conditions of optimality. Let us consider the regularized control problem $(\mathrm{CP})^\mu_\varepsilon$. We define the adjoint state as a solution of the problem $(\mathrm{AP})^\mu_\varepsilon$.

$$\begin{cases} \big(D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}(t))\, p'_{\mu\varepsilon}(t), z\big) - a\big(p_{\mu\varepsilon}(t), z\big) = \big(y'_{\mu\varepsilon}(t) - \vartheta_d(t), z\big) & \text{(4.12a)} \\ \qquad\qquad \text{for all } z \in V, \text{ a.a. } t \in [0, T], & \\ p_{\mu\varepsilon}(T) = 0 \quad \text{in } \Omega, & \text{(4.12b)} \end{cases}$$

where $y'_{\mu\varepsilon} = \Xi^\mu_\varepsilon(u)$.

Problem (4.12) has unique solution $p_{\mu\varepsilon} \in L^\infty(0, T; V) \cap H^1(0, T; H)$. Optimal solutions to $(\mathrm{CP})^\mu_\varepsilon$ can be given the following characterization.

THEOREM 4.2. *Assume that* $\bar{\varrho}_\mu$, $\varepsilon > 0$. *Let* $\hat{u}_{\mu\varepsilon} \in \mathscr{U}$ *be an arbitrary optimal control for* $(\mathrm{CP})^\mu_\varepsilon$ *and* $\hat{y}'_{\mu\varepsilon} = \Xi^\mu_\varepsilon(\hat{u}_{\mu\varepsilon})$ *represent the corresponding optimal state. Then there exists a function* $\hat{p}_{\mu\varepsilon} \in L^\infty(0, T; V) \cap H^1(0, T; H)$ *which satisfies the adjoint problem* $(\mathrm{AP})^\mu_\varepsilon$ *corresponding to* $\hat{y}'_{\mu\varepsilon}$, *together with the boundary conditions*

$$\hat{p}_{\mu\varepsilon}|_\Sigma = \alpha \hat{u}_{\mu\varepsilon} \quad \text{if } \mathscr{U} = \mathscr{U}^0, \tag{4.13}$$

$$\hat{\hat{p}}_{\mu\varepsilon}|_\Sigma = \alpha \hat{u}'_{\mu\varepsilon}, \quad \hat{\hat{p}}_{\mu\varepsilon}(0)|_\Gamma = \alpha \hat{u}_{\mu\varepsilon}(0) \quad \text{if } \mathscr{U} = \mathscr{U}^1, \tag{4.14}$$

where

$$\hat{\hat{p}}_{\mu\varepsilon}(t) = \int_t^T \hat{p}_{\mu\varepsilon}(s)\, ds.$$

Proof. If $\mathcal{U} = \mathcal{U}^0$, then the Gateaux differential of I_ε^μ is characterized by

$$(4.15)\quad DI_\varepsilon^\mu(u)v = (\Xi_\varepsilon^\mu(u)-\vartheta_d,\ D\Xi_\varepsilon^\mu(u)v)_{L^2(Q)}+\alpha(u,v)_{L^2(\Sigma)},\qquad \text{for all } u,\ v\in\mathcal{U}^0.$$

But in view of (4.2), (4.3) and (4.12), we have

$$(4.16)\quad (\Xi_\varepsilon^\mu(u)-\vartheta_d,\ D\Xi_\varepsilon^\mu(u)v)_{L^2(Q)}$$
$$= (\Xi_\varepsilon^\mu(u)-\vartheta_d,\ \zeta'_{\mu\varepsilon})_{L^2(Q)}$$
$$= \int_0^T \left[\left(D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}(t))\,p'_{\mu\varepsilon}(t),\ \xi'_{\mu\varepsilon}(t)\right)-a\left(p_{\mu\varepsilon}(t),\ \xi'_{\mu\varepsilon}(t)\right)\right]dt$$
$$= \int_0^T \left[\left(D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}(t))\,\xi'_{\mu\varepsilon}(t),\ p'_{\mu\varepsilon}(t)\right)+a\left(\xi_{\mu\varepsilon}(t),\ p'_{\mu\varepsilon}(t)\right)\right]dt$$
$$= \int_0^T \left(\int_0^t v(s)\,ds,\ p'_{\mu\varepsilon}(t)\right)_\Gamma dt = -(p_{\mu\varepsilon},\ v)_{L^2(\Sigma)}.$$

By virtue of (4.15) and (4.16), the optimality condition $DI_\varepsilon^\mu(\hat{u}_{\mu\varepsilon}) = 0$ directly implies (4.13). If $\mathcal{U} = \mathcal{U}^1$ then due to (4.16), upon integrating by parts we get

$$DI_\varepsilon^\mu(u)v = (\Xi_\varepsilon^\mu(u)-\vartheta_d,\ D\Xi_\varepsilon^\mu(u)v)_{L^2(Q)}+\alpha(u',v')_{L^2(\Sigma)}+\alpha(u(0),v(0))_\Gamma$$
$$= (\tilde{p}'_{\mu\varepsilon},\ v)_{L^2(\Sigma)}+\alpha(u',v')_{L^2(\Sigma)}+\alpha(u(0),v(0))_\Gamma$$
$$= (-\tilde{p}_{\mu\varepsilon}+\alpha u',\ v')_{L^2(\Sigma)}+(-\tilde{p}_{\mu\varepsilon}(0)+\alpha u(0),\ v(0))_\Gamma,$$
$$\text{for all } u,\ v\in\mathcal{U}^1.$$

Hence, relations (4.14) follow what completes the proof. ∎

4.3. Convergence of the regularized control problems. We are going to show that the regularization procedure we have proposed is correct, i.e., the regularized control problems in a certain sense approximate the original one. To begin with, let us consider the parabolic case, with regularization reduced to the smoothing procedure ($\varepsilon > 0$ and $\mu = 0$).

THEOREM 4.3. *Consider control problem* (CP) *in the parabolic case* ($\bar{\varrho} > 0$), *with* $\mathcal{U} = \mathcal{U}^0$ *or* $\mathcal{U}^1$. *Let* (A1), (A2)^0, (A3)–(A6) *hold. Let* $\{\hat{u}_\varepsilon\}\subset\mathcal{U}$ *be any sequence of optimal controls for problems* $(\mathrm{CP})_\varepsilon$. *Then, for a subsequence, as* $\varepsilon\to 0$,

(4.17) $\quad \hat{u}_\varepsilon\to\hat{u}$ *strongly in* $\mathcal{U}$,

(4.18a) $\quad \Xi_\varepsilon(\hat{u}_\varepsilon)\to\Xi(\hat{u})$ *weakly in* $L^2(0,T;V)$ *and strongly in* $L^2(Q)$ *if* $\mathcal{U} = \mathcal{U}^0$,

(4.18b) $\quad$ *weakly-* in* $L^\infty(0,T;V)$ *and weakly in* $H^1(Q)$ *if* $\mathcal{U} = \mathcal{U}^1$,

(4.19) $\quad \hat{I}_\varepsilon\to\hat{I},\quad I(\hat{u}_\varepsilon)\to\hat{I}$ *with the rate* $O(\varepsilon^{1/2})$ *of convergence,*

where $\hat{u}$ *is an optimal control for* (CP); $\hat{I}_\varepsilon = I_\varepsilon(\hat{u}_\varepsilon)$, $\hat{I} = I(\hat{u})$.

Proof. We first show that $\{\hat{u}_\varepsilon\}$ is bounded in $\mathcal{U}$. Indeed, since

$$(4.20)\qquad I_\varepsilon(\hat{u}_\varepsilon) = J\big(\Xi_\varepsilon(\hat{u}_\varepsilon), \hat{u}_\varepsilon\big) \leqslant J\big(\Xi_\varepsilon(\hat{u}), \hat{u}\big),$$

by virtue of (4.11) we get

$$(4.21)\qquad \limsup_{\varepsilon\to 0} I_\varepsilon(\hat{u}_\varepsilon) \leqslant J\big(\Xi(\hat{u}), \hat{u}\big) = \hat{I}.$$

Thus, $\|\hat{u}_\varepsilon\|_{\mathcal{U}} \leqslant C$ with constant C independent of ε, and, for a subsequence, $\hat{u}_\varepsilon \to \hat{u}$ weakly in $\mathcal{U}$ as $\varepsilon \to 0$. By virtue of (2.14a, b), we conclude the suitable bounds on $\hat{y}'_\varepsilon = \Xi_\varepsilon(\hat{u}_\varepsilon)$. Hence, at $\mathcal{U} = \mathcal{U}^0$,

$$(4.22a)\qquad \hat{y}_\varepsilon \to \hat{y} \quad \text{weakly-* in } W^{1,\infty}(0, T; H), \quad \text{weakly in } H^1(0, T; V);$$

whereas at $\mathcal{U} = \mathcal{U}^1$,

$$(4.22b)\qquad \hat{y}_\varepsilon \to \hat{y} \quad \text{weakly-* in } W^{1,\infty}(0, T; V), \quad \text{weakly in } H^2(0, T; H).$$

In order to prove that $\hat{y}' = \Xi(\hat{u})$, we pass to the limit as $\varepsilon \to 0$ in (VI) including $\hat{u}_\varepsilon$. To this end, we reduce $(\mathrm{VI})_\varepsilon$ to the form (2.8c) with $\mu = 0$ and f_0, F substituted by $f_{0\varepsilon}$, F_ε, respectively. By arguments reminiscent of those used for proving Theorem 3.1(i), on account of (4.22) and the properties (2.7) of F_ε (see Remark 2.3), we can conclude that $\hat{y}$ actually satisfies (VI) corresponding to $\hat{u}$. In addition, observe that because of the obvious relation

$$\|\Xi_\varepsilon(\hat{u}_\varepsilon) - \Xi(\hat{u})\|_{L^2(Q)} \leqslant \|\Xi_\varepsilon(\hat{u}_\varepsilon) - \Xi(\hat{u}_\varepsilon)\|_{L^2(Q)} + \|\Xi(\hat{u}_\varepsilon) - \Xi(\hat{u})\|_{L^2(Q)},$$

by virtue of (4.11) and Theorem 3.1(ii), $\Xi_\varepsilon(\hat{u}_\varepsilon) \to \Xi(\hat{u})$ strongly in $L^2(Q)$ as $\varepsilon \to 0$. By weak l.s.c. of the norm,

$$(4.23)\qquad J\big(\Xi(\hat{u}), \hat{u}\big) \leqslant \liminf_{\varepsilon\to 0} J\big(\Xi_\varepsilon(\hat{u}_\varepsilon), \hat{u}_\varepsilon\big).$$

At the same time, by (4.21) we get the inequality $J\big(\Xi(\hat{u}), \hat{u}\big) \leqslant \hat{I}$. This implies optimality of the control $\hat{u}$ for problem (CP). Assertion (4.18) has been proved. In view of (4.21) and (4.23), $\hat{I}_\varepsilon \to \hat{I}$ as $\varepsilon \to 0$.

To conclude (4.17), observe that the strong convergence of $\{\hat{u}_\varepsilon\}$ in $\mathcal{U}$ follows from (4.18) in the same way as in the proof of Theorem 3.3.

Finally, to prove (4.19), let us observe that by virtue of (4.11) and due to the uniform bounds (2.14a) on $\Xi_\varepsilon(\hat{u})$ and $\Xi(\hat{u})$,

$$(4.24)\qquad |\hat{I} - I_\varepsilon(\hat{u})| \leqslant C\,\|\Xi(\hat{u}) - \Xi_\varepsilon(\hat{u})\|_{L^2(Q)} \leqslant C\varepsilon^{1/2},$$

with positive constant C independent of ε. Similarly,

$$(4.25)\qquad |I(\hat{u}_\varepsilon) - \hat{I}_\varepsilon| \leqslant C\varepsilon^{1/2},$$

with C independent of ε, because $\{\hat{u}_\varepsilon\}$ is uniformly bounded in $\mathcal{U}$. By (4.24), (4.25) and in view of the obvious inequalities $\hat{I} \leqslant I(\hat{u}_\varepsilon)$, $\hat{I}_\varepsilon \leqslant I_\varepsilon(\hat{u})$, we have

(4.26) $$0 \leqslant I(\hat{u}_\varepsilon) - \hat{I} \leqslant |I(\hat{u}_\varepsilon) - \hat{I}_\varepsilon| + |I_\varepsilon(\hat{u}) - \hat{I}| \leqslant C\varepsilon^{1/2},$$

with C independent of ε. Eventually, according to (4.25) and (4.26),

$$|\hat{I}_\varepsilon - \hat{I}| \leqslant |\hat{I}_\varepsilon - I(\hat{u}_\varepsilon)| + |I(\hat{u}_\varepsilon) - \hat{I}| \leqslant C\varepsilon^{1/2}.$$

The above inequality completes the proof. ■

In the degenerate case ($\bar{\varrho} = 0$) there are two parameters of regularization: μ, $\varepsilon > 0$. Thus, it is then of interest to examine both iterative and joint convergences of the solutions of the regularized control problems $(\mathrm{CP})^\mu_\varepsilon$ with respect to μ and ε.

THEOREM 4.4. *Consider control problems* $(\mathrm{CP})^\mu_\varepsilon$, $\mu, \varepsilon \in (0, 1]$, *in the degenerate case* ($\bar{\varrho} = 0$), *with* $\mathscr{U} = \mathscr{U}^0$ *or* $\mathscr{U}^1$. *Assume that* (A1), (A2)^0, (A3)–(A6) *hold. Let* $\{\hat{u}_{\mu\varepsilon}\} \subset \mathscr{U}$ *be a sequence of optimal controls for problems* $(\mathrm{CP})^\mu_\varepsilon$.

(I) *Iterative convergence*: $\varepsilon \to 0$, $\mu \to 0$.

(i) *Assume* $\mu > 0$ *to be fixed. Then there exists a subsequence of* $\{\hat{u}_{\mu\varepsilon}\}$, *such that* $\hat{u}_{\mu\varepsilon} \to \hat{u}_\mu$ *strongly in* $\mathscr{U}$ *as* $\varepsilon \to 0$, *assertions* (4.18) *and* (4.19) *on the convergences of the optimal states* $\Xi^\mu_\varepsilon(\hat{u}_{\mu\varepsilon}) \to \Xi^\mu(\hat{u}_\mu)$ *and the minimal values of the cost functionals,* $\hat{I}^\mu_\varepsilon \to \hat{I}^\mu$, $I^\mu(\hat{u}_{\mu\varepsilon}) \to \hat{I}^\mu$ *are true, with* $\hat{u}_\mu$ *representing an optimal control for problem* $(\mathrm{CP})^\mu$, $\hat{I}^\mu = I^\mu(\hat{u}_\mu)$.

(ii) *Let* $\{\hat{u}_\mu\}$ *be a sequence of optimal controls for problems* $(\mathrm{CP})^\mu$. *Then there exists a subsequence of* $\{\hat{u}_\mu\}$, *such that as* $\mu \to 0$,

(4.27) $$\hat{u}_\mu \to \hat{u} \quad \textit{strongly in } \mathscr{U},$$

(4.28) $$\Xi^\mu(\hat{u}_\mu) \to \Xi(\hat{u}) \quad \textit{weakly in } L^2(0, T; V),$$

(4.29) $$\hat{I}^\mu \to \hat{I},$$

where $\hat{u}$ *is an optimal control for problem* (CP); $\hat{I} = I(\hat{u})$.

(II) *Joint convergence*: $\mu, \varepsilon \to 0$. *Assume that*

(4.30) $$\varepsilon \leqslant \varepsilon_0 \mu^{2+\delta}, \quad \textit{where } \varepsilon_0,\ \delta > 0.$$

Then, for a subsequence of $\{\hat{u}_{\mu\varepsilon}\}$, *as* $\mu, \varepsilon \to 0$, *the assertions* (4.27)–(4.29) *hold for* $\hat{u}_{\mu\varepsilon}$, $\Xi^\mu_\varepsilon(\hat{u}_{\mu\varepsilon})$ *and* $\hat{I}^\mu_\varepsilon$, *respectively.*

Proof. (I) Assertion (i) follows directly from Theorem 4.3. In the proof of (ii) we make use of the following convergences established in Theorem 2.4:

(4.31a) for any $u \in \mathscr{U}$, $\Xi^\mu(u) \to \Xi(u)$ weakly in $L^2(0, T; V)$ and strongly in $L^2(Q)$ as $\mu \to 0$,

where $\Xi(u) = y'$, with y being the solution of (VI). Apart from (4.31a), we shall take advantage of the following property:

(4.31b) if $u_\mu \to u$ weakly in $\mathscr{U}$ as $\mu \to 0$, then $\Xi^\mu(u_\mu) \to \Xi(u)$ weakly in $L^2(0, T; V)$.

To prove (4.31b), observe that since $\{u_\mu\}$ is bounded in $\mathcal{U}$, estimates (2.14a) (see Theorem 2.2) imply uniform bounds on y_μ in $H^1(0, T; V)$, with $y'_\mu = \Xi^\mu(u_\mu)$. Thus, for a subsequence, $y_\mu \to y$ weakly in $H^1(0, T; V)$ as $\mu \to 0$, and $y(0) = 0$ in Ω. In order to prove that $y' = \Xi(u)$, take the upper limit at $\mu \to 0$ in the inequality (2.8c) with u_μ. By arguments similar to those used in the proof of Theorem 3.1(i), due to the properties (2.7) of F^μ, we can conclude that y satisfies (VI) including u. This proves (4.31b).

By virtue of the convergences (4.31a,b), we can argue as in the proof of Theorem 4.3, to show that:

(a) $\{\hat{u}_\mu\}$ is uniformly bounded in $\mathcal{U}$, thus, for a subsequence, $\hat{u}_\mu \to \hat{u}$ weakly in $\mathcal{U}$, and $\Xi^\mu(\hat{u}_\mu) \to \Xi(\hat{u})$ weakly in $L^2(0, T; V)$ as $\mu \to 0$;

(b) $\hat{u}$ is an optimal control for problem (CP) (i.e., $I(\hat{u}) = \hat{I}$), and $\hat{I}^\mu \to \hat{I}$ as $\mu \to 0$;

(c) $\hat{u}_\mu \to \hat{u}$ strongly in $\mathcal{U}$.

The above conclusions yield (4.27)–(4.29).

(II) To study the joint convergence as $\mu, \varepsilon \to 0$, we first show the following properties:

(4.32a) for any $u \in \mathcal{U}$,
$\Xi^\mu_\varepsilon(u) \to \Xi(u)$ weakly in $L^2(0, T; V)$ as $\mu, \varepsilon \to 0$;

(4.32b) if $u_{\mu\varepsilon} \to u$ weakly in $\mathcal{U}$ as $\mu, \varepsilon \to 0$, then
$\Xi^\mu_\varepsilon(u_{\mu\varepsilon}) \to \Xi(u)$ weakly in $L^2(0, T; V)$.

Let $y'_{\mu\varepsilon} = \Xi^\mu_\varepsilon(u_{\mu\varepsilon})$. By the uniform bound (2.14a) on $y_{\mu\varepsilon}$ (independent of μ, ε, because of the uniform boundedness of $\{u_{\mu\varepsilon}\}$ in $\mathcal{U}$),

$$y_{\mu\varepsilon} \to y \quad \text{weakly in } H^1(0, T; V) \text{ as } \mu, \varepsilon \to 0.$$

To prove that $y' = \Xi(u)$, we pass to the limit as $\mu, \varepsilon \to 0$ in $(\mathrm{VI})^\mu_\varepsilon$ given the form (2.8c), with F^μ, f_μ substituted with F^μ_ε and $f_{\mu\varepsilon}$, respectively. We again follow the arguments of Theorem 3.1(i), this time exploiting the properties (2.7) of F^μ_ε at $\mu, \varepsilon \to 0$ (see Remark 2.3), as well as the convergence

$$f_{\mu\varepsilon} \to f_0 \quad \text{strongly in } L^2(Q) \text{ as } \mu, \varepsilon \to 0$$

(consequence of (2.3) and (2.11b)), to conclude that y satisfies (VI) including u. This proves (4.32b). In a similar way, (4.32a) can be shown. We shall now prove that for any $u \in \mathcal{U}$,

(4.32c) $$\Xi^\mu_\varepsilon(u) \to \Xi(u) \quad \text{strongly in } L^2(Q) \text{ as } \mu, \varepsilon \to 0,$$

provided that ε and μ are related to each other by (4.30). To this purpose, let us use the inequality

(4.33) $$\|\Xi^\mu_\varepsilon(u) - \Xi(u)\|_{L^2(Q)} \leqslant \|\Xi^\mu_\varepsilon(u) - \Xi^\mu(u)\|_{L^2(Q)} + \|\Xi^\mu(u) - \Xi(u)\|_{L^2(Q)}.$$

By virtue of (4.11), with $\bar{\varrho}$ to be substituted by μ (since the problem has been regularized parabolically), for every $u \in \mathscr{U}$

$$\|\Xi^{\mu}(u) - \Xi^{\mu}_{\varepsilon}(u)\|_{L^2(Q)} \leqslant C\frac{\varepsilon^{1/2}}{\mu} \leqslant C\varepsilon_0^{1/2}\mu^{\delta/2}, \tag{4.34}$$

with constant C independent of μ, ε and u. On account of (4.31a) and (4.34), estimate (4.33) implies (4.32c).

At last, with the properties (4.32a–c) of Ξ^{μ}_{ε} shown, we can again apply the same arguments as in the proof of Theorem 4.3, to conclude assertion (II). ∎

Remark 4.1. How to construct necessary conditions of optimality for control problem (CP) in a straightforward way, remains an open question. This is so due to the lack of any regularity of the free boundary, nonlocal in time. Were it at least of zero Lebesgue measure in Q (what is unknown and rather questionable, in general), a construction due to Tiba (cf., [T1]) would apply in the case of the parabolic two-phase Stefan problem. No such constructions are known for the problems with degeneration.

5. Comments on the related numerical realizations

The regularization techniques that form kernel of this paper can be applied in the construction of discrete approximations to the problems under consideration, including equally the free boundary problem (1.2) and related problems of control.

Concerning discrete approximations to problem (1.2) in variational inequality formulation, we refer to [P3]. Discrete approximations to the related control problems were constructed in [P2], [P3]. The construction of an algorithm for numerical solving control problems, as exposed in [P3], splits into two stages:

(i) regularization of the variational inequality (VI), comprising parabolic regularization and smoothing; the former is of primary importance for the degenerate (parabolic-elliptic) problems, the latter provides differentiability of the state observation mappings and, consequently, differentiability of the cost functional;

(ii) discretization of the resulting regularized optimal control problems by using finite elements in space and finite differences in time.

The regularization procedure we have applied turns out useful in an analysis of the constructed discrete problems. On account of some additional regularity of solutions to the regularized problems $(\mathrm{VI})^{\mu}_{\varepsilon}$ and, at the same time, due to error estimates for parabolic regularization and smoothing effects we have established, an application of the regularizing procedure makes it posible to evaluate a convergence rate of the discrete approximations (cf., [P3]).

Moreover, as regularization delivers the differentiability of the cost functional, it gives rise to optimality conditions in an explicit form. Thus, it gives rise to a constructive method for solving the class of control problems of our concern (cf., [P2], [P3]).

References

[A1] N. Ahmed, K. Teo, *Optimal Control of Distributed Parameter Systems*, North-Holland, New York 1981.

[B1] V. Barbu, *Optimal Control of Variational Inequalities*, Pitman, Boston 1984.

[B2] A. Bermudez, C. Saguez, *Optimal control of variational inequalities; optimality conditions and numerical methods*, in A. Bossavit, A. Damlamian, M. Frémond (eds.), *Free Boundary Problems — Applications and Theory, IV*, Pitman, Boston 1985, 478–487.

[B3] —, —, *Optimal control of variational inequalities*, Control Cybernet. 14 (1985), 9–30.

[B4] A. Butkovskii, *Methods of Control for Distributed Parameter Systems*, Nauka, Moscow 1975 (in Russian).

[C1] M. G. Crandall, A. Pazy, *Semigroups of nonlinear contractions and dissipative sets*, J. Funct. Anal. 3 (1969), 376–418.

[E1] C. M. Elliott, J. R. Ockendon, *Weak and Variational Methods for Moving Boundary Problems*, Pitman, Boston 1982.

[H1] K.-H. Hoffmann, M. Niezgódka, J. Sprekels, *Feedback control of multidimensional two-phase Stefan problems via thermostats*, Nonlinear Anal. — Theory, Methods and Applications, in print.

[H2] U. Hornung, *A parabolic-elliptic variational inequality*, Manuscripta Math. 39 (1982), 155–172.

[L1] J. Langer, *Instabilities and pattern formation in solidification*, Rev. Modern Phys. 52 (1980), 1–28.

[L2] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin 1971.

[L3] K. A. Lur'e, *Optimal Control in Problems of Mathematical Physics*, Nauka, Moscow 1975 (in Russian).

[N1] P. Neittaanmäki, D. Tiba, *A finite element approximation of the boundary control of two-phase Stefan problems*, Report No. 4, Lappeenranta Univ. Tech., 1983.

[N2] M. Niezgódka, I. Pawlow, *A mathematical model for artificial freezing of geologic formations*, in K.-H. Hoffmann, J. Sprekels (eds.), *Free Boundary Problems — Theory and Applications*, Vols. V–VI, Pitman Research Notes in Mathematics, Longman, London, in print.

[N3] M. Niezgódka, I. Pawlow, A. Żochowski, *An application of the mathematical model of the freezing processes in geologic formations*, Part 2, Technical Report No. 110/U/10/87, Systems Research Institute, Warsaw 1987 (in Polish).

[P1] I. Pawłow, *A variational inequality approach to generalized two-phase Stefan problem in several space variables*, Ann. Mat. Pura Appl. (4) 131 (1982), 333–373.

[P2] —, *Optimal control of two-phase Stefan problems — numerical solutions*, in K.-H. Hoffmann, W. Krabs (eds.), *Optimal Control of Partial Differential Equations* II, Birkhäuser, Basel 1987, 179–206.

[P3] —, *Analysis and Control of Evolution Multi-Phase Problems with Free Boundaries*, Ossolineum, Wrocław 1987.

[R1] L. Rubinstein, *Stefan Problem*, Zvaigzne, Riga 1967 (in Russian).

[S1] C. Saguez, *Contrôle optimal de systèmes à frontière libre*, Thèse, Univ. Technol., Compiègne 1980.

[T1] D. Tiba, *Boundary control for a Stefan problem*, in K.-H. Hoffmann, W. Krabs (eds.), *Optimal Control for Partial Differential Equations*, Birkhäuser, Basel 1984, 229–242.

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