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# On Nevanlinna's proximity functions of a meromorphic function

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**1.** Suppose that $f(z)$ is a meromorphic function in the entire complex plane $\mathfrak{C}$. For the sake of brevity we do not mention the explicit meanings of the symbols $m(r,f)$, $m(r,1/f)$; $n(r,f)$, $n(r,1/f)$; $N(r,f)$, $N(r,1/f)$; $T(r,f)$, etc. occurring frequently in the Nevanlinna theory of meromorphic functions (see Hayman [1]). If $\varphi$ denotes any of the auxiliary functions, we write $\varphi(r) = \varphi(r,f)+\varphi(r,1/f)$. Our purpose in this paper is to investigate the estimations of the $m(r)$ relative to $n(r)$ and $r^{\varrho(r)}$, where $\varrho(r)$ is a proximate order of $f(z)$, $\varrho$ being the usual order of $f(z)$ in terms of $T(r,f)$.

**2.** In this section we collect the necessary lemmas of which we make use in our investigations.

LEMMA A. *Let $f(z)$ be a meromorphic function of order $\varrho$ $(0<\varrho<1)$. Then*

$$m(r) \leqslant \int_0^r \frac{n(x)}{x}\,dx + r\int_r^\infty \frac{n(x)}{x^2}\,dx + \tag{2.1}$$

$$+\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\left\{\frac{1}{r^{2m}}\int_0^r \frac{n(x)}{x^{-2m+1}}\,dx + r^{2m+2}\int_r^\infty \frac{n(x)}{x^{2m+3}}\,dx\right\} + O(1).$$

The proof of Lemma A follows by putting $p=0$ in a result obtained earlier by one of us (Kamthan [3], Theorem C).

From lemma A we deduce

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**Lemma B.** *Let $f(z)$ be a meromorphic function of order $\varrho$ $(0<\varrho<1)$ and let $0\leqslant a<1$. Then*

$$\int_0^r \frac{m(t)}{t^{1+a}}\,dt \leqslant \frac{1}{1-a}\int_0^r \frac{N(t)}{t^{1+a}}\,dt+\frac{r^{1-a}}{1-a}\int_r^\infty \frac{N(t)}{t^2}\,dt+$$

$$+\frac{8}{\pi}\sum_{m=0}^{\infty}\left\{\frac{a}{(2m+a)(2m+2-a)}\int_0^r \frac{N(t)}{t^{1+a}}\,dt+\right.$$

$$\left.+\frac{1}{2m+1}\left[\frac{m}{(2m+a)r^{2m+a}}\int_0^r \frac{N(t)}{t^{1-2m}}\,dt+\frac{m+1}{2m+2-a}r^{2m+2-a}\int_r^\infty \frac{N(t)}{t^{2m+3}}\,dt\right]\right\}+$$

$$+O(\bar{r}^{a}),$$

*where the last term $O(r^{-a})$ is to be written as $O(\log r)$ when $a=0$.*

Proof of Lemma B. We have

$$(2.2)\qquad \int_0^r \frac{m(t)}{t^{1+a}}\,dt \leqslant \int_0^r \frac{dt}{t^{1+a}}\int_0^r \frac{n(x)}{x}\,dx+\int_0^r \frac{dt}{t^a}\int_t^\infty \frac{n(x)}{x^2}\,dx+$$

$$+\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\left\{\int_0^r \frac{dt}{t^{2m+1+a}}\int_0^t \frac{n(x)}{x^{-2m+1}}\,dx+\right.$$

$$\left.+\int_0^r t^{2m+1-a}\,dt\int_t^\infty \frac{n(x)}{x^{2m+3}}\,dx\right\}+\int_0^r O(t^{-1-a})\,dt$$

$$=I_1+I_2+\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}(I_3+I_4)+I_5\qquad \text{(say)}.$$

Clearly

$$I_1=\int_0^r \frac{N(t)}{t^{1+a}}\,dt.$$

Now, as the order $\varrho$ is less than 1, one has

$$I_2=\int_0^r \frac{dt}{t^a}\left\{-\frac{N(t)}{t}+\int_t^\infty \frac{N(x)}{x^2}\,dx\right\}=\frac{a}{1-a}\int_0^r \frac{N(t)}{t^{1+a}}\,dt+\frac{r^{1-a}}{1-a}\int_r^\infty \frac{N(t)}{t^2}\,dt.$$

Again, changing the order of integration in the integrals in $I_3$, we have

$$I_3=\frac{a}{2m+a}\int_0^r \frac{N(t)}{t^{1+a}}\,dt+\frac{2m}{2m+a}\int_0^r \frac{t^{2m-1}}{r^{2m+a}}N(t)\,dt.$$

Further, we have

$$I_4 = -\int_0^r \frac{N(t)}{t^{1+a}}\,dt + (2m+2)\int_0^r t^{2m+1-a}\,dt \int_t^\infty \frac{N(x)}{x^{2m+3}}\,dx$$

$$= -\int_0^r \frac{N(t)}{t^{1+a}}\,dt + (2m+2)\left\{\int_0^r \frac{N(x)}{x^{2m+3}}\,dx \int_0^x t^{2m+1-a}\,dt + \right.$$

$$\left. + \int_r^\infty \frac{N(x)}{x^{2m+3}}\,dx \int_0^r t^{2m+1-a}\,dt\right\}$$

$$= \frac{a}{2m+2-a}\int_0^r \frac{N(t)}{t^{1+a}}\,dt + \frac{2(m+1)}{2m+2-a}\,r^{2m+2-a}\int_r^\infty \frac{N(t)}{t^{2m+3}}\,dt,$$

and

$$I_5 = \begin{cases} O\left(\dfrac{1}{ar^a}\right) & \text{if } a > 0, \\ O(\log r) & \text{if } a = 0. \end{cases}$$

The proof of the lemma follows on substituting the values of $I_1$-$I_5$ in (2.2).

LEMMA C. *This is Lemma* B *for* $a = 0$.

**3.** We now proceed to state and prove the results we promised in section 1 of this paper.

THEOREM 1. *Suppose* $f(z)$ *is a meromorphic function of order* $\varrho$ $(0 < \varrho < 1)$. *Then*

$$\varliminf_{r\to\infty} \frac{m(r)}{n(r)} \leqslant A(\varrho), \tag{3.1}$$

*where*

$$A(\varrho) = \frac{1}{\varrho(1-\varrho)} + \frac{2}{1-\varrho}\cot\frac{\pi\varrho}{2}.$$

Proof. If $\eta > 0$, then it is clear that

$$\varlimsup_{t\to\infty} \frac{N(t)}{t^{\varrho-\eta}} = \infty, \qquad \lim_{t\to\infty} \frac{N(t)}{t^{\varrho+\eta}} = 0.$$

Therefore, by a lemma on Pólya-peaks (Hayman [1], p. 101), there exist arbitrarily large values of $r$, say $\{r_n\}$, $r_n \to \infty$ as $n \to \infty$, such that

$$\frac{N(t)}{t^{\varrho-\eta}} \leqslant \frac{N(r_n)}{r_n^{\varrho-\eta}} \quad (0 \leqslant t \leqslant r_n), \tag{3.2}$$

$$\frac{N(t)}{t^{\varrho+\eta}} \leqslant \frac{N(r_n)}{r_n^{\varrho+\eta}} \quad (t \geqslant r_n). \tag{3.3}$$

Hence on making use of Lemma C, we obtain

$$\int_0^{r_n} \frac{m(t)}{t} dt \leqslant N(r_n)\Big\{\frac{1}{r_n^{\varrho-\eta}} \int_0^{r_n} t^{\varrho-\eta-1} dt + \frac{1}{r_n^{\varrho+\eta-1}} \int_{r_n}^{\infty} t^{\varrho+\eta-2} dt +$$

$$+\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\Big[\frac{1}{r_n^{2m+\varrho-\eta}}\int_0^{r_n} t^{2m-1+\varrho-\eta} dt + r_n^{2m+2-\varrho-\eta}\int_{r_n}^{\infty} t^{-2m-3+\varrho+\eta} dt\Big]\Big\} +$$

$$+ O(\log r_n)$$

$$= N(r_n)\Big\{\frac{1-2\eta}{(\varrho-\eta)(1-\varrho-\eta)} +$$

$$+\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{2}{2m+1}\cdot\frac{2m+1-2\eta}{(2m+\varrho-\eta)(2m+2-\varrho-\eta)}\Big\} + O(\log r_n).$$

Since $\eta$ is arbitrary, therefore, letting $\eta \to 0$, one finds

$$\int_0^{r_n} \frac{m(t)}{t} dt \leqslant A(\varrho) N(r_n)\big(1+o(1)\big). \tag{3.4}$$

Now, for $t \geqslant t_0$ let

$$m(t) > A(\varrho) n(t)\big(1+o(1)\big).$$

Then

$$\int_0^{r} \frac{m(t)}{t} dt > A(\varrho) N(r)\big(1+o(1)\big),$$

for all large $r$, which contradicts (3.4), and so for arbitrarily large values of $r$

$$m(r) \leqslant A(\varrho) n(r)\big(1+o(1)\big),$$

and this completes the result.

Remark. The above result is not necessarily true for functions of integral order. Take for instance $f(z) = e^z$. Then $m(r) = 2r/\pi$ for all $r > 0$, $n(r) = 0$ for all $r$. Hence the left-hand expression in (3.1) is $\infty$.

Finally we prove

THEOREM 2. *Let $f(z)$ be a meromorphic function of proximate order $\varrho(r)$, order $\varrho$ $(0 < \varrho < 1)$. Let*

$$\varliminf_{r\to\infty} \frac{n(r)}{r^{\varrho(r)}} = \delta, \quad 0 < \delta < \infty;$$

$$\varlimsup_{r\to\infty} \frac{T(r,f)}{r^{\varrho(r)}} = \sigma, \quad 0 < \sigma < \infty.$$

*Then*
$$\overline{\lim_{r\to\infty}} \frac{m(r)}{r^{\varrho(r)}} \leqslant (2\varrho e\sigma - \delta) A(\varrho),$$
*where $A(\varrho)$ means the quantity in Theorem* 1.

Proof. Following the lines of proof of Theorem 7 in [2], p. 12, we obtain for $k > 1$ the quantity
$$\overline{\lim_{r\to\infty}} \frac{n(r)}{r^{\varrho(r)}} \leqslant \frac{2\varrho\sigma k^{\varrho} - \delta}{\varrho \log k}.$$
Therefore, (2.1) gives
$$m(r) \leqslant \mathrm{B}\Big\{ \int_{r_0}^{r} x^{\varrho(x)-1}dx + r\int_{r}^{\infty} x^{\varrho(x)-2}dx +$$
$$+\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\Big[\frac{1}{r^{2m}}\int_{r_0}^{r} x^{2m-1+\varrho(x)}dx + r^{2m+2}\int_{r}^{\infty} x^{-2m-3+\varrho(x)}dx +$$
$$+O(r^{-2m})\Big]\Big\} + O(1),$$
for $r \geqslant r_0$, where
$$B = \frac{2\varrho\sigma k^{\varrho} - \delta}{\varrho\log k} + \varepsilon, \qquad \varepsilon > 0.$$
Hence
$$\overline{\lim_{r\to\infty}} \frac{m(r)}{r^{\varrho(r)}} \leqslant A(\varrho)\,\frac{2\varrho\sigma k^{\varrho} - \delta}{\varrho\log k},$$
which, on putting $\log k = 1/\varrho$, yields the result.

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