ANNALES
POLONICI MATHEMATICI
XLIII (1983)

Monotonic solution of the translation equation

by A. Gregorczyk, J. Tabor (Kraków)

Abstract. Let X be a linearly ordered set and G a linearly ordered group. We shall consider functions $F\colon X \times G \to X$ satisfying the translation equation

$$F(F(x, \alpha), \beta) = F(x, \beta\alpha), \quad x \in X, \ \alpha, \beta \in G,$$

and the identity condition

$$F(x, 1) = x, \quad x \in X.$$

In the paper we give the general form of transitive monotonic (in each variable) solutions of the translation equation X and G may be also viewed as topological spaces endowed with topologies defined by the order. Then every transitive monotonic solution of the translation equation is continuous.

In various considerations it is convenient to look at a given group G as a group of transformations of a certain non-empty set X. For this purpose we need a function $F\colon X \times G \to X$ satisfying the following conditions (cf. [4]):

$$F(F(x, \alpha), \beta) = F(x, \beta\alpha) \quad \text{for } x \in X, \ \alpha, \beta \in G, \tag{1}$$

$$F(x, 1) = x \quad \text{for } x \in X. \tag{2}$$

Equation (1) is called the *translation equation*, condition (2) the *identity condition*. A solution F of equation (1) is called *transitive* if for every $x, y \in X$ there exists $\alpha \in G$ such that $F(x, \alpha) = y$.

If we want to make an ordered group into an ordered group of monotonic transformations (cf. [1]) we must additionally assume that the functions $F(\cdot, \alpha)$, $\alpha \in G$, are monotonic and that the following condition holds for all $\alpha, \beta \in G$:

If $\alpha \leqslant \beta$, then

$$F(\cdot, \alpha) \leqslant F(\cdot, \beta) \quad (F(\cdot, \alpha) \geqslant F(\cdot, \beta), \text{ respectively}),$$

where an order "$\leqslant$" in the family of monotonic functions $F(\cdot, \alpha)$ is defined as follows: $F(\cdot, \alpha) \leqslant F(\cdot, \beta)$ iff $F(x, \alpha) \leqslant F(x, \beta)$ for all $x \in X$. These assumptions mean that the function F is monotonic with respect to the first and the second variable.

1. Throughout the paper X will denote a linearly ordered set and G a linearly ordered group.

DEFINITION. A solution $F\colon X\times G \to X$ of the translation equation (1) will be called *monotonic* if for every $\alpha \in G$, $x \in X$ the functions $F(\cdot, \alpha)$ and $F(x, \cdot)$ are monotonic.

Suppose that G^* is a convex subgroup of G. Denote by $G|G^*$ left cosets of G modulo G^*. The set $G|G^*$ can be linearly ordered by the following relation:

(3) $A \leqslant B$, $A, B \in G|G^*$ iff there exist $\alpha \in A$, $\beta \in B$ such that $\alpha \leqslant \beta$.

Now and further on we assume that the set $G|G^*$ is ordered by this relation. It is clear that then for every $A \in G|G^*$, $\beta \in G$ the mappings $G \ni \alpha \to \alpha A$ and $G|G^* \ni A \to \beta A$ are increasing, i.e. the following relations hold:

(4) If $\alpha_1 \leqslant \alpha_2$, $\alpha_1, \alpha_2 \in G$, then $\alpha_1 A \leqslant \alpha_2 A$.

(5) If $A_1 \leqslant A_2$, $A_1, A_2 \in G|G^*$, then $\alpha A_1 \leqslant \alpha A_2$.

As it is known (cf. [2]), the general transitive solution of equation (1) is given by the formula [1]

(6) $$F(x, \alpha) = g^{-1}(\alpha g(x)) \quad \text{for } x \in X,\ \alpha \in G,$$

where g is a bijection of X onto left cosets of G (modulo) some subgroup G^*. One can take for G^* any stability subgroup of F, i.e. a subgroup of the form

$$G_x = \{\alpha \in G\colon F(x, \alpha) = x\},$$

where x is a fixed element of X.

We shall prove a similar result for monotonic solutions of equation (1).

THEOREM 1. *The general transitive and monotonic solution of equation* (1) *is given by formula* (6), *where it is additionally required that G^* be convex and g be monotonic.*

Proof. It is easy to observe, in view of (4), (5) and the monotonicity of g, that a function F of form (6) is monotonic. Now assume that F is a transitive and monotonic solution of equation (1). We shall show that for every $x \in X$ the group

$$G_x = \{\alpha \in G\colon F(x, \alpha) = x\}$$

is convex. Consider arbitrary $\alpha, \gamma \in G_\alpha$, $\alpha \leqslant \gamma$ and suppose that $\alpha \leqslant \beta \leqslant \gamma$, $\beta \in G$. We then have

$$x = F(x, \alpha) \leqslant F(x, \beta) \leqslant F(x, \gamma) = x \quad (\text{if } F(x, \cdot) \text{ is increasing}),$$

[1] Of course, in this case we do not need any order in X or in G.

or

$$x = F(x, \alpha) \geqslant F(x, \beta) \geqslant F(x, \gamma) = x \quad (\text{if } F(x, \cdot) \text{ is decreasing}),$$

and hence $F(x, \beta) = x$, i.e. $\beta \in G_x$, which proves the convexity of G_x. Fixing x in formula (6) and applying (4), we observe that g^{-1} is monotonic and hence g is also monotonic.

Remark 1. If F is a transitive and monotonic solution of (1) then, in view if Theorem 1, we may assume that the functions g and g^{-1} are strictly monotonic. By (5) the function $G|G^* \ni A \to \alpha A$ is strictly increasing. So, by (6), the function $F(\cdot, \alpha)$ is strictly increasing. Since every solution of the translation equation satisfying identity condition is a disjoint union of transitive solutions on suitable sets, it follows that if a solution F of equation (1) satisfying condition (2) is monotonic, then the function $F(\cdot, \alpha)$ must be strictly increasing.

Let $\tilde{H}$ be a group and H a subgroup of $\tilde{H}$. Assume that $F\colon Y \times H \to Y$ is a transitive solution of equation (1) and denote by H^* any stability subgroup of F. Z. Moszner proved (cf. [3]) that F can be extended to a transitive solution $\tilde{F}\colon Y \times \tilde{H} \to Y$ of equation (1) iff there exists a subgroup $\tilde{H}_1$ of $\tilde{H}$ such that the following relations hold:

$$\tilde{H}_1 \cdot H = \tilde{H}, \quad \tilde{H}_1 \cap H = H^*.$$

A similar result is also valid for a monotonic solution. Namely, we have

THEOREM 2. *Let $\tilde{G}$ be a linearly ordered group and G a subgroup of $\tilde{G}$. Suppose that $F\colon X \times G \to X$ is a monotonic transitive solution of equation* (1) *and denote by G^* any stability subgroup of the function F. The solution F can be extended to a monotonic transitive solution $\tilde{F}\colon X \times \tilde{G} \to X$ of equation* (1) *iff there exists a convex subgroup $\tilde{G}_1$ of $\tilde{G}$ such that*

$$\tilde{G}_1 \cdot G = \tilde{G}, \tag{7}$$

$$\tilde{G}_1 \cap G = G^*. \tag{8}$$

Proof. Suppose that a transitive monotonic solution $\tilde{F}\colon X \times \tilde{G} \to X$ of equation (1) is an extension of a transitive monotonic solution $F\colon X \times G \to X$ of this equation. There exists $x \in X$ such that

$$G^* = \{\alpha \in G\colon F(x, \alpha) = x\}.$$

Put

$$\tilde{G}_1 = \{\alpha \in \tilde{G}\colon \tilde{F}(x, \alpha) = x\}.$$

In virtue of Theorem 1, $\tilde{G}_1$ is convex. Furthermore, as it has been proved in [3], conditions (7) and (8) hold.

Conversely, suppose that a convex subgroup $\tilde{G}_1$ of $\tilde{G}$ satisfies (7) and (8). As in [3] we put

$$\tilde{g}(a\tilde{G}_1) = g(aG^*) \quad \text{for } a \in G. \tag{9}$$

We only have to show that $\tilde{g}$ is monotonic. Suppose that $a\tilde{G}_1 \leqslant b\tilde{G}_1$, $a, b \in G$. Then there exist $a_1 \in a\tilde{G}_1$, $b_1 \in b\tilde{G}_1$ such that $a_1 \leqslant b_1$ (see (3)). But then $a_1 G^* \leqslant b_1 G^*$ and as g is monotonic (say increasing), we obtain applying (9):

$$\tilde{g}(a\tilde{G}_1) = \tilde{g}(a_1 \tilde{G}_1) = g(a_1 G^*) \leqslant g(b_1 G^*) = \tilde{g}(b_1 \tilde{G}_1) = \tilde{g}(b\tilde{G}_1). \ \square$$

2. Now we are going to consider the interrelation between monotonicity and continuity of solutions of the translation equation.

Let Y be a linearly ordered set and let $x, y \in Y$. We shall use the following notation:

$$]x, y[\, = \{z \in Y\colon x < z < y\}, \quad]x, \infty[\, = \{z \in Y\colon x < z\},$$
$$]-\infty, x[\, = \{z \in Y\colon z < x\}.$$

As is well known, the family of the sets:

$$\{]x, y[\colon x, y \in X\} \cup \{]x, \infty[\colon x \in X\} \cup \{]-\infty, x[\colon x \in X\} \cup \{X\}$$

is a base of topology in Y, which will be called the topology induced by order. Further on we shall regard X and G as topological spaces with topologies induced by order. The group G is made into a topological group. It is easy to observe that if the order in G is not dense, then the induced topology in G is discrete.

Before proving the next theorem we prove three simple lemmas.

LEMMA 1. *If F is a monotonic solution of equation* (1) *satisfying the identity condition and*

$$G_x\colon = \{\alpha \in G\colon F(x, \alpha) = x\} = \{1\},$$

then the function $F(x, \cdot)$ is strictly monotonic.

Proof. Let, say, $F(x, \cdot)$ be increasing and let $\alpha < \beta$, $\alpha, \beta \in G$. Then $F(x, \alpha) \leqslant F(x, \beta)$. If $F(x, \alpha) = F(x, \beta)$, then we would have $F(x, \beta^{-1}\alpha) = x$, i.e. $\beta^{-1}\alpha \in G_x$, and hence $\alpha = \beta$, which is a contradiction. $\square$

LEMMA 2. *Let F be a monotonic and transitive solution of equation* (1). *If the order in G is dense and $G_x = \{1\}$ for some $x \in X$, then the order in X is dense.*

Proof is obvious.

LEMMA 3. *If F is a monotonic solution of equation* (1) *satisfying condition* (2), *then the following conditions hold*:

$$F^{-1}(]-\infty, t[) = \bigcup_{a \in G}]-\infty, F(t, a^{-1})[\times \{a\}, \tag{10}$$

(11) $$F^{-1}(]t, \infty[) = \bigcup_{a\in G}]F(t, a^{-1}), \infty[\times \{a\},$$

(12) $$F^{-1}(]s, t[) = \bigcup_{a\in G}]F(s, a^{-1}), F(t, a^{-1})[\times \{a\}.$$

Proof. We shall prove equality (10) only (the proof of conditions (11) and (12) is similar).

Suppose that $(x, a)\ F^{-1}(]-\infty, t[)$, i.e. $F(x, a) < t$. Since the function $F(\cdot, a)$ is strictly increasing (see Remark 1), we have

$$x = F(F(x, a), a^{-1}) < F(t, A^{-1})$$

which means that $(x, a) \in]-\infty, F(t, a^{-1})[\times \{a\}$.

Now suppose that $(x, a) \in \bigcup_{b\in G}]-\infty, F(t, b^{-1})[\times \{b\}$, i.e. $(x, a) \in$ $\in]-\infty, F(t, a^{-1})[\times \{a\}$. Then $x < F(t, a^{-1})$ and consequently

$$F(x, a) < F(F(t, a^{-1}), a) = F(t, a\cdot a^{-1}) = t,$$

which means that $(x, a) \in F^{-1}(]-\infty, t[)$. □

We now prove the following

THEOREM 3. *Let F be a monotonic solution of equation* (1) *satisfying condition* (2). *If the order in G is not dense, then F is continuous.*

Proof. It is sufficient to prove that the inverse image by F of any element of the base in X is open. Since the order in G is not dense, the topology in G is discrete and hence every one-element set is open. Sets of the form $]-\infty, F(t, a^{-1})[$, $]F(t, a^{-1}), \infty[$, $]F(s, a^{-1}), F(t, a^{-1})[$, X are open (as they belong to the base of topology in X). Consequently sets of the form (10), (11), (12) are open and so is the set $F^{-1}(X) = \bigcup_{a\in G} X \times \{a\}$. □

The question of continuity of F is already solved in the case where the order in G is not dense. We now consider the case of a dense order in G. First we prove two lemmas.

LEMMA 4. *If the order in X is not dense and F is a transitive monotonic solution of* (1), *then the topology in X is discrete.*

Proof. We shall show that every one-element set is open. Consider any $x \in X$. There exist $s, t \in X$ such that $s < t$ and $]s, t[= \emptyset$. There also exist $a, b \in G$ such that $x = F(s, a)$, $x = F(t, b)$. We have

$$F(s, b) < F(t, b) = x = F(x, a) < F(t, a),$$

and consequently

$$]F(s, b), F(t, a)[=]F(s, b), x[\cup \{x\} \cup]x, F(t, a)[.$$

We shall prove that $]F(s, b), F(t, a)[= \{x\}$.

Suppose that $]F(s, b), x[\neq \emptyset$, i.e. there exists $y \in X$ such that $F(s, b) < y < x = F(t, b)$. But then we would have

$$s = F(F(s, b), b^{-1}) < F(y, b^{-1}) < F(F(t, b), b^{-1}) = t,$$

which contradicts the condition $]s, t[= \emptyset$. Hence $]F(s, b), x[= \emptyset$. In a similar way it can be proved that $]x, F(t, a)[= \emptyset$, which together with the last equality implies the equality $]F(s, b), F(t, a)[= \{x\}$.

LEMMA 5. *If the order in G is dense, the order in X is not dense and F is a monotonic and transitive solution of* (1), *then for every $x, t \in X$ the set*

$$A_{x,t} := \{a \in G: F(x, a) = t\}$$

is open in G.

Proof. $A_{x,t}$ is a left coset of G modulo the subgroup $G_x = \{a \in G: F(x, a) = x\}$. Since G is a topological group, it is sufficient to prove that G_x is open in G. We are going to prove that

$$G_x = \bigcup_{a \in G_x, a \geq 1}]a^{-1}, a[. \tag{13}$$

Suppose that $b \in G_x$. By Lemma 2, $G_x \neq \{1\}$ and hence there exists $c \in G_x$, $c > 1$. Obviously $bc \in G_x$ and $(bc)^{-1} \in G_x$. If $b \geq 1$, then $1 \leq b < bc$ and consequently $(bc)^{-1} < 1 \leq b < bc$, which implies that

$$b \in](bc)^{-1}, bc[\quad \text{with } bc \in G_x, bc > 1.$$

In a similar way, if $b \leq 1$, then $bc^{-1} < b \leq 1 < (bc^{-1})^{-1}$, which means that $b \in](bc^{-1}), (bc^{-1})^{-1}[$ with $bc^{-1} \in G_x$, $(bc^{-1})^{-1} > 1$. Now suppose that $b \in]a^{-1}, a[$ for some $a \in G$, $a \geq 1$. Then $a^{-1} < b < a$ and $a^{-1}, a \in G_x$. But G_x is convex and hence $b \in G_x$. We have proved equality (13). Every set of the form $]a^{-1}, a[$ is open and hence by (13) G_x is open.

We are going to prove

THEOREM 4. *If the order in G is dense, then every transitive and monotonic solution F of equation* (1) *is continuous.*

Proof. Suppose that the order in X is not dense. Then by Lemma 4 the topology in X is discrete. We have also

$$F^{-1}(\{t\}) = \{(x, a) \in X \times G: F(x, a) = t\} = \bigcup_{x \in X} \{x\} \times \{a \in G: F(x, a) = t\} = \bigcup_{x \in X} \{x\} \times A_{x,t}.$$

But by Lemma 5, $A_{x,t}$ is open. Hence $F^{-1}(\{t\})$ is open.

Now assume that the order in X is dense. It is sufficient to prove that for every $t, s \in X$ the sets $F^{-1}(]-\infty, t[)$, $F^{-1}(]s, t[)$, $F^{-1}(]t, \infty[)$, $F^{-1}(x)$ are open. We are going to show that the set $F^{-1}(]s, t[)$ is open. Assume that

$(x, a) \in F^{-1}(]s, t[)$, i.e. $s < F(x, a) < t$. Since the order in X is dense, there exist $p, q \in X$ such that

$$s < p < F(x, a) < q < t.$$

Further there exist $b, c \in G$ such that $p = F(x, b)$, $q = F(x, c)$. Thus we have

$$s < F(x, b) < F(x, a) < F(x, c) < t. \tag{14}$$

Assume that the function $F(x, \cdot)$ is increasing (the proof in the case when $F(x, \cdot)$ is decreasing is similar). Then in view of (14)

$$b < a < c. \tag{15}$$

Put $U =]F(s, b^{-1}), F(t, c^{-1})[\times]b, c[$. By (14) and (15) $(x, a) \in U$. This means that U is a neighbourhood of (x, a). We shall prove that $U \subset F^{-1}(]s, t[)$. Suppose that $(y, d) \in U$, i.e. $y \in]F(s, b^{-1}), F(t, c^{-1})[$ and $d \in]b, c[$. Then

$$s < F(y, b) \leqslant F(y, d) \leqslant F(y, c) < t,$$

which together with the condition $d \in]b, c[$ means that $(y, d) \in F^{-1}(]s, t[)$. Hence $U \subset F^{-1}(]s, t[)$. We have thus proved that the set $F^{-1}(]s, t[)$ is open. In a similar way one can prove that the sets $F^{-1}(]-\infty, t[)$, $F^{-1}(]t, \infty[)$, $F^{-1}(X)$ are open. □

Remark 2. In the case where the order in G is dense the question whether every monotonic solution of equation (1) satisfying condition (2) must be continuous is still open.

A function F satisfying (1) and (2) may be transitive and continuous but not monotonic. To show this consider the following

EXAMPLE. Let $G = \mathbf{Z}$ be the ordered group of integers and let $X = \{0, 1\}$ with $0 < 1$. Then the topologies in G and in X are discrete. Put

$$F(x, a) = \begin{cases} 0 & \text{if } x+a \text{ is an even integer,} \\ 1 & \text{if } x+a \text{ is odd.} \end{cases}$$

It is clear that F satisfies (1), (2) and F is transitive. Obviously F is continuous (since the topologies in G and in X are discrete). But we have

$$F(0, 3) = 1, \quad F(0, 4) = 0$$

and

$$F(0, 2) = 0, \quad F(0, 3) = 1,$$

which means that F is not monotonic.

References

[1] A. G. Kurosz, *Algebra ogólna* (*General algebra*, Polish translation from the Russian), Warszawa 1965.

[2] Z. Moszner, *Structure de l'automate plein, réduit et inversible*, Aeq. Math. 9 (1973), p. 46–59.

[3] —, *Sur le prolongement des objects géométriques transitifs*, Tensor, N. S. 26 (1972), p. 239–242.

[4] L. S. Pontriagin, *Grupy topologiczne* (*Topological groups*, Polish translation from the Russian), Warszawa 1961.

Reçu par la Rédaction le 2.04.1979