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Projectivity, injectivity and duality

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INTRODUCTION

There are striking analogies between certain theorems in various branches of mathematics. For instance, let us mention the following ones:

1. In homological algebra: *Every left Λ-module is a quotient of a left Λ-module with a basis.*

2. In general topology: *Every compact (Hausdorff) space is a continuous image of $\beta(N_t)$ where N_t is a discrete space.*

3. In functional analysis: *Every Banach space is a quotient of a space* $l(N_t)$.

4. In the theory of Boolean algebras: *Every Boolean algebra is a quotient of a free Boolean algebra.*

These analogies and some related ones were pointed out recently by some authors (Gleason [1], Halmos [1] and [2], Nachbin [2], Rainwater [1] and others). A purpose of this paper [1] is to show that they are actually special cases of a general theorem. In different theories they are expressed differently and we are going to adopt the terminology and methods of homological algebra.

In homological language an object P is called *projective* (respectively, an object I is called *injective*) if every diagram

$$\begin{tikzcd} & P \arrow[d, "\alpha"] \\ B \arrow[r, "\pi", "\text{onto}"'] & A \end{tikzcd} \qquad \begin{tikzcd} I & \\ A \arrow[u, "\beta"] \arrow[r, "\sigma", "\text{embedding}"'] & B \end{tikzcd}$$

can be embedded into a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "\alpha'"'] \arrow[dr, "\alpha"] & \\ B \arrow[rr, "\pi"'] & & A \end{tikzcd} \qquad \begin{tikzcd} & I & \\ A \arrow[ur, "\beta"] \arrow[rr, "\sigma"'] & & B \arrow[ul, "\beta'"'] \end{tikzcd}$$

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respectively. These diagrammatic definitions can be mutatis mutandis formulated for each of the categories quoted above.

In standard terminology this means that an object I (a group, a topological space) is injective if and only if for every object B and subobject A every admissible map $\beta: A \to I$ (a homomorphism, a continuous map etc.) can be extended to an admissible map β' from B to I (i.e. $\beta'\sigma = \beta$). This problem has been investigated intensively in various contexts. E.g. injective objects in the category of compact spaces and continuous maps are just the absolute retracts, injective Boolean algebras are those which are complete, injective objects in the category of Banach spaces and linear operators with $\|T\| \leqslant 1$ are just the spaces equivalent to spaces $C(S)$ with S extremally disconnected.

Similarly, under certain assumptions of existence of kernels and quotients, an object P is projective if any admissible map α from P into any quotient space $B/C = A$ can be lifted to a map $\alpha': P \to B$ so that for each $x \in P$ the element $\alpha'(x)$ belongs to the coset of $\alpha(x)$ (i.e. $\pi\alpha' = \alpha$ where $\pi: B \to B/C$ is the quotient map).

A systematization and a unified approach to the theorems of the kind mentioned above is the subject of the paper; the starting point is of course the notion of a category of objects and maps introduced by Eilenberg and MacLane (2) or rather that of a bicategory introduced by MacLane [1]. Concrete categories may yield concrete bicategories in various ways and none of them covers all special cases that we need. The crucial point here is the notion of a subobject; the general definition considered by Grothendieck [3] and adopted in [KLS] does not fit some cases that we are interested in (e.g. according to that definition, in the category of locally compact abelian groups and continuous homomorphisms, the additive group Z of integers is considered as a subobject of the compact circle group T). Thus the axiomatic treatment of bicategories seems to be the best way. Specifically, we shall assume Isbell's ([1], [2]) modification of MacLane's axioms.

The main notion introduced in the paper is that of a pair of objects dually defined in primary terms of category, namely a *basic direct object* and a *basic free object* (abbreviations: b. d. o. and b. f. o.). Roughly speaking, a b. d. o. is the smallest object D such that for each object A the set of all morphisms $\varphi: A \to D$ separates A; an exact definition is in § 6. Both the objects are unique (up to equivalence) whenever they exist and are exemplified by the compact circle group T (b. d. o.) and the group Z of integers (b. f. o.) in the category of locally compact abelian groups

(2) For quick reference the paper by Eilenberg and MacLane [1] will be quoted as [E-McL] and the expository article by Kurosh, Livshits and Shulgeifer [1] will be quoted as [KLS]. All notions in Part I are defined and the knowledge of the theory of categories is not assumed.

and continuous homomorphisms, a closed interval I (b. d. o.) and a one-point space (b. f. o.) in the category of compact Hausdorff spaces and continuous maps, a two-element algebra $\{0, 1\}$ (b. d. o.) and a four-element algebra $\{0, 1, A, A'\}$ (b. f. o.) in the category of Boolean algebras and homomorphisms, the real line R (both b. d. o. and b. f. o.) in the category of Banach spaces and linear operators of norm $\|U\| \leqslant 1$.

An object is called *direct* if it is a direct join of a set of copies of a b. d. o. and is called *free* if it is a free join of a set of copies of a b. f. o. This yields such objects as toroidal groups $T^{\mathfrak{m}}$ (direct) and free abelian groups (free) in the category of locally compact abelian groups, Tikhonov cubes $I^{\mathfrak{m}}$ (direct) and Stone-Čech compactifications $\beta(N_t)$ of isolated spaces (free) in the category of compact spaces, fields of all subsets (direct) and free Boolean algebras (free) in the category of Boolean algebras, spaces $m(N_t)$ of bounded functions (direct) and spaces

$$l(N_t) = \Big\{x\colon x = x(u),\ \|x\| = \sum_{u\in N_t} |x(u)| < \infty\Big\}$$

(free) in the category of Banach spaces.

Unfortunately, a ring Λ need not be a basic free object in the category of left Λ-modules and Λ-homomorphisms though it has one free generator. We do not use the standard definition of a free object as an object with a set of free generators because we want all these notions to be defined in terms of admissible maps, without the notion of an element ([3]). Similarly we prove that any object is a subobject of a direct object but we have to quit the canonical embedding $a \to \{a^*(a)\}_{a^*\in A^*}$ into the direct product indexed by elements of a dual A^* ([4]).

The main result, called the Universality Theorem (§ 6), states that the free and direct objects are universal with respect to maps onto and embeddings, respectively.

Specifically, we consider an abstract bicategory $\mathscr{K}$ and we assume the existence of a basic direct object D and direct joins of arbitrary many copies of D; we also assume that D is injective and a stronger separation property of $\mathrm{Map}(A, D)$. Then *each object is a subobject of a direct object* and *an object is injective if and only if it is a retract of a direct object*. The dual statement concerns relations between free and projective objects:

([3]) Thus, the notion "free" defined in this paper does not coincide necessarily with the notion "free" used in general algebra, cf. 6.2 and 9.5.

In concrete categories, if the notion of an element is given, we may define the notion of a set of free generators (cf. § 14) assuming that every map from such a free basis to another object can be extended uniquely to a morphism.

Nevertheless, the Universality Theorem does apply to the bicategory of left Λ-modules, because it is actually true if F is any projective strict coseparator [D is any injective strict separator], not necessarily the smallest one (cf. 5.4).

([4]) The duality of $a \to \{a^*(a)\}$ type is investigated by Ohkuma [1].

every object is an image of a free object, an object is projective if and only if it is a retract of a free object. This, in turn, enables us to characterize projective and injective objects as absolute quotient retracts and absolute subretracts, respectively. A *retraction* is a surjection $\alpha: A \to B$ which admits a *cross-section* $\beta: B \to A$ in the category, i.e. a map β such that $\alpha\beta$ is the identity on B. If it is the case, B is called a *retract* of A. *Absolute subretract* is an object which is a retract of any object containing it as a subobject; an *absolute quotient retract* is an object which is a retract of any object that can be mapped onto it.

Part II deals with several special bicategories; it contains a systematization and a review of known results for some bicategories, and some contributions to the others. The most interesting thing in this part seems to be the table of 12 selected bicategories which contains description of objects, morphisms, constructions of free and direct joins, description of basic free and direct objects and so forth. In this way analogies and differences are laid out.

The notion of formal duality appears throughout the paper. The paper, however, is also devoted to the justification and explanation why some well-known categories have unique "natural" objects serving the duality; we are going to account for the fact that the dual of a Banach space is defined as the set of all linear maps to the real line (functionals) while for locally compact abelian groups the best are homomorphisms to the circle (characters), in Stone's duality representation of Boolean algebras we use $(0,1)$-homomorphisms and in the theory of compact spaces we consider the continuous maps into the unit interval. The answer to this question is considered in the paper, and it follows implicitly from the presented analysis that *this "natural" object turns out to be a basic direct object.*

The paper does not exhaust this problem; the discussion and analysis of relations between various aspects of the notion of duality as well as the discussion of the notion of a free object (from algebraic and categorical point of view) are not complete.

We shall use the notation $A \twoheadrightarrow B$ for surjections and $A \rightarrowtail B$ for injections (for more details, see § 2); this is particularly convenient in diagrams. The symbols $A \twoheadrightarrow B$ and $A \rightarrowtail B$ may be considered as remains of short exact sequences $A \longrightarrow B \longrightarrow 0$ and $0 \longrightarrow A \longrightarrow B$, respectively, which are not admissible without additional assumptions about the category (cf. Heller [1]).

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I. PROJECTIVITY AND INJECTIVITY IN ABSTRACT BICATEGORIES

§ 1. Categories and bicategories

1.1. We give here a short review of basic notions and establish the terminology which is founded on [E-McL] and [KLS] and on the papers by MacLane [1], Grothendieck [3] and Isbell [1], [2], [3]. Whenever our terminology disagrees essentially with one of those papers, it is mentioned in footnote. Terminological problems make several troubles because one of the tasks of the theory of categories is to unify various concepts investigated in different theories, but more serious are conceptual difficulties: Each author has an individual point of view depending on his major subject outside of categories. This influences particularly the option of definitions.

Eilenberg and MacLane explain possible ways of avoiding antinomies while using terms like "the category of (all) groups". One of them is to assume the axiomatic theory of sets and classes due to Bernays and von Neumann and distinguish carefully the notions of a "class" and a "set". The most important restriction is that no class may be a member of any class.

Specifically, we shall assume the axioms listed by Gödel [1]; they yield existence theorems for categories (cf. Chapter II of Gödel's paper). By axiom of choice we shall mean the strong form of this axiom listed by Gödel.

1.2. A *map* is a triple (φ, A, B), written $\varphi: A \to B$, where A and B are sets and φ is a single-valued function defined on A and with values in B. Thus, if $\varphi(A) \subset C \subset B$ and $C \neq B$, then $\varphi: A \to B$ and $\varphi: A \to C$ are considered as different maps though they are determined by the same correspondence $a \to \varphi(a)$ for all $a \in A$. The *composition* of $\varphi: A \to B$ and $\psi: B \to C$ is denoted (5) by $\psi\varphi: A \to C$. The *identity* map is denoted by $\varepsilon_A: A \to A$ or shortly by ε_A. The set A is called the *domain* and B is called the *range* of $\varphi: A \to B$.

(5) In [KLS] the notation $\varphi\psi: A \to C$ is used; it is more convenient while considering diagrams (first "acting of φ", then "acting of ψ") but we will keep the notation used in analysis $(\psi\varphi)(x) = \psi[\varphi(x)]$. Whenever we write $\psi\varphi$, we assume that this product exists. If $B_1 \subset B_2$ but $B_1 \neq B_2$, then the composition of $\alpha: A \to B_1$ and $\beta: B_2 \to C$ is assumed not to exist, but we may consider $\beta\varepsilon_{B_2B_1}\alpha$ where $\varepsilon_{B_2B_1}: B_1 \to B_2$ is the embedding map.

Given a (non-empty) class $\mathscr{A}$ of sets, a *concrete category over* $\mathscr{A}$ is a class $\mathscr{K}$ of maps satisfying the following three conditions:

(i) If $\varphi: A \to B$ belongs to $\mathscr{K}$, then $A \in \mathscr{A}$ and $B \in \mathscr{A}$.

(ii) If $A \in \mathscr{A}$, then the identity $\varepsilon_A: A \to A$ belongs to $\mathscr{K}$.

(iii) If $\varphi: A \to B$ and $\psi: B \to C$ belong to $\mathscr{K}$, then so does $\psi\varphi: A \to C$.

The elements of $\mathscr{A}$ are called *objects* and the elements of $\mathscr{K}$ are called *morphisms* ([6]). $\mathrm{Map}(A, B)$ or $\mathrm{Map}(A, B; \mathscr{K})$ will denote the set of all morphisms $\varphi: A \to B$.

An *abstract category* is a class $\mathscr{K}$ of elements, called morphisms, whose product (composition) is the subject to some axioms ([7]).

A *monomorphism* is a morphism φ which is left cancellable (i.e. $\varphi\alpha = \varphi\beta$ implies $\alpha = \beta$ for all morphisms α, β); an *epimorphism* is a right cancellable morphism. Every one-one morphism is a monomorphism and every morphism onto is an epimorphism, but the converse implications are not true without additional assumptions. A morphism is a *bimorphism* if it is a monomorphism and an epimorphism simultaneously.

An equivalence $\varphi: A \to B$ might be defined as a one-one morphism onto B such that the inverse map $\varphi^{-1}: B \to A$ belongs to $\mathscr{K}$ as well ([8]), but this definition uses point-map terminology; another definition, in terms of morphisms only, is as follows: a morphism $\varphi: A \to B$ is an *equivalence* if there exists a morphism $\psi: B \to A$ such that $\varphi\psi = \varepsilon_B$ and $\psi\varphi = \varepsilon_A$.

We shall say that a category has the *inversion property* if every bimorphism is an equivalence. This is a rather strong assumption; e.g. the category of Banach spaces and linear maps does not have this property (despite Banach's inversion theorem, cf. § 12).

A *subcategory* $\mathscr{K}_1$ of $\mathscr{K}$ is a category over a subclass $\mathscr{A}_1$ of $\mathscr{A}$ such that every morphism $\varphi: A \to B$ of $\mathscr{K}_1$ belongs to $\mathscr{K}$. $\mathscr{K}_1$ is called a *full subcategory* of $\mathscr{K}$ if every morphism of $\mathscr{K}$ with domain and range in $\mathscr{A}_1$ belongs to $\mathscr{K}_1$.

1.3. A *bicategory* is a category with an axiomatic notion of a subobject. Technically, we add two primary notions of an *injection* and of a *surjection* ([9]). Even if the category is concrete, these notions are defined

([6]) The common term is "map" or "mapping". The term "morphism" is used by Grothendieck [3]. We reserve the term "map" for general use, i.e. every concrete morphism is a map but not conversely.

In [KLS] an (abstract) category isomorphic to a concrete one is also called concrete.

([7]) Actually, axioms (A_0)-(A_4) of § 15 are the axioms of a category.

([8]) Another common term is "isomorphism".

([9]) MacLane [1] and Isbell [1], [2] use the term "projection" for any morphism onto, but according to the terminology adopted in functional analysis a "projection" means a linear idempotent operator. In order to avoid confusions we do not use the term "projection" at all, using "surjection" and "retraction", respectively. In the sequel we shall mean bicategories in Isbell's sense (rather than in MacLane's).

axiomatically and (if the category is given) they are not unique (unless the category has the inversion property, cf. Isbell [2], p. 576), though either of these notions can be expressed by the other (cf. Isbell [1], p. 116, [2], p. 573.) The class $\mathcal{J}$ of injections and the class $\mathcal{S}$ of surjections are assumed to satisfy the following axioms:

(i) $\mathcal{J}$ and $\mathcal{S}$ are subcategories of $\mathcal{K}$.

(ii) $\mathcal{J} \cap \mathcal{S}$ consists exactly of all equivalences.

(iii) The injections are monomorphisms and the surjections are epimorphisms.

(iv) $\mathcal{J}$ and $\mathcal{S}$ generate $\mathcal{K}$ in the following sense: Every morphism φ can be written uniquely (up to commuting equivalence) in the form $\varphi = \sigma\pi$ where $\sigma \in \mathcal{J}$ and $\pi \in \mathcal{S}$.

We ought also to add an axiom guaranteeing that the injections to a fixed object and surjections from a fixed object can be divided to a set (not a proper class) of equivalence classes, but this is not necessary here.

As a typical example let us consider the category of topological groups and continuous homomorphisms. We may define injections either as one-one homomorphisms into and surjections as open homomorphisms onto or injections as bicontinuous homomorphisms onto closed subgroups and surjections as homomorphisms onto dense subgroups (10).

1.4. A morphism $\varphi: A \to B$ is called a *retraction* (11) if there exists a *cross-section* $\psi: B \to A$ such that $\varphi\psi = \varepsilon_B$. This implies automatically that φ must be a surjection and ψ an injection. Indeed, let us write $\varphi = \varphi_i\varphi_s$ and $\psi = \psi_i\psi_s$ where φ_i and ψ_i are injections and φ_s and ψ_s are surjections; then $\varphi_i\varphi_s\psi_i\psi_s = \varepsilon_B$ whence φ_i and ψ_s are equivalences by a proposition of Isbell ([2], p. 116).

The notion of a retract does not depend on the way the category becomes a bicategory (12).

1.5. Let $\{A_t\}_{t \in T}$ be a set of objects. An object A is called a *free join* [a *direct join*] of $\{A_t\}$ if there exist monomorphisms $\sigma_t: A_t \to A$ [epimorphisms $\pi_t : A \to A_t$] with the following property: For every object B and every set of morphisms $\varphi_t: A_t \to B$ [morphisms $\varphi_t: B \to A_t$] there exists a unique morphism $\varphi: A \to B$ [$\varphi: B \to A$] such that $\varphi_t = \varphi\sigma_t$ [$\varphi_t = \pi_t\varphi$] for all $t \in T$. If a free [direct] join exists, it is unique and the

(10) Cf. § 14 - § 15. Isbell [5] discusses the question of realizations of the notions of surjection and injection as: 1° Mappings onto and embeddings, 2° Epimorphisms and polar monomorphisms, 3° Polar epimorphisms and monomorphisms, 4° Pure epimorphisms and copure monomorphisms.

(11) We do not assume that if B is a retract of A, then $B \subset A$; any object equivalent to a retract is a retract.

(12) In particular, if a category admits several definitions of a surjection-injection decomposition, then the retractions are surjections in the strongest sense.

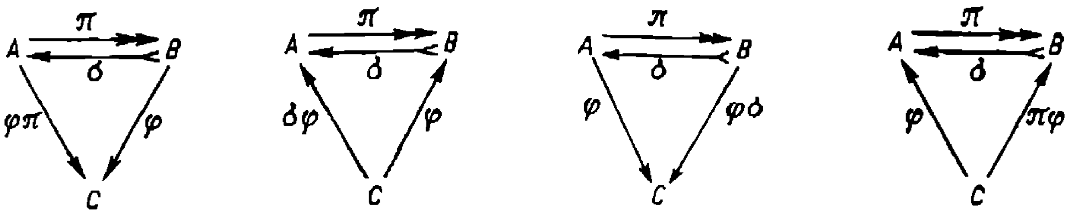
monomorphisms σ_t [epimorphisms π_t] are unique (up to commuting equivalence, cf. [KLS], § 12). We shall use the symbols $\sum A_t$ and $\prod A_t$ for free and direct join, respectively [13].

§ 2. Arrow notation and the duality principle

We shall use the following notation. Arbitrary morphisms will be denoted by $a: A \longrightarrow B$ (or $A \xrightarrow{a} B$), injections $a: A \rightarrowtail B$, surjections $a: A \twoheadrightarrow B$, equivalences $a: A \rightarrowtail\!\!\!\twoheadrightarrow B$, identities in diagrams $A = A$ (the only case when the direction does not matter). We shall use two notations for retractions: either $a: A \mapsto B$ (retraction of A onto B) or $A \underset{\sigma}{\overset{\pi}{\rightleftarrows}} B$ where π is a surjection and σ is an injection such that $\pi\sigma = \varepsilon_B$.

The symbol $a: A \to B \to C$ will mean that a is a morphism from A to C which is (or: can be) decomposed into two morphisms: one from A to B and the other from B to C.

Whenever a diagram is drawn, it is assumed that it commutes in the sense that any two morphisms from an object A to an object B of the diagram, which are compositions of morphisms of the diagram, coincide; we do not assume, however, that if $a_1: A \to A_2$, $a_2: A_2 \to A_3, \ldots,$ $a_n: A_n \to A_1$ are morphims of the diagram, then $a_n a_{n-1} \ldots a_2 a_1$ is necessarily the identity ε_{A_1} (e.g. in the case of a retraction and its cross-section where we have only partial commutativity ($\sigma\pi$ need not be the identity), but we assume that all other commutations hold. Thus, the first two of the following diagrams (where $\pi\sigma = \varepsilon_B$)



are admissible according to our convention but the other two need not be. Another convention is drawing $\dashrightarrow$ instead of $\longrightarrow$ ($\rightarrowtail$ dashed instead of $\rightarrowtail$, etc.); continuous arrows belong to the primary diagram or are products of primary maps while dotted arrows are constructed or assumed to exist in the next step of the reasoning. This convention enables us to leave the diagram without any comment as a proof of a simple theorem.

Any statement which can be expressed in primary terms of a bicat-

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[13] In any bicategory such that $\mathrm{Map}(A, B) \neq \emptyset$ for any two objects, all σ_t must be injections and all π_t must be surjections. The proof of this is essentially the same as the proof of 3.7. Σ and Π are also called *sum* and *product* in $\mathscr{K}$.

egory and any such diagram have their duals. Technically, in order to find the dual statement we use the following "dictionary" (cf. McLane [1]): $\alpha: A \longrightarrow B$ is to be replaced by $\alpha: B \longrightarrow A$, $\alpha\beta$ by $\beta\alpha$, $\alpha: A \twoheadrightarrow B$ by $\alpha: B \rightarrowtail A$ and conversely. Retraction is self-dual: $A \underset{\leftarrowtail}{\overset{\twoheadrightarrow}{\rightleftarrows}} B$ is replaced by $B \underset{\twoheadleftarrow}{\overset{\rightarrowtail}{\rightleftarrows}} A$, i.e. $A \mapsto B$ by $A \mapsto B$.

Definitions and theorems will be stated together with their duals (duals being in brackets) but dual proofs will be omitted.

§ 3. Singletons

An object S of a category $\mathscr{K}$ will be called a *singleton* [a *cosingleton*] if the following two conditions are satisfied ([14]):

(i) For every object A there exists at least one [exactly one] morphism $\alpha: S \to A$.

(ii) For every object B there exists exactly one [at least one] morphism $\beta: B \to S$.

An object is called a *null object* if it is a singleton and a cosingleton simultaneously (cf. [KLS], § 7).

3.1. *If S is a singleton or a cosingleton, then* Map(S, S) *consists exactly of one map* ε_S.

3.2. *All singletons are equivalent; all consingletons are equivalent; if the category contains a singleton and a cosingleton, they are equivalent, too.*

3.3. *If a concrete category has a singleton or a cosingleton, then the empty set cannot be an object (unless the category consists of only one object).*

We omit the proofs.

3.4. ***Let S be a singleton or a cosingleton in a bicategory $\mathscr{K}$. Then all morphisms $\alpha: A \longrightarrow S$ are surjections and all morphisms $\beta: S \longrightarrow A$ are injections.***

Proof. Let $\alpha: A \longrightarrow S$ and $\beta: S \longrightarrow A$ be any morphisms; then $\alpha\beta = \varepsilon_S$ (by 3.1). This means that any map of Map (S, A) is a cross-section of any map of Map (A, S). Hence 3.4 follows from 1.4.

Suppose that S is a singleton and denote by ν_A the only morphism in Map(A, S). By the axiom of choice we can select exactly one morphism ω_A of each set Map(S, A). Let us define: $\omega_{AB} = \omega_A \nu_B: B \to A$.

Now, suppose that S is a cosingleton and denote by ω_A the only morphism in Map(S, A) and by ν_A any morphism of Map(A, S) and $\omega_{AB} = \omega_A \nu_B$. In either case:

([14]) Isbell has called objects satisfying (ii) [or the dual of (i)] left-zero objects [right-zero objects, respectively]; cf. [5].

3.5. *For any objects A, B, C we have $\omega_{AB}\omega_{BC} = \omega_{AC}$.*

Proof. $\omega_{AB}\omega_{BC} = \omega_A \nu_B \omega_B \nu_C = \omega_A \varepsilon_S \nu_C = \omega_{AC}$.

3.6. *If the category has a singleton or a cosingleton and if A is a free [direct] join of $\{A_t\}_{t\in T}$, then all monomorphisms σ_t [epimorphisms π_t] admit retractions π_t [cross-sections σ_t] such that $\pi_t\sigma_t = \varepsilon_{A_t}$ and $\pi_t\sigma_u = \omega_{A_tA_u}$ for $t \neq u$. In particular, if the category is a bicategory, all σ_t are injections [π_t are surjections].*

Proof ([16]). Let A be a free join of $\{A_t\}$. Fix t and consider $\varphi_u\colon A_u \to A_t$ defined as follows: $\varphi_t = \varepsilon_{A_t}$ and $\varphi_u = \omega_{A_tA_u}$ for $u \neq t$. Then there exists a (unique) morphism $\varphi\colon A \to A_t$ such that $\varphi_u = \varphi\sigma_u$ for all $u \in T$. This means that $\pi_t = \varphi$ is the desired retraction.

§ 4. Projective and injective objects ([16])

An object P [object I] is said to be *projective* [*injective*] in $\mathscr{K}$ if for every $\alpha\colon A \twoheadrightarrow B$ and $\beta\colon P \longrightarrow B$ [$\alpha\colon B \rightarrowtail A$ and $\beta\colon B \longrightarrow I$] there exists a commuting morphism $\gamma\colon P \longrightarrow A$ [$\gamma\colon A \longrightarrow I$], i.e. if respectively

$$\begin{tikzcd} & P \arrow[dl, dashed, "\gamma"'] \arrow[dr, "\beta"] & \\ A \arrow[rr, two heads, "\alpha"'] & & B \end{tikzcd} \qquad \begin{tikzcd} & I & \\ A \arrow[ur, dashed, "\gamma"] & & B \arrow[ul, "\beta"'] \arrow[ll, tail, "\alpha"] \end{tikzcd}$$

4.1. *Let P be projective [I be injective] and let $\varrho\colon P \mapsto P_1$ [$\varrho\colon I \mapsto I_1$] be a retraction. Then P_1 is projective [I_1 is injective].*

Proof.

$$\begin{tikzcd} P \arrow[r, two heads, shift right, "\varrho"'] \arrow[d, dashed, "\gamma"'] \arrow[dr, "\beta\varrho"] & P_1 \arrow[l, tail, shift right, "\delta"'] \arrow[d, "\beta"] \arrow[dl, dashed, "\gamma\delta"] \\ A \arrow[r, two heads, "\alpha"'] & B \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, two heads, shift left, "\varrho"] & I_1 \arrow[l, tail, shift left, "\delta"] \\ A \arrow[u, dashed, "\gamma"] \arrow[ur, dashed, "\varrho\gamma"] & B \arrow[l, tail, "\alpha"] \arrow[u, "\beta"'] \arrow[ul, "\delta\beta"'] \end{tikzcd}$$

4.2. *If P is projective [I is injective] and if $\alpha\colon A \twoheadrightarrow P$ is any surjection [$\beta\colon I \rightarrowtail B$ is any injection], then α admits a cross-section α' [β admits a retraction β'].*

([15]) The proof is analogous to that in [KLS] in the case when S is a null object.

([16]) Propositions 4.1-4.3 are well known in many special cases and the proofs are essentially standard; we present them for the sake of completeness. Concerning the proofs, cf. the convention assumed in § 2.

Proof.

$$
\begin{tikzcd}
& P \arrow[dl, dashed, "\alpha'"'] \arrow[dr, equal] & \\
A \arrow[rr, two heads, "\alpha"'] & & P
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dr, equal] & \\
B \arrow[ur, dashed, "\beta'"] & & I \arrow[ll, tail, "\beta"]
\end{tikzcd}
$$

4.3. *The free [direct] join of any set of projective [injective] objects is projective [injective], if it exists.*

Proof. Let $P = \sum P_u$. Fix t and consider the diagram

$$
\begin{tikzcd}
P_t \arrow[r, "\sigma_t"] \arrow[d, dashed, "\gamma_t"'] \arrow[dr, "\beta\sigma_t"] & P \arrow[d, "\beta"] \\
A \arrow[r, two heads, "\alpha"'] & B
\end{tikzcd}
$$

for each index t. By the definition of a free join, there exists $\gamma: P \longrightarrow A$ such that $\gamma\sigma_t = \gamma_t$. Hence $\alpha\gamma\sigma_t = \alpha\gamma_t = \beta\sigma_t$; $\alpha\gamma: P \longrightarrow B$ has the property that $\alpha\gamma\sigma_t = \beta\sigma_t$ for all t, hence by the uniqueness of free-join extensions we get $\alpha\gamma = \beta$.

4.4. *Let $\mathscr{K}$ admits a singleton or a cosingleton. Then if a free [direct] join is projective [injective], each term must be projective [injective], too.*

Proof. It follows from 3.6 and 4.1.

4.5. *Singletons are injective [cosingletons are projective].*

4.6. *If $\mathscr{K}$ admits at least one projective [injective] object, then every singleton is projective [cosingleton is injective].*

4.5 is obvious, 4.6 follows from 4.1.

Now, let M be a fixed object and $A^* = \mathrm{Map}(A, M)$ for any object A. Any morphism $\alpha: A \to B$ induces the map $\alpha^*: B^* \to A^*$ defined by $\alpha^*(b^*) = b^*\alpha$ for $b^* \in B^*$; then $(\alpha\beta)^* = \beta^*\alpha^*$. The following statement yields an obviously equivalent and well-known definition of injectivity.

4.7. *M is injective if and only if the conjugate map $\alpha^*: B^* \to A^*$ of any injection $\alpha: A \to B$ is onto A^*.*

§ 5. Separators and generators

We shall say that $\mathrm{Map}(C, M)$ *separates* C [$\mathrm{Map}(M, C)$ *coseparates* C] if, for each object A and each pair of morphisms $\alpha: A \longrightarrow C$ and $\beta: A \longrightarrow C$ [$\alpha: C \longrightarrow A$ and $\beta: C \longrightarrow A$], the following condition holds: If $\gamma\alpha = \gamma\beta$ for all $\gamma \in \mathrm{Map}(C, M)$ [if $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathrm{Map}(M, C)$], then $\alpha = \beta$. Thus, if $\mathrm{Map}(C, M)$ separates C and if we consider the conjugate maps

$$\alpha^*: \mathrm{Map}(C, M) \longrightarrow \mathrm{Map}(A, M) \quad \text{and} \quad \beta^*: \mathrm{Map}(C, M) \longrightarrow \mathrm{Map}(A, M),$$

then $\alpha^* = \beta^*$ implies $\alpha = \beta$. Using diagrams we may say that $\mathrm{Map}(C, M)$ separates C if the following implication holds:

$$\text{If} \quad \begin{tikzcd} C \arrow[r, "\gamma"] & M \\ A \arrow[u, "\alpha"] \arrow[r, "\beta"'] & C \arrow[u, "\gamma"'] \end{tikzcd} \quad \text{for all } \gamma \in \mathrm{Map}(C, M), \text{ then} \quad \begin{tikzcd} C \arrow[r, "\gamma"] \arrow[dr, equal] & M \\ A \arrow[u, "\alpha"] \arrow[r, "\beta"'] & C \arrow[u, "\gamma"'] \end{tikzcd} .$$

An object M will be called a *separator* [*coseparator*] (in the given category) if $\mathrm{Map}(C, M)$ separates [$\mathrm{Map}(M, C)$ coseparates] each object C.

5.1. *Let M and B be objects of a concrete category $\mathcal{K}$ such that for each pair a, b of distinct elements of B there exists a morphism $\varphi: B \longrightarrow M$ such that $\varphi(a) \neq \varphi(b)$. Then* $\mathrm{Map}(B, M)$ *separates B.*

The proof is trivial. Now, an object M will be called [17] a *generator* [*cogenerator*] (in a given bicategory) if it has the following property: Given $\alpha: A \rightarrowtail B$ [given $\alpha: B \twoheadrightarrow A$], if for each $\beta: M \longrightarrow B$ [for each $\beta: B \longrightarrow M$] there exists $\gamma: M \longrightarrow A$ such that $\alpha\gamma = \beta$ [$\gamma: A \longrightarrow M$ such that $\gamma\alpha = \beta$], i.e. if respectively

$$\begin{tikzcd} B & M \arrow[l, "\beta"'] \arrow[dl, dashed, "\gamma"] \\ A \arrow[u, tail, "\alpha"] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, "\alpha"'] & M \\ A \arrow[ur, dashed, "\gamma"'] & \end{tikzcd}$$

(for all β), then α is an epimorphism [a monomorphism]. Analogously we define a *strict generator* [*strict cogenerator*] by the requirement that α be an equivalence; in other words, M is a strict generator [strict cogenerator] if the following condition holds: If $\alpha: B \rightarrowtail A$ and the induced map $\alpha': \mathrm{Map}(M, B) \longrightarrow \mathrm{Map}(M, A)$ is onto [if $\alpha: A \twoheadrightarrow B$ and $\alpha^*: \mathrm{Map}(B, M) \longrightarrow \mathrm{Map}(A, M)$ is onto], then α is an equivalence.

5.2. *Each coseparator is a generator* [*each separator is a cogenerator*].

Proof. Let M be a coseparator, let $\alpha: C \rightarrowtail A$ and let $\alpha': \mathrm{Map}(M, C) \longrightarrow \mathrm{Map}(M, A)$ be onto. We have to show that α is an epimorphism.

---

[17] The term "generator" has been used by Grothendieck ([3], p.134) in a different sense (coinciding with that assumed above whenever the bicategory has the inversion property).

Let $\beta_1: A \longrightarrow B$, $\beta_2: A \longrightarrow B$ and $\beta_1\alpha = \beta_2\alpha$. Then $\beta_1\alpha\gamma = \beta_2\alpha\gamma$ for each $\gamma \in \mathrm{Map}(M, C)$. Since α' is onto, $\beta_1\delta = \beta_2\delta$ for all $\delta \in \mathrm{Map}(M, A)$, whence $\beta_1 = \beta_2$. Thus α is right cancellable.

5.3. *If the bicategory has the inversion property, then every generator [cogenerator] is a strict generator [strict cogenerator].*

The last statement is obvious. An object M will be called a *strict separator* [*strict coseparator*] if it is a separator [coseparator] and a strict cogenerator [strict generator].

5.4. *Suppose that M is a strict generator [strict cogenerator] and the free [direct] join of any set of copies of M exists in $\mathscr{K}$. Then for each object B there exists a surjection $\varphi: \sum(M)_\iota \longrightarrow\!\!\!\!\rightarrow B$ [an injection $\varphi: B \rightarrowtail \prod(M)_\iota$].*

Proof [16]. Let $T = \mathrm{Map}(M, B)$ and $L = \sum\limits_{\gamma \in T}(M)_\gamma$. By the definition of a free join, there exists a morphism $\varphi: L \longrightarrow B$ such that $\varphi\sigma_\gamma = \gamma$ for each $\gamma \in T$, where σ_γ is a monomorphism $\sigma_\gamma: M \longrightarrow L$ corresponding to "γ-th coordinate", i.e. to the morphism $\gamma: (M)_\gamma \longrightarrow B$ which is at the same time an index and a morphism from γ-th copy of M. Then φ must be a surjection. Indeed, let $\varphi = \alpha\beta$ where $\alpha: A \rightarrowtail B$ and $\beta: L \twoheadrightarrow A$, then

$$\begin{tikzcd} L \arrow[r, two heads, "\beta"] & A \arrow[r, tail, "\alpha"] & B \\ & M \arrow[ul, "\delta_\gamma"] \arrow[u, dashed, "\beta\delta_\gamma"'] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

whence α is an equivalence (as M is a strict generator).

5.5. *If M is a coseparator [separator] and if there exists a surjection $L \twoheadrightarrow M$ [injection $M \rightarrowtail L$] then L is also a coseparator [separator]. Analogous statements are valid for strict coseparators and generators and their duals.*

The proof is trivial.

5.6. *If $\mathscr{K}$ has a non-zero object, then a singleton cannot be a separator [cosingleton cannot be a coseparator].*

Indeed, if A is not a cosingleton, then there are morphisms $\alpha: A \longrightarrow B$ and $\beta: A \longrightarrow B$ with $\alpha \neq \beta$ and $\gamma\alpha = \gamma\beta$ for any $\gamma: B \rightarrow S$.

---

[16] The idea of this proof (known in special cases) is the following: Definition of a strict generator means that if A is a subobject of B and A contains the images of all morphisms $\gamma: M \rightarrow B$, then $A = B$, i.e. that the union of all images generates B (cf. § 14). If we assume only that M is a generator, we conclude that φ is an epimorphism (cf. Grothendieck [3], p. 134). If M is strict, then the image of φ contains the images of all γ's.

5.7. *A non-null singleton need not be a coseparator.*

EXAMPLE. Let the objects be pairs (A, B) where A is a compact space and B is a group, and let morphisms be pairs (α, β) where $\alpha: A_1 \longrightarrow A_2$ is a continuous map and $\beta: B_1 \longrightarrow B_2$ is a homomorphism. Singletons are of the form $(S, 0)$ where S is a one-point set and 0 is a null group, and are not coseparators.

§ 6. Free and direct objects

An object M will be called a *basic free object* [*basic direct object*] if it is the smallest coseparator [separator]; more precisely, M is a basic free [direct] object if the following conditions are satisfied:

(i) *M is a coseparator* [*separator*].

(ii) *If M' is a retract of M and M' is a coseparator* [*separator*], *then M' is equivalent to M.*

(iii) *If M'' is a coseparator* [*separator*], *then M is a retract of M''.*

We shall use the abbreviations: b. f. o. [b. d. o.]. Evidently, such an object is unique up to equivalence, if it exists ([19]). This enables us to denote a selected representant by F [by D] and call it the b. f. o. [the b. d. o.]. Examples are discussed in Part II of the paper.

A b. f. o. [b. d. o.] will be called *strict* if it is a strict coseparator [strict separator].

Given a cardinal number $\mathfrak{m}$, let $F_\mathfrak{m}$ [$D_\mathfrak{m}$] be a free [direct] join of $\mathfrak{m}$ copies of F [of D]. $F_\mathfrak{m}$ [$D_\mathfrak{m}$] is determined by $\mathfrak{m}$ uniquely up to equivalence if it exists. An object will be called *free* [*direct*] if it is equivalent to some $F_\mathfrak{m}$ [$D_\mathfrak{m}$]. We shall say that $\mathscr{K}$ admits free [direct] objects if $F_\mathfrak{m}$ [$D_\mathfrak{m}$] exist for all cardinals ([20]).

6.1. (UNIVERSALITY THEOREM). *Suppose that $\mathscr{K}$ admits free* [*direct*] *objects and that F is strict and projective* [*D is strict and injective*]. *Then* ([21])

(i) *Every free* [*direct*] *object is projective* [*injective*].

(ii) *Every object is an image of a free object* [*a subobject of a direct object*], *i.e. for each A there exists a cardinal $\mathfrak{m}$ and a surjection $F_\mathfrak{m} \twoheadrightarrow A$* [*an injection $A \rightarrowtail D_\mathfrak{m}$*].

([19]) Let us notice that b. f. o. and b. d. o. are defined in terms of a category only, and they do not depend on the way the category becomes a bicategory.

([20]) Sometimes it is convenient to restrict the considerations to all cardinals smaller than a given one (e.g. to $\mathfrak{m} < \aleph_0$ or $\mathfrak{m} \leqslant \aleph_0$ in case of separable spaces, or to $\mathfrak{m} < \aleph_{\mathrm{in}}$ where $\aleph_{\mathrm{in}}$ is the first innaccessible cardinal); this yields obvious variants of the Universality Theorem.

([21]) Actually the theorem is valid if F is any projective strict coseparator [D is any injective strict separator], but it seems to be better to use the notions of a free and direct object; cf. 5.4, 6.2, and footnote ([8]).

(iii) *An object is projective [injective] if and only if it is a retract of a free [direct] object.*

(iv) *An object A is projective [injective] if and only if it is an absolute quotient retract [absolute subretract], i.e. if for each object B each surjection $\alpha: B \twoheadrightarrow A$ has a cross-section [each injection $\alpha: A \rightarrowtail B$ has a retraction].*

This follows from 4.1, 4.2, 4.3, and 5.4.

6.2. There is no obvious reason why the definition of the basic free and direct objects introduced in this section is the proper one. One might prefer the definition: A *basic free object* is an object with one free generator (i.e. with an element $a \in F$ such that for any object B and $b \in B$ there exists a unique morphism $\varphi: F \to B$ such that $\varphi(a) = b$) but this involves the notion of element of an object and is not acceptable here.

The following requirements concering b. f. o. and b. d. o. seem to be reasonable:

(1) B. f. o. and b. d. o. coincide with the examples mentioned in the introduction and some other natural ones, cf. Part II.

(2) B. f. o. and b. d. o. be defined in primary notions of bicategory.

(3) Definitions of b. d. o. and b. f. o. be mutually dual.

(4) B. f. o. and b. d. o. be unique (up to equivalence) whenever they exist.

(5) Every object be an image of the free join of a number of copies of b. f. o. and every object be a subobject of the direct join of a number of copies of b. d. o.

(6) B. f. o. be projective and b. d. o. be injective.

Thus, if somebody wanted to find another definition, his task would be to look for suitable properties of the examples we have in mind and choose some satisfying these conditions; in particular, the duality requirement makes us discard many of them. The definitions assumed in this section satisfy (2)-(5). Condition (6) is not assumed explicitly, but it is satisfied by almost all examples in Part II.

Unfortunately, the condition (1) is not satisfied in one fairly important case, namely for the category of left modules over a ring (see the remarks in the introduction and in 9.5).

Perhaps it is not possible to give a definition satisfying (1)-(5) according to which Λ is always a b. f. o. in the category of left Λ-modules. J. P. Jans has pointed out the following example: Let Λ be a finite-dimensional algebra over a field K, which is a quasi-Frobenius algebra but not a Frobenius one (cf. Nakayama [1], p. 624). Consider two contravariant functors: $M^T = \mathrm{Hom}_K(M, K)$ and $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$. Both $M \to M^T$ and $M \to M^*$ are one-one and map the class of finitely generated left Λ-modules onto the class of right ones and conversely;

moreover, both functors map the class of homomorphism of finitely generated Λ-modules onto that of right ones and are exact on finitely generated modules ($M \to M^T$ is always exact, $M \to M^*$ is exact if and only if Λ is quasi-Frobenius). Hence the composition $M \to (M^*)^T$ is a one-one exact covariant functor mapping the set of finitely generated left Λ-modules onto itself, though Λ is mapped onto a module which is not isomorphic to Λ. Thus, Λ cannot be characterized in terms of the category of finitely generated left Λ-modules.

Requirements (1)-(6) have some methodological meaning. We may also discuss the definition of a b. d. o. from the point of view of functional duality (in the sense of MacLane [1]) determined by an object M. The following statement gathers previous considerations and explains the role of asumptions of the Universality Theorem (note that if M is injective then M is a subobject of A if and only if it is a retract of A).

6.3. Given M, consider the contravariant functor $A \to A^* = \mathrm{Map}(A, M)$ and $a \to a^*$ (cf. 4.7). Then

(i) M is a separator if and only if the functor is one-one.

(ii) M is a strict cogenerator if and only if the condition "a^* is onto" implies "a is an injection".

(iii) M is injective if and only if the condition "a is an injection" implies "a^* is onto".

(iv) If a is an epimorphism, then a^* is one-one.

(v) If a^* is one-one and M is a separator, then a is an epimorphism.

II. SOME SPECIAL BICATEGORIES

§ 7. Table of examples

In the following table we have 12 examples of various bicategories (definitions of objects, morphisms, injections etc.). Such a parallel presentation points out analogies and differences. Of course, the table does not exhaust all special cases in which projective and injective objects have been investigated (explicitly or implicitly), but it contains bicategories in which these notions seem to be especially interesting.

If an empty place is left, this means that the answer is not known (this concerns, in particular, characterization of injective and projective objects if the only known characterization is that following from the Universality Theorem).

More detailed discussion, explanation of terms and symbols and some proofs are in § 8 - § 13.

Four columns (1, 2, 7 and 9) are devoted to various categories of topological spaces. Analogies between Columns 1 and 7 are justified by Stone's representation theorem. Columns 7 and 8 are mutually dual as well as Columns 9 and 10. Column 10 concerns a subcategory of the category considered in Column 11 and this, in turn, is a subcategory of that considered in Column 12.

Columns 3 - 6 form an independent group; Columns 5 and 6 concern the same category but different bicategories.

§ 8. Topological spaces

8.1. There are several natural categories of topological spaces; we shall consider only some full subcategories of the category of Hausdorff spaces and continuous maps. The simplest case occurs when we consider only discrete spaces and all possible maps; this category is presented in Column 1 of the table and does not need any comment.

8.2. In the category of completely regular spaces(22) free and direct joins coincide with disjoint unions and Cartesian products, respectively.

(22) This is the smallest category of topological spaces which contains an interval and is closed with respect to subsets and direct joins. For general references, see Kelley [2].

	1	2	3
objects	sets	completely regular topological spaces	groups
morphisms	arbitrary maps	continuous maps	homomorphisms
equivalences	one-one maps onto	homeomorphisms onto	one-one onto (isomorphisms)
inversion property	yes	no	yes
injections	one-one maps into	homeomorphisms into	one-one into
surjections	maps onto	onto	onto
singleton, cosingleton	singleton: one-point set	singleton: one point space	null object
free join	disjoint union	disjoint union	free product
direct join	Cartesian product	Cartesian product	Cartesian product
basic free object F	singleton	singleton	infinite cyclic group Z
basic direct object D	two-element set	closed interval	none
free objects	everyone	discrete spaces	free groups
direct objects	sets of power 2^{m}	cubes I^{m}	none
projective objects	everyone	the same as free	the same as free
injective objects	everyone	singletons only	null object only
Map(A, D)	space of subsets of A	$\{x \in C(A) : 0 < x < 1\}$	—

	4	5	6
objects	abelian groups	locally compact abelian groups	locally compact abelian groups
morphisms	homomorphisms	continuous homomorphisms	continuous homomorphisms
equiv-alences	one-one onto (isomorphisms)	bicontinuous onto	bicontinuous onto
inversion property	yes	no	no
injections	one-one into	bicontinuous into	one-one into
surjections	onto	onto a dense subset	open onto (quotient maps)
singleton, cosingle-ton	null object	null object	null object
free join	restricted direct sum	restricted direct sum if loc. compact	restricted direct sum if loc. compact
direct join	unrestricted direct sum	Cartesian product if locally compact	Cartesian product if locally compact
basic free object F	infinite cyclic group Z	infinite cyclic group Z	infinite cyclic group Z
basic direct object D	Q/Z (rationals modulo integers)	compact circle group T	compact circle group T
free objects	free abelian group	free abelian groups (discrete)	free abelian groups (discrete)
direct objects	"rational-toroidal" groups $\prod_t (Q/Z)_t$	toroidal groups (compact)	toroidal groups (compact)
projective objects	the same as free	null object only	$\sum (Z)_t \oplus R^n$ (n finite)
injective objects	divisible	$T^m \oplus R^n$ (n finite)	null object only
Map (A, D)	Hom $(A, Q/Z)$	the character group	the character group

	7	8	9
objects	0-dimensional compact spaces	Boolean algebras	compact Hausdorff spaces
morphisms	continuous maps	Boolean homomorphisms	continuous maps
equivalences	homeomorphisms onto	one-one onto	homeomorphisms onto
inversion property	yes	yes	yes
injections	homeomorphisms into	one-one into	homeomorphisms into
surjections	onto	onto	onto
singleton, cosingleton	singleton: one-point set	cosingleton: $(0, 1)$-algebra	singleton: one-point set
free join	Stone-Čech β of disjoint union	product [= sum]	Stone-Čech β of disjoint union
direct join	Cartesian product	direct union [= product]	Cartesian product
basic free object F	singleton	four-element $(0, 1, A, A')$-algebra	singleton
basic direct object D	two-point set	cosingleton	closed interval
free objects	βN_a or finite	algebras with free generators	βN_a or finite
direct objects	Cantor sets $2^{\mathfrak{m}}$	algebras of all subsets	cubes $I^{\mathfrak{m}}$
projective objects	extremally disconnected		extremally disconnected
injective objects		complete algebras	
$\operatorname{Map}(A, D)$	algebra of closed-open subsets	the Stone space of A	the convex set $\{x \in C(A): 0 \leqslant x \leqslant 1\}$

	10	11	12
objects	spaces $C(S)$ with S compact	normed linear spaces	normed linear spaces
morphisms	ring homomorphisms preserving units	linear operators with $\|T\| \leqslant 1$	linear operators with $\|T\| < \infty$
equiv-alences	ring isomorphisms onto	linear isometries onto	isomorphisms (in Banach's sense)
inversion property	yes	no	no
injections	one-one into	linear isometries into	isomorphisms into
surjections	onto	onto	onto
singleton, cosingle-ton	cosingleton: the real line	null object	null object
free join	$C(\Pi S_t)$	l-join	Cartesian product (for finite families only)
direct join	m-join	m-join	Cartesian product (for finite families only)
basic free object F	$C(J)$, J = interval	real line R	real line R
basic direct object D	cosingleton	real line R	real line R
free objects	$C(J^{\mathfrak{m}})$	$l(N_a)$ or l_n	finite dimensional spaces
direct objects	$m(N_a) = C(\beta N_a)$ or c_n	$m(N_a)$ or c_n	finite dimensional spaces
projective objects		the same as free	
injective objects	spaces $C(S)$ with S extremally discon.	spaces $C(S)$ with S extremally discon.	
Map (A, D)	the set of Gelfand homomorphisms	the unit cell of the conjugate space	the conjugate space

This category does not have the inversion property and there are several natural ways to define a surjection-injection factorization. For our purposes it will be the most convenient to assume injections to be homeomorphisms into and surjections to be maps onto. Then the closed interval I is a strict b. d. o. Indeed, I is a separator by 5.1, every retract of I is either a point or is homeomorphic to I and finally, if a set A is a separator, then there is a non-trivial continuous map from I into A; consequently, A contains a Peano curve and hence contains an arc as a retract (cf. Kuratowski [1], § 24a, VIa, p. 237 and [2], § 45. II, pp. 185-186 and § 48. II). I is strict because if $\varphi: A \to B$ is onto and $\varphi^*: \mathrm{Map}(B, I) \to \mathrm{Map}(A, I)$ is onto, then φ is one-one (by separation properties of continuous functions) and hence we may consider A and B as the same set with two completely regular topologies τ_1, τ_2 leading to the same space of real-valued bounded continuous functions, whence $\tau_1 = \tau_2$.

Thus the direct objects in this bicategory are Tikhonov cubes and the Universality Theorem yields Tikhonov's compactification theorem. A one-point space is a strict projective b. f. o. I is not injective because Tietze's theorem applies only to closed subsets and normal spaces ([23]).

8.3. An interesting example is the category of real-complete spaces ([24]) and continuous maps. A space E is called *real-complete* if it is completely regular and for every point $u \in \beta E \setminus E$ there exists an (unbounded) continuous function $f: E \to R$ that cannot be continuously extended to $E \cup (t)$ (in the topology induced by βE); an extensive exposition of the theory of these spaces can be found in the book by Gillman and Jerison [1]. Let us define injections as homeomorphic embeddings onto closed subsets and consequently surjections as maps onto dense subsets. Then the interval is a separator but not a strict separator (no compact space is a strict separator, because if we consider the Stone-Čech embedding $E \to \beta E$ as a surjection and take any function $\varphi: E \to K$, where K is compact, then φ can be extended to $\varphi_1 : \beta E \to K$).

The half-straight line $R_+ = \{t \in R: t \geqslant 0\}$ is a strict separator. Indeed, let φ be a map of A onto a dense subset of B and let $\varphi^*: \mathrm{Map}(B, R_+) \to \mathrm{Map}(A, R_+)$ be onto. Then φ must be one-one. Let $\varphi = \varphi_2\varphi_1$, where $\varphi_1: A \to \varphi(A)$ is one-one and $\varphi_2: \varphi(A) \to B$ is the embedding map (φ_1 and φ_2 need not be in the category, because $\varphi(A)$ need not be real complete; we consider them simply as continuous maps). Then $\varphi^* = \varphi_1^*\varphi_2^*$, whence

([23]) We may consider the bicategory of normal spaces with injections being homeomorphisms onto closed subsets. Then I is injective.

([24]) The term "real-complete" is due to M. Jerison; many other terms are in use, in particular "Q-spaces", "real-compact", "functionally closed" (cf. Gillman and Jerison [1]). We assume for simplicity that any cardinal number is non-measurable (cf. ibidem); according to recent results this assumption is much weaker than non-existence of an inaccessible aleph (Scott [1], Tarski [1]).

both φ_1^* and φ_2^* are onto. Thus φ_1 must be a homeomorphism and $\varphi(A)$ is real-complete and dense in the topology of B. If $\varphi(A) \neq B$, there would exist, by well-known properties of real-complete spaces, a continuous function $\alpha: \varphi(A) \to R_+$ without a continuous extension to the whole of B and this would mean that $\varphi^*[\mathrm{Map}(B, R_+)] \neq \mathrm{Map}(A, R_+)$; hence φ is a homeomorphism onto B. Thus, R_+ is a strict separator and any retract of R_+ which is a strict separator must be homeomorphic to R_+ ([25]).

If we apply 5.4 to this case, we get a well-known homeomorphism from a real-complete space onto a closed subset of the Cartesian product of a set of copies of R_+.

8.4. The category of metrisable and separable spaces admits countable free and direct joins only. If we define injections as bicontinuous embeddings and surjections as maps onto, then a one-point set is a strict b. f. o. and a closed interval I is a strict b. d. o. (strictness follows from the fact that if A is a dense subset of a separable metric space B and $t_0 \in B \setminus A$, then there exists a bounded continuous real-valued function on A with positive oscillation at t_0). Thus the modified Universality Theorem (with some countability restrictions) gives Urysohn's embedding into the Hilbert cube ([26]) and Borsuk's theorems on absolute retracts as retracts of the Hilbert cube. No characterization of absolute retracts is known (it is only known that in the full subcategory of finite-dimensional spaces absolute retracts are characterized as contractible and locally contractible compact spaces, cf. Borsuk [1], Kuratowski [2], p. 289, Dugundji [1]). The unit interval is not injective unless we define injections as mappings onto closed subsets. Then, however, surjections are maps onto dense subsets and discrete spaces are not projective.

The categories of compact spaces and 0-dimensional compact spaces are discussed separately in § 11.

§ 9. Groups. Abelian groups. Modules over a ring

9.1. The following categories will be considered in this section: $\mathscr{G}$—the category of groups and homomorphisms, $\mathscr{G}_Z$—that of abelian groups and, more generally, $\mathscr{G}_\Lambda$—the category of left modules over a ring Λ with a unit and Λ-homomorphisms. All of them have the inversion property and are bicategories in a unique natural sense.

The Cartesian product (= direct product, unrestricted direct product, unrestricted direct sum, complete direct product—according to various terminologies) is a direct join in each of these categories. Free

([25]) The question arises if R_+ is the smallest strict separator in the following sense: For any strict separator A there exists a retraction $\pi: A \to R_+$.

([26]) This is a homeomorphic embedding, not necessarily onto a closed subspace, though I is a strict b. d. o. in the second bicategory considered in 8.4 ($I^{\aleph_0}$ is too small in this case and if we want to consider $I^{\aleph_\tau}$, we have to consider non-metrisable spaces and I is no longer strict).

product is a free join in $\mathscr{G}$ and the direct sum (= restricted direct sum, weak direct product) is a free join in each $\mathscr{G}_A$.

9.2. The null group is the only injective object in $\mathscr{G}$ (Baer [2], Theorem 2), namely a group is a direct factor of every group containing it as a subgroup if and only if it consists of the unit only. At the same time, a group is a direct factor of every group containing it as a normal subgroup if and only if it is complete, i.e. without center and outer automorphisms (ibidem, Theorem 1; cf. also Kurosh [1], § 13). Some consequences of this theorem resemble the Universality Theorem, but we cannot say that complete groups are injective. There are several reasons for it: Firstly, we might want to make $\mathscr{G}$ a bicategory assuming injections to be embeddings of normal subgroups and surjections to be homomorphisms onto subgroups generating given group as the smallest normal subgroup, but then surjections would not be epimorphisms. J. R. Isbell has pointed out that the analgamated product method (cf. Kurosh [2], § 35) yields, for any subgroup G_0 of G, the existence of two homomorphisms $\varphi: G \to H$ and $\psi: G \to H$ such that $G_0 = \{x: \varphi(x) = \psi(x)\}$. Secondly, if H_1 is a normal subgroup of H_2 and H_2 is a normal subgroup of H_3, then H_1 need not be a normal subgroup of H_3; if H_1 is not normal in H_3, then the natural projection $\pi: H_2 \to H_2/H_1$ cannot be extended to the whole of H_3.

9.3. In either of the categories $\mathscr{G}$ and $\mathscr{G}_Z$, if an object is a coseparator, then it can be mapped onto Z and this homomorphism must have a cross-section. On the other hand, Z is a coseparator and has no non-trivial retracts, whence it is a b. f. o.

Consequently, free groups are free objects in $\mathscr{G}$ and free abelian groups are free objects in $\mathscr{G}_Z$. In both cases projective objects are the same as free because any subgroup of a free group is free (Nielsen-Schreier Theorem, cf. Kurosh [2], § 35 and § 36) and any subgroup of a free abelian group is free abelian (ibidem, § 19).

9.4. The discrete circle group T_d is an injective separator in $\mathscr{G}_Z$ but it is not the smallest one. We shall show that the group Q/Z of the rationals modulo integers is a b. d. o. in $\mathscr{G}_Z$.

It is known (cf. Baer [1], MacLane [1], Kurosh [1], § 23) that an abelian group is injective if and only if it is divisible (sometimes also called complete). It is true that not only direct but also free joins and quotients of injective abelian groups are injective (this is an example of statements which are valid in $\mathscr{G}_Z$ but not valid in other categories).

Divisible groups have also the following nice characterization (cf. Kurosh [1], § 23): Every divisible abelian group G is decomposable into direct sums of a set of groups isomorphic to Q and p^∞-groups (p^∞ denotes the multiplicative group of all roots of unity whose degrees are powers of p, p being a prime number) and any two such decompositions are essentially unique, i.e. the powers of the sets of copies of each group Q, $2^\infty, 3^\infty, 5^\infty, \ldots$ do not depend on the choice of decomposition.

Thus, the group $Q/Z = \sum p^\infty$ is injective. Further, let us notice the following two facts:

(i) Q/Z is a separator. Indeed, by 5.1 it is enough to show that if A is any abelian group, $a \in A$ and $a \neq 0$, then there exists a homomorphism $\alpha: A \to Q/Z$ such that $\alpha(a) \neq 0$. Let B be the cyclic group generated by a. If B is of finite order $p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$, then (by decomposition properties of cyclic groups) there exists

$$\alpha_0: B \rightarrowtail p_1^\infty \oplus \ldots \oplus p_n^\infty \rightarrowtail Q/Z.$$

The homomorphism α_0 can be extended to $\alpha: B \to Q/Z$ and $\alpha(a) = \alpha_0(a) \neq 0$. If B is infinite, then it is free and can be mapped into Q/Z so that a is mapped onto any given element.

(ii) No non-trivial retract of Q/Z is a separator. Indeed, any retract of Q/Z is of the form $\sum p_n^\infty$ where $p_1, p_2, \ldots$ is a finite or infinite sequence of distinct primes. Since every homomorphism of p_0^∞ into $\sum\limits_{p \neq p_0} p^\infty$ is the zero-homomorphism, the only retract of Q/Z which is a separator is Q/Z itself. The proof that every separator contains Q/Z is analogous (the homomorphic image of p^∞ is either zero or is isomorphic to p^∞); since Q/Z is injective, if $Q/Z \subset A$, then Q/Z is a retract of A.

9.5. The ring Λ is obviously a coseparator in $\mathscr{G}_\Lambda$ but it need not be the smallest one. I owe J. P. Jans the following example. Let Λ be the ring of all 2×2 matrices with real-valued entries. Then

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} + \begin{pmatrix} 0 & u \\ 0 & v \end{pmatrix}$$

yields the decomposition of Λ into two left ideals L_1 and L_2 which are mutually Λ-isomorphic and are projective b.f. objects in $\mathscr{G}_\Lambda$. Thus, L_1 is free according to the definition of § 6, but it is not free according to the standard terminology (a free left Λ-module is a free join of copies of Λ).

It is known that $Q = \mathrm{Hom}_Z(\Lambda, Q/Z)$ is a left Λ-module and is injective (cf. Cartan and Eilenberg [1], p. 30, Prop. 6.1a, Jans [1], Section 6). Straight-forward argument shows that it is a separator, but not necessarily the smallest one.

9.6. No group is a separator in $\mathscr{G}$. J. R. Isbell has pointed out the following example: Let G be any group and let H be an alternating group of power greater than that of G; then any homomorphism $h: H \to G$ is zero because H does not have any non-trivial normal subgroup (cf. Hall [1], p. 61). Thus $\mathscr{G}$ does not have any b. d. o.

§ 10. Locally compact abelian groups

10.1. Let $\mathscr{K}$ be the category of locally compact abelian groups and continuous homomorphisms. If $\{G_t\}$ is a family of such groups and all but a finite number are compact then the Cartesian product $\prod G_t$ is locally compact and is a direct join in $\mathscr{K}$. Similarly if all but a finite number of G_t are discrete, then the direct sum is locally compact and is a free join in $\mathscr{K}$; both correspond mutually in Pontryagin's duality ([27]).

10.2. We may consider $\mathscr{K}$ as a bicategory with two natural definitions of surjection-injection decomposition: 1° Injections are bicontinuous isomorphisms into (necessarily onto closed subgroups) and surjections are onto dense subgroups. 2° Injections are one-one continuous homomorphisms into and surjections are open continuous homomorphisms onto, i.e. quotient maps. Let us denote by $\mathscr{K}_1$ and $\mathscr{K}_2$ the bicategories obtained in this way. They correspond mutually in Pontryagin's duality (if $\alpha: G \to H$ is a $\mathscr{K}_1$-injection, then $\alpha^*: \hat{H} \to \hat{G}$ is a $\mathscr{K}_2$-surjection and so on).

Perhaps the best approach is to consider $\mathscr{K}$ as a tricategory (cf. § 15) with the canonical decomposition of $\alpha: G \to H$ as

$$G \xrightarrow[\text{quotient}]{} G/\mathrm{Ker}\,\alpha \xrightarrow[\text{bimorphism}]{} \overline{\mathrm{Im}\,\alpha} \xrightarrow[\text{embedding}]{} H$$

where $\mathrm{Ker}\,\alpha = \alpha^{-1}(0)$ and $\mathrm{Im}\,\alpha = \alpha(G) \subset H$.

10.3. The discrete cyclic group Z is a b. f. o. in $\mathscr{K}$ since it is a coseparator without non-trivial retracts and if H is a coseparator, then there exists a homomorphism φ of H onto Z (because every non-zero subgroup of Z is isomorphic to Z). If $\varphi: H \to Z$ is onto, then we can find a cross-section $\psi: Z \to H$ mapping the number 1 (generator of Z) onto any element of $\varphi^{-1}(1)$, whence φ is a retraction.

Hence, by Pontryagin's duality theorem, the compact circle group T is a b. d. o. in $\mathscr{K}$. This can be also checked immediately (every quotient of T is equivalent to T and every homomorphic image is a quotient). Thus the toroidal compact groups T^{m} are direct objects and the discrete free abelian groups (duals of toroidal groups) are free in $\mathscr{K}$.

10.4. The group T is injective in $\mathscr{K}_1$, but not injective in $\mathscr{K}_2$; Z is projective in $\mathscr{K}_2$, but it is not projective in $\mathscr{K}_1$. Moreover, T is a strict b. d. o. in $\mathscr{K}_2$ (by extension and separation properties of characters, cf. Hewitt and Ross [1], (24.12)) but it is not strict in $\mathscr{K}_1$; dually Z is strict in $\mathscr{K}_1$ but not strict in $\mathscr{K}_2$. Indeed, if T were strict in $\mathscr{K}_1$, every

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([27]) The question arises if this is the only case when a family $\{G_t\}$ has a direct join in $\mathscr{K}$. Cf. Hewitt and Ross [1], 6.4, 23.18 - 23.22.

locally compact abelian group would be bicontinuously isomorphic to a locally compact (hence closed and compact) subgroup of $T^{\mathfrak{m}}$. The following argument explains better this phenomenon: Let G be any locally compact non-compact abelian group; then the dual $\hat{G}$ is not discrete. Let $(\hat{G})_d$ be $\hat{G}$ equipped with the discrete topology and let bG be the compact dual of $(\hat{G})_d$. Then bG is the Bohr compactification of G (cf. ibidem (26.11)-(26.14)) and the natural embedding $\varkappa: G \to bG$ is one-one continuous onto a dense subgroup (whence a $\mathscr{K}_1$-surjection) and the dual map $\varkappa^*: (\hat{G})_d \to \hat{G}$ is one-one and onto though $\varkappa$ is not an equivalence ($\varkappa: G \to bG$ is not bicontinuous unless G is compact). Thus, the Universality Theorem gives only a one-one continuous embedding of G into $T^{\mathfrak{m}}$ ([20]). Cf. Hewitt and Ross [1], 25.30-25.36.

10.5. *Every injective object in $\mathscr{K}_1$ is of the form $T^{\mathfrak{m}} \oplus R^n$ where $\mathfrak{m}$ is any cardinal number and n is finite.* Indeed, R is injective (Hewitt and Ross [1], Theorem (24.36)), whence $T^{\mathfrak{m}} \oplus R^n$ is injective. In order to prove the converse statement let us assume that G is injective in $\mathscr{K}_1$ and consider three cases: 1° G is compact. Then G is a closed subgroup of $T^{\mathfrak{m}}$ and hence a retract of $T^{\mathfrak{m}}$, by 4.2. Thus the dual of G is discrete and is a subgroup of the free abelian dual $\sum(Z)_t$ of $T^{\mathfrak{m}}$, whence it is also free (cf. Kurosh [1], § 19) and consequently G is a toroidal group. 2° G is connected. Then G is compactly generated and can be represented in the form $G = G_1 \oplus R^n$ where G_1 is compact an connected and n is finite (cf. Hewitt and Ross [1], Theorem (9.14)); if G is injective, so is G_1, whence, by the case 1°, we get $G = T^{\mathfrak{m}} \oplus R^n$. 3° If G is not connected, it cannot be injective. Indeed, let H be the component of 0 in G; it is a closed subgroup. Suppose that $x \in G \setminus H$ and let L be the group generated by x. Let $a(1) = x$ determine a homomorphism $a: Z \to L$. If G were injective, there would exist an extension $\gamma: R \to G$ of a, the image $\gamma(R)$ would be a connected subgroup of G, whence the group generated by $\gamma(R)$ and H would also be connected, contradicting the definition of H.

§ 11. Boolean algebras. Compact spaces

11.1. The columns 7-10 of the table concern two dual pairs of closely related categories having the inversion property. Duality between columns 7 and 8 is established by Stone's representation theorem. It is clear that free joins of Boolean algebras ("products" according to Sikorski [2] and "sums" according to Halmos [1]) correspond to direct joins of 0-dimensional compact spaces (called also Boolean spaces) and direct joins of Boolean algebras ("direct unions" according to Sikorski

([20]) Glicksberg [1] has proved that if $a: G \to H$ is a one-one continuous homomorphism from G onto H then either a is bicontinuous or $a^*: H \to G$ is not onto; this is a kind of strictness of T but in much weaker form.

and "products" according to Halmos), correspond to free joins of the corresponding compact spaces. The fact that $\beta(\bigcup S_t)$ is a free join of $\{S_t\}$ follows from Stone's extension theorem (cf. Gillman and Jerison [1], p. 86) as $\beta(\bigcup S_t)$ is 0-dimensional if all S_t are 0-dimensional.

The Universality Theorem is known for Boolean algebras (cf. Sikorski [2], pp. 40, 44, 46 and Halmos [1], § 30-32). It yields, in particular, statements like: every 0-dimensional compact metric space is homeomorphic to a subset of $2^{\aleph_0}$, every Boolean algebra can be embedded into the algebra of all subsets of a set. Characterization of injective Boolean algebras is due to Sikorski [1]. No characterization of projective Boolean algebras is known. It is only known that every countably generated algebra is projective; this follows immediately from the theorem that any closed subset of $2^{\aleph_0}$ is its retract (Sierpiński [1], cf. also Kuratowski [1], p. 169).

Let us notice that Stone's duality between Boolean algebras and Boolean spaces is obtained in either direction by $A \to \mathrm{Map}(A, D)$ where D is a b. d. o.

11.2. Characterization of projective compact spaces is due to Gleason [1]; a simplified proof using methods of the Universality Theorem is due to Rainwater [1]. He calls βN_t a free compact space and characterizes extremally disconnected compact spaces as retracts of free compact spaces.

A closed interval I is a b. d. o. and the proof of this is analogous to that of 8.2; thus, injective spaces are retracts of cubes I^m, but the problem of internal topological characterization is open.

11.3. The free product in the category of spaces $C(S)$ and ring homomorphisms preserving units can be expressed by the duality as $C\big(\prod S_t\big)$ but no simple immediate construction is known (this perhaps is the reason why Banach's problem if $C(I)$ is isomorphic to its free square is yet unsolved, see Banach [1], p. 185). Direct products are just the m-products defined in next section. Characterization of (conditionally) lattice-complete spaces as spaces $C(S)$ with S extremally disconnected is due to Nakano [1], (cf. also Stone [2] and [3], Kantorovitch, Vulih and Pinsker [1], Chapter XIII) and they are precisely the injective spaces $C(S)$.

Let us point out, however, that a space $C(S)$ may be injective in the category of Banach spaces without being injective in the category of spaces $C(S)$ (see Amir [1], Isbell and Semadeni [1]).

The space $C(S)$ is not exactly $\mathrm{Map}(S, I)$, so the duality between Columns 9 and 10 is not obtained with a b. d. o., but we may easily get this if we consider the spaces $C^I(S) = \{x \in C(S): 0 \leqslant x \leqslant 1\}$ with restricted ring homomorphisms; then $C^I(F) = I$ where F is a one-point set, and the set $\mathrm{Map}\big(C^I(S), I\big)$ represents exactly the space S in the

Stone-Gelfand theory; the homomorphism equal identically to zero is excluded because all the homomorphisms of the category are assumed to satisfy $h(e) = e_1$, e and e_1 being the units of the corresponding rings.

§ 12. Banach spaces. Linear topological spaces

12.1. Banach spaces form two natural categories: If we admit all linear [29] operators as morphisms, we get a category in which two objects are equivalent if and only if there exists a linear bicontinuous operator from one onto the other. If we admit only linear operators with $\|T\| \leqslant 1$, then two objects are equivalent if and only if they are equivalent in Banach's sense, i.e. if there exists a linear isometry from one onto the other. We shall call the first of these categories the *category of Banach spaces* and the other the *category of Banach cells* (as it may be considered as the category of the unit cells of Banach spaces and linear operators from unit cells into unit cells).

If $\| \|_1$ is a coarser non-equivalent norm in $\langle X, \| \|\rangle$, then the map $a(x) = x$ from $\langle X, \| \|\rangle$ to $\langle X, \| \|_1\rangle$ is a bimorphism but not equivalence, so neither of the categories defined above has the inversion property. We have two standard surjection-injection factorizations of a linear map $a: X \to Y$, namely map onto a dense subset and embedding of a closed subspace or the quotient map and one-one map into. For our purposes, however, the most convenient is the simple onto-embedding factorization which requires considering a larger category in which such a factorization is possible, so in the sequel we shall consider the bicategory of normed linear spaces and that of the unit cells, with surjections being onto and injections being isomorphical or isometrical embeddings, respectively.

12.2. Given a family $\{X_t\}$ of normed linear spaces, the *m-product* [the *l-product*] is the space X of all functions $x = \{x_t\}_{t\in T}$ with $x_t \in X_t$ (i.e. X is a subset of the Cartesian product) such that

$$\|x\|_m = \sup_{t\in T} \|x_t\| < \infty, \qquad \|x\|_l = \sum_{t\in T} \|x_t\| < \infty,$$

respectively (in the second case all terms with except of a countable set are equal to zero).

By straight-forward argument we show that m-product is a direct join and l-product is a free join in the category of unit cells, but the category of normed linear spaces admits neither direct nor free join of any infinite set of objects. Indeed, in the definition of a free join we have

[29] Throughout this paper linear operators are assumed to be continuous.

$a\sigma_\iota = a_\iota$, whence $\|a_\iota\| = O(\|\sigma_\iota\|)$ is a necessary condition in order that such extension of the a_ι's be possible. Since $\{\|\sigma_\iota\|\}$ is fixed, we can always find $\{a_\iota\}$ such that the quotients $\|a_\iota\|/\|\sigma_\iota\|$ are unbounded. Nevertheless, some questions concerning the category of normed linear spaces can be reduced to that of unit cells because if $T \neq 0$ is any linear operator, $T/\|T\|$ is of norm 1.

12.3. The real line R is obviously a b. f. o. and b. d. o. simultaneously, moreover it is strict in both cases. Strictness of R as a b. f. o. is obvious, strictness of R as a b. d. o. can be shown as follows: Let $a: X \to Y$ be a linear map onto and let the dual map $a^*: Y^* \to X^*$ be onto. Then a must be one-one by separation properties of linear functionals and it must be an isomorphism (cf. Mazur and Orlicz [1], p. 138), whence an equivalence morphism in the category of normed linear spaces; an analogous argument works in the case of unit cells because then the map $a^*: \{y^* \in Y^*: \|y^*\| \leqslant 1\} \to \{x^* \in X^*: \|x^*\| \leqslant 1\}$ must be an isometry.

Let $l(N_\iota)$ and $m(N_\iota)$ be the space of respectively absolutely summable and bounded real functions on a set N_ι of power $\aleph_\iota$, i.e. a free and direct join of $\aleph_\iota$ copies of a b. f. o. and b. d. o.; we should call them a *free Banach space* and a *direct Banach space*, respectively.

12.4. The injective part of the Universality Theorem for normed linear spaces is due to Phillips [1]. Characterization of injective objects in the category of unit cells is due to Nachbin [1], Goodner [1], Kelley [1] and others, cf. also Day [1], Chapter V, § 4. Several characterizations are known; the most important is the following: an object is injective in the bicategory of unit cells if and only if it is equivalent to the space of bounded real-valued continuous functions on a compact extremally disconnected space S.

The class of injective objects in the bicategory of normed linear spaces is usually denoted by $\mathfrak{P}$. Such spaces must be retracts of direct Banach spaces, but no characterization of them is known. There are numerous papers on this subject and related questions on Boolean algebras and so on, for the bibliography of this see Nachbin [2], Isbell and Semadeni [1], Semadeni [3]. Let us notice that there exist injective Banach spaces which are not direct (cf. Pełczyński [1], p. 223).

12.5. The problem of lifting linear operators was considered by Grothendieck [2] and Nachbin [2]. No characterization of projective objects in the bicategory of normed linear spaces is known, but for the unit cells the situation is simple, namely every projective object is free in this case ([30]). Indeed, by the Universality Theorem it is enough to show that if X is a subspace of a space $Y = l(N_\iota)$ and there exists a projection

([30]) The projectivity of $l(N_\iota)$ is due to Köthe, the converse theorem to Grothendieck ([2], p. 558).

$\alpha: Y \to X$ with $\|\alpha\| = 1$, then X is also free, i.e. X is equivalent to a space $l(N_s)$. If $\alpha: Y \to X$ has norm 1, then $\alpha^*: X^* \to Y^*$ is an isometry admitting a retraction $\gamma: Y^* \to X^*$ of norm 1 as well, defined as the restriction of functionals of Y^* to the subspace X. Since $Y^* = m(N_t)$, X^* must be injective in the bicategory of unit cells whence, by a theorem of Grothendieck ([2], p. 554), X is an L-space. On the other hand, Y has Schur's property (i.e. weak and strong convergences are equivalent in Y, cf. Banach [1], p. 137) whence so does X. Since any L-space with Schur's property is equivalent to a space $l(N_s)$, X is free.

The fact that every Banach space is a quotient of a free Banach space is an immediate generalization of a theorem of Banach and Mazur ([1], p. 111) that every separable Banach space is a linear image of $l = l(N_0)$. Every separable projective Banach space is a retract of l, whence either it is finite dimensional or it is isomorphic to l (cf. Pełczyński [1], p. 213).

12.6. Let $\mathscr{L}$ be the category of locally convex (Hausdorff) linear topological spaces. The Cartesian product is a direct join in $\mathscr{L}$ and the direct sum ([31]) is a free join.

The real line R is a b. f. o. and a b. d. o. simultaneously, but it is not a strict b. d. o. (under the usual onto-into definition of the bicategorical decomposition) because different locally convex topologies may lead to the same class of linear functionals. R is injective and projective simultaneously.

R is a strict b. d. o. in the subcategory of those spaces $\langle X, \tau\rangle$ for which the topology τ is identical with the weak topology $\sigma(X, X^*)$ where X^* is the dual of $\langle X, \tau\rangle$; such spaces form pairs weakly dual in the sense of Dieudonné [1].

Free objects are the linear spaces with the finest topology (all additive operators being continuous) and direct objects are product of lines with the product topology (free and direct objects are mutually dual here).

Similar reasonings are valid for a locally convex space with a partial order determined by a cone (injectivity of R has been proved by M. Krein [1], cf. also H. Bauer [1]).

§ 13. Two-norm spaces and linear spaces with mixed topology

13.1. *A linear space with mixed topology* (abbreviation: MT-*space*) is a triple $\langle X, \|\ \|, \tau^*\rangle$ in which $\langle X, \|\ \|\rangle$ is a normed space and τ^* is a locally convex (Hausdorff) topology in X such that the topology of $\|\ \|$ is finer than τ^* (i.e. the identity map $\langle X, \|\ \|\rangle \to \langle X, \tau^*\rangle$ is continuous)

([31]) By the direct sum of $\{X_t\}$ we mean the space X consisting of all functions $x = \{x_t\}$, $x_t \in X_t$, $t \in T$, such that $\{t: x_t \neq 0\}$ is finite for each x; the sets $\{x: \Sigma\|x_t\|_{S_t} < 1\}$, where $\|\ \|_{S_t}$ are any pseudonorms in $\langle X_t, \tau_t\rangle$ form a basis of neighbourhoods in X.

and the function $a(x) = \|x\|$ is lower semicontinuous on $\langle X, \tau^*\rangle$, i.e. the cell $S = \{x \in X: \|x\| \leqslant 1\}$ is closed in $\langle X, \tau^*\rangle$. If τ^* is metrisable, then we can introduce a norm $\| \|^*$ in X which gives the same topology on S as τ^*; this yields a *γ-normal two-norm space* (cf. Alexiewicz and Semadeni [2] and [3]).

An operator φ from $\langle X, \| \|, \tau^*\rangle$ to a linear topological space $\langle Y, \tau\rangle$ is called *γ-linear* if it is additive and homogeneous and its restriction $\varphi \mid S$ is continuous as a map from $\langle S, \tau^*\rangle$ to $\langle Y, \tau\rangle$. An operator φ from $\langle X_1, \| \|_1, \tau_1^*\rangle$ to $\langle X_2, \| \|_2, \tau_2^*\rangle$ is called *γ-γ-linear* if it is linear as an operator from $\langle X_1, \| \|_1\rangle$ to $\langle X_2, \| \|_2\rangle$ and the restricted map $\varphi \mid S_1: \langle S_1, \tau_1^*\rangle \to \langle X_2, \tau_2^*\rangle$ is continuous, where $S_1 = \{x \in X_1: \|x\|_1 \leqslant 1\}$.

Let us denote by Ξ the space conjugate to $\langle X, \| \|\rangle$ with $\|\xi\| = \sup\{\xi(x): x \in S\}$, by Ξ^γ the class of all γ-linear functionals on $\langle X, \| \|, \tau^*\rangle$ and by Ξ^* the class of all linear functionals on $\langle X, \tau^*\rangle$. Obviously, $\Xi^* \subset \Xi^\gamma \subset \Xi$.

Let us recall the following theorems:

13.2. (Wiweger's Theorem). *There exists a unique locally convex topology τ_γ in X with the following properties:*

(i) *τ_γ is finer than τ^* and coarser than $\| \|$.*

(ii) *τ^* and τ_γ coincide on S.*

(iii) *For any locally convex space $\langle Y, \tau\rangle$, an additive operator $\varphi: X \to Y$ is γ-γ-continuous if and only if it is continuous from $\langle X, \tau_\gamma\rangle$ to $\langle Y, \tau\rangle$* (cf. Wiweger [1] and [2]).

13.3. *The dual space of $\langle X, \tau_\gamma\rangle$ coincide with Ξ^γ and the strong topology $\beta(\Xi^\gamma; \langle X, \tau_\gamma\rangle)$ is just the topology of the norm $\| \|$ induced by Ξ* (cf. Alexiewicz and Semadeni [3], p. 280).

13.4. *For every $x \in X$ the identity*

$$\|x\| = \sup\{\xi(x): \xi \in \Xi^*, \|\xi\| \leqslant 1\}$$

holds (cf. Alexiewicz and Semadeni [2], p. 124; this can be also deduced from some general theorems on representing lower semicontinuous convex functionals as upper envelopes of families of linear functionals).

13.5. (Grothendieck's Theorem) *Ξ^γ is the closure of Ξ^* in $\langle \Xi, \| \|\rangle$* (cf. Grothendieck [4], Bourbaki [1], Chap. IV, § 3, No. 4, Ex. 2, Alexiewicz and Semadeni [1] and [2], p. 130).

An MT-space X is called *γ-compact* if $\langle S, \tau^*\rangle$ is compact; X is called *γ-reflexive* if every linear functional $\mathfrak{z}$ on $\langle \Xi^\gamma, \| \|\rangle$ is of the form $\mathfrak{z}(\xi) = \xi(z)$ for some $z \in X$.

13.6. *Every γ-compact* MT-*space is γ-reflexive* (cf. Alexiewicz and Semadeni [3], p. 286).

Proof. Let $\mathfrak{z}$ be a linear functional on $\langle \Xi^\gamma, \|\ \|\rangle$, $\|\mathfrak{z}\| < 1$ and let $A(\xi_1, \ldots, \xi_n) = \bigcap_{k=1}^{n} \{x \in S: \xi_k(x) = \mathfrak{z}(\xi_k)\}$. By a theorem of Helly (cf. Day [1], p. 38), each set $A(\xi_1, \ldots, \xi_n)$ is non-empty and closed for any finite sequence $\xi_1, \ldots, \xi_n \in \Xi^\gamma$ (Helly's theorem may be applied to a Hahn-Banach extension of $\mathfrak{z}$ to Ξ). $\langle S, \tau^*\rangle$ being compact and $A(\xi_1, \ldots, \xi_n) = \bigcap_{k=1}^{n} A(\xi_k)$, the intersection of all $A(\xi_1, \ldots, \xi_n)$ contains an element z such that $\mathfrak{z}(\xi) = \xi(z)$ for all $\xi \in \Xi^\gamma$.

13.7. Given a set $\langle X_t, \|\ \|_t, \tau_t^*\rangle$, $t \in T$, of MT-spaces, *l_γ-join* of $\{X_t\}$ is the set X of all functions $x = \{x_t\}_{t\in T}$ such that $x_t \in X_t$ and

$$\|x\| = \sum_{t\in T} \|x_t\|_t < \infty,$$

with the topology τ^* determined by the system of neighborhoods of zero of the form

$$\Big\{x \in X: \sum_t \|x_t\|_{s_t}^* < 1\Big\}$$

where $\|\ \|_{s_t}^*$ is any pseudonorm of the topology τ_t^* such that $\|x_t\|_{s_t}^* \leqslant \|x_t\|_t$ for all $x_t \in X_t$. Let us show that S is τ^*-closed in X. Suppose that $y = \{y_t\} \notin S$, i.e. $\sum \|y_t\|_t = 1 + 2\delta$, $\delta > 0$. Then we can select a finite system $t_1, \ldots, t_n$ of indices such that $\|y_{t_1}\|_{t_1} + \ldots + \|y_{t_n}\|_{t_n} \geqslant 1 + \delta$. If y were the limit of a Moore-Smith sequence $(y^{(\sigma)})$ of elements of S, then y_{t_k} would be the limit of $(y_{t_k}^{(\sigma)})$ for $k = 1, \ldots, n$, whence

$$\varliminf_\sigma \|y^{(\sigma)}\| \geqslant \varliminf \sum_{k=1}^{n} \|y_{t_k}^{(\sigma)}\|_{t_k} \geqslant \sum_{k=1}^{n} \varliminf \|y_{t_k}^{(\sigma)}\|_{t_k} \geqslant \sum_{k=1}^{n} \|y_{t_k}\|_{t_k} \geqslant 1 + \delta.$$

The *m_γ-join* of $\{X_t\}$ is defined as the space of all bounded functions $x = \{x_t\}_{t\in T}$ with $\|x\| = \sup\{\|x_t\|_t: t \in T\}$ and the topology τ^* induced by the product topology of $\prod \langle X_t, \tau_t^*\rangle$.

13.8. *The m_γ-join of any family of γ-reflexive* MT-*spaces is γ-reflexive.*

Proof. Let $\langle X, \|\ \|, \tau^*\rangle$ be the m_γ-join of the family $\langle X_t, \|\ \|_t, \tau_t^*\rangle$, $t \in T$. Then Ξ^* is the free join of Ξ_t^* in the sense of 12.6, i.e. each $\xi \in \Xi^*$ is of the form $\xi = \sum_{t\in T} \xi_t$ where $\xi_t \in \Xi_t^*$ and the summation is actually finite. Further, $\|x\| = \sup \|x_t\|_t$ yields $\|\xi\| = \sum_t \|\xi_t\|_t$, whence the completion of $\langle \Xi^*, \|\ \|\rangle$ is the l-join of the completions of $\langle \Xi_t^*, \|\ \|_t\rangle$, i.e. $\langle \Xi^\gamma, \|\ \|\rangle$ is a free join of the spaces $\langle \Xi_t, \|\ \|_t\rangle$, in the category of Banach cells. Thus, any linear functional $\mathfrak{z}$ on $\langle \Xi, \|\ \|\rangle$ is determined by $\mathfrak{z}_t$, $t \in T$, where $\mathfrak{z}_t$ denotes the restriction of $\mathfrak{z}$ to Ξ_t. Since $\mathfrak{z}_t(\xi) = \xi(z_t)$ for $\xi \in \Xi_t^\gamma$ with some $z_t \in X_t$ and $\sup \|z_t\| = \sup \|\mathfrak{z}_t\| = \|\mathfrak{z}\| = \|z\|$, the function $z = \{z_t\}_{t\in T}$ belongs to X and $\mathfrak{z}(\xi) = \xi(z)$ for all $\xi \in \Xi^\gamma$.

In the sequel we shall consider the following categories: $\mathscr{A}$—that of all MT-spaces and γ-γ-linear maps, $\mathscr{A}_r$, $\mathscr{A}_c, \mathscr{A}_m$—full subcategories of γ-reflexive, γ-compact MT-spaces and of γ-normal two-norm spaces, respectively, $\mathscr{A}_{rm} = \mathscr{A}_r \cap \mathscr{A}_m$, etc. Further, $\mathscr{A}^1, \mathscr{A}^1_r, \ldots$ will denote the subcategories of $\mathscr{A}, \mathscr{A}_r, \ldots$ with the restriction that all γ-γ-linear maps in the question be of norm $\leqslant 1$, respectively. Injections in $\mathscr{A}, \mathscr{A}_r, \ldots$ will be γ-isomorphisms onto γ-closed subspaces (i.e. linear maps φ such that φ and $\varphi^{-1}: \varphi(X) \to X$ are γ-γ-linear); injections in $\mathscr{A}^1, \mathscr{A}^1_r, \ldots$ will be γ-isomorphisms onto γ-closed subspaces, which are isometries with respect to $\| \|$. (X_0 is γ-closed in X if $X_0 \cap S$ is closed in $\langle X, \tau^* \rangle$).

13.9. *The l_γ-join is a free join and the m_γ-join is a direct join in $\mathscr{A}^1$.*

We omit the straight-forward proof. Let us notice that for $\mathscr{A}$ we have the same difficulties as for Banach spaces.

13.10. *The real line R is not injective in $\mathscr{A}_m$.* In other words, there exists a normal two-norm space X and a γ-closed subspace X_0 of X such that some γ-linear functionals on X_0 cannot be extended to γ-linear functionals on X (Semadeni [1]).

13.11. *The real line is injective in $\mathscr{A}_r$.* Specifically, if X_0 is a γ-closed subspace of a γ-reflexive MT-space and ξ_0 is any γ-linear functional on X_0, then for every $\varepsilon > 0$ there exists a γ-linear fuctional ξ on X such that $\xi(x) = \xi_0(x)$ for $x \in X_0$ and $\|\xi\| \leqslant (1+\varepsilon)\|\xi_0\|$. The proof is quite analogous to that in the case when τ^* is metrisable (Semadeni [2], Theorem 6, cf. also Wiweger [2], Theorem 26.5 and [3]) ([32]).

13.12. *The real line is not injective in $\mathscr{A}^1_{cm}$.* Specifically, there exists a normal γ-compact two-norm space X and a γ-closed subspace X_0 of X such that certain γ-linear functionals do not admit any γ-linear norm-preserving extensions to the whole of X. In other words, the number ε is indispensable in 13.11.

Proof. Let $\langle X, \| \| \rangle$ be any separable Banach space with non-separable conjugate $\langle \Xi, \| \| \rangle$. There exists an equivalent norm $\| \|_1$ in X such that $\langle \Xi, \| \|_1 \rangle$ is rotund (i.e. strictly convex, cf. Klee [1], p. 56). There exists $\| \|^*$ such that $\langle X, \| \|_1, \| \|^* \rangle$ is normal and γ-precompact, whence $\langle \Xi^\gamma, \| \|_1 \rangle$ is separable and $\Xi^\gamma \neq \Xi$ (cf. Orlicz and Pták [1], p. 63-64). By a theorem of Bishop and Phelps [1], there exist $\xi_0 \in \Xi \setminus \Xi^\gamma$ and $x_0 \in X$ such that $\|\xi_0\| = \|x_0\| = \xi_0(x_0) = 1$. Let $\langle X_1, \| \|_1, \| \|^* \rangle$ be the γ-completion of $\langle X, \| \|_1, \| \|^* \rangle$; it is normal and γ-compact and ξ_0 has a unique norm-preserving extension to a γ-linear functional ξ_1 on $\langle X_1, \| \|_1, \| \|^* \rangle$ (cf. Alexiewicz and Semadeni [3], p. 282 and p. 288). Let X_0 be the one-dimensional subspace generated by x_0 and let η_0 be the restriction of

([32]) A generalization of 13.11 was published by A. Persson, *A generalization of two-norm spaces*, Arkiv för Mat. 5 (1963), p. 27-36.

ξ_1 to X_0. Obviously, η_0 is γ-linear (on X_0). Let η_1 be any norm preserving extension of η_0 to the whole of X_1 and let η be the restriction of η_1 to X. Then ξ_0 and η are two norm preserving extensions of η_0 to X. Since $\langle \Xi, \| \|_1 \rangle$ is rotund, all Hahn-Banach extensions in $\langle X, \| \|_1 \rangle$ are unique (cf. A. E. Taylor [1], Phelps [1]), whence $\xi_0 = \eta$ which prove that η and η_1 are not γ-linear.

Let $l(N_t)$ and $m(N_t)$ denote the l_γ-join and m_γ-join of $\aleph_t$ copies of R, respectively.

13.13. *$m(N_a)$ is injective in $\mathscr{A}_\gamma$.*

In the proof we cannot apply 4.3 because m_γ-join is not the direct join in $\mathscr{A}_\gamma$ (nor is R injective in $\mathscr{A}_\gamma^1$), but we may modify slightly the usual proof for Banach spaces because 13.11 gives a uniform estimation of the increase of the norm; details are omitted.

13.14. *R is a b. f. o. and a b. d. o. in all the categories considered in this section; it is a strict b. d. o. in $\mathscr{A}_c$ but it is not strict in $\mathscr{A}$.*

First and third of these statements are trivial; second is a consequence of the inversion theorem for γ-compact spaces (cf. Alexiewicz and Semadeni [3], p. 288).

13.15. *Each γ-compact* MT*-space is γ-isomorphic to a γ-closed subspace of a space $m(N_a)$. The space $m(N_0)$ is universal for normal γ-compact two-norm spaces.*

First statement follows from 3.14 and 5.4; the proof of the other is similar, see Semadeni [4].

13.16. *A γ-compact* MT*-space is injective in $\mathscr{A}_c$ if and only if it is a γ-retract of a space $m(N_a)$.*

13.17. *Let $\langle X_1, \| \|_1, \tau_1^* \rangle$ and $\langle X_2, \| \|_2, \tau_2^* \rangle$ be two γ-compact* MT*-spaces and let Ξ_1^γ and Ξ_2^γ be the corresponding sets of γ-linear functionals. Then every linear operator $T\colon \langle \Xi_1, \| \|_1 \rangle \to \langle \Xi_2, \| \|_2 \rangle$ is induced by a γ-γ-linear operator $U\colon X_2 \to X_1$.*

We omit the proofs.

13.18. *The categories $\mathscr{A}_c, \mathscr{A}_{mc}, \mathscr{A}_c^1, \mathscr{A}_{mc}^1$ are dual to the categories of all Banach spaces and of separable Banach spaces, with $\|T\| < \infty$ and $\|T\| \leqslant 1$, respectively. This duality is obtained by $\langle X, \| \|, \tau^* \rangle \to \langle \Xi^\gamma, \| \| \rangle$ is one direction, and by $\langle X, \| \| \rangle \to \langle \Xi, \| \|, \sigma(\Xi, X) \rangle$ in the other.*

This follows from our previous considerations. Indeed, if $\langle \Xi, \| \| \rangle$ is the conjugate of $\langle X, \| \| \rangle$ and $\mathfrak{z}$ is a linear functional on $\langle \Xi, \| \| \rangle$ which is continuous on $\Sigma = \{\xi \in \Xi\colon \|\xi\| \leqslant 1\}$ with respect to $\sigma(\Xi, X)$ and if $H = \{\xi \in \Xi\colon \mathfrak{z}(\xi) = 0\}$, then $H \cap \Sigma$ is closed in $\langle \Xi, \sigma(\Xi, X) \rangle$, whence H is closed in $\langle \Xi, \sigma(\Xi, X) \rangle$ (cf. Day [1], p. 43) and $\mathfrak{z}(\xi) = \xi(z)$ for some $z \in X$. Thus, all theorems about categories of Banach spaces have their duals valid for γ-compact MT-spaces.

APPENDIX

§ 14. Remarks on subobjects and injections

14.1. The purpose of this section is to discuss the following question: Given a concrete category $\mathscr{K}$, how to define injections and surjections in order to get a bicategory. The class of surjections is determined uniquely by the class of injections, so we have only to discuss injections. Natural definition of an injection $A \rightarrowtail B$ is an equivalence $A \rightarrowtail\!\!\twoheadrightarrow A_1$ onto a subobject A_1 of B composed with the embedding of A_1 into B. This means that we should precise the notion of a subobject; in general, this notion may be optionable. E.g. in the category of topological groups a subobject may mean any subgroup or any closed subgroup.

The following definition seems to be reasonable (though it is not the only acceptable). B is called a *subobject* of A in a concrete category $\mathscr{K}$ over $\mathscr{A}$ if the following conditions are satisfied:

(i) $B \subset A$, $A \in \mathscr{A}$, $B \in \mathscr{A}$.

(ii) The inclusion embedding $\varepsilon_{AB}\colon B \to A$ belongs to $\mathscr{K}$.

(iii) If $\varphi\colon C \to A$ belongs to $\mathscr{K}$ and $\varphi(C) \subset B$, then $\varphi\colon C \to B$ also belongs to $\mathscr{K}$.

(iv) If $x \in A \setminus B$, then there existan object C and morphisms $\alpha\colon A \to C$ and $\beta\colon A \to C$ such that $\alpha(u) = \beta(u)$ for $u \in B$ and $\alpha(x) \neq \beta(x)$.

14.2. The last condition may seem to be too restrictive. It eliminates e.g. the non-closed subsets in the category of topological spaces. The following example, however, shows that conditions (i)-(iii) yield subobjects that we do not want to consider as subobjects ([33]).

Consider the category of Banach spaces and linear operators and spaces L_p with $\|x\|_p = (\int_0^1 |x(t)|^p dt)^{1/p}$, $p \geqslant 1$. The embedding $\langle L_2, \| \|_2\rangle \to \langle L_1, \| \|_1\rangle$ is continuous and (i) and (ii) are obvious for $A = L_1$ and

([33]) There is a natural embedding of L_2 into L_1 based on the fact that the condition $\|x\|_2 < \infty$ implies $\|x\|_1 < \infty$, whence $L_2 \subset L_1$, but L_2 is not a subspace of L_1, because not only L_2 is not closed in L_1 but also the topology induced in L_2 by L_1 does not coincide with that of L_2.

We can find, however, another embedding of L_2 onto a closed subspace of L_1 (applying Rademacher functions and Khinchin inequality, cf. Kaczmarz and Steinhaus [1], p. 132) and in this sense L_2 is a subspace of L_1.

$B = L_2$. In order to prove (iii), let us consider any linear operator α from a Banach space $\langle X, \|\ \|\rangle$ to $\langle L_1, \|\ \|_1\rangle$ with values actually in L_2. We have to prove that α is continuous as operator from $\langle X, \|\ \|\rangle$ to $\langle L_2, \|\ \|_2\rangle$. Let $Y = \alpha^{-1}(0)$ be the kernel of α. $Z = X/Y$ is a Banach space and α induces a linear operator $\gamma: \langle Z, \|\ \|_Z\rangle \to \langle L_1, \|\ \|_1\rangle$ defined by $\gamma(x+Y) = \alpha(x)$; γ is one-one and continuous map onto $\alpha(X) = E$. Define, for $f \in E$,

$$\|f\|_Z = \|\gamma^{-1}(f)\|_Z = \inf\{\|x\|: \alpha(x) = f\}$$

and $\|f\|^* = \|f\|_Z + \|f\|_2$. Both norms $\|\ \|_Z$ and $\|\ \|_2$ are stronger than $\|\ \|_1$ on E and $\|\ \|_Z$ is complete on E. We shall show that $\langle E, \|\ \|^*\rangle$ is also complete. Let $\|f_p - f_q\|^* \to 0$ as $p, q \to \infty$; then $\|f_p - f_q\|_Z \to 0$, whence $\|f_p - f'\|_Z \to 0$ for some $f' \in E$. On the other hand, $\|f_p - f_q\|_2 \to 0$, whence $\|f_p - f''\|_2 \to 0$ as $p \to \infty$. Since $\|f_p - f'\|_1 \to 0$ and $\|f_p - f''\|_1 \to 0$, we infer $f' = f''$ and $\|f_p - f'\|^* \to 0$.

Thus, both $\langle E, \|\ \|_Z\rangle$ and $\langle E, \|\ \|^*\rangle$ being complete, by Banach's inversion theorem both norms are equivalent, whence $\gamma: Z \to \langle E, \|\ \|_2\rangle$ is continuous and $\alpha: X \to \langle E, \|\ \|_2\rangle$ is continuous as well.

14.3. We shall assume that $\mathscr{K}$ satisfies the following conditions:

(i) If B is a subobject of A and C is a subobject B, then C is a subobject of A.

(ii) If B is a subobject of A and $\varphi: A \to C$ is an equivalence, then $\varphi(B)$ is a subobject of C.

(iii) The intersection of any family of subobjects is either a subobject or it is empty.

(iv) If $\varphi: A \to B$ and $\psi: A \to B$ are any morphisms in $\mathscr{K}$, then $\{t \in A: \varphi(t) = \psi(t)\}$ is a subobject of A or is empty.

If W is any subset of an object A, then the intersection $M_A(W)$ of all subobjects of A containing W will be called the *subobject generated by* W. Of course, $M_A(M_A(W)) = M_A(W)$.

14.4. *If W generates an object A and if two morphisms $\varphi: A \to B$ and $\psi: A \to B$ coincide on W, then $\varphi = \psi$.*

This follows from condition (iv) of 14.3.

14.5. *If $\varphi: A \to B$ is a morphism and W is any subset of A, then $\varphi(M_A(W)) \subset M_B[\varphi(W)]$.*

Proof. Let $a \in M_A(W)$ and $b = \varphi(a)$. If b were not in $M_B[\varphi(W)]$, there would exist an object C and morphisms $\beta_1 : B \to C$ and $\beta_2: B \to C$ such that $\beta_1(u) = \beta_2(u)$ for $u \in \varphi(W)$ and $\beta_1(b) \neq \beta_2(b)$. Let $\alpha_1 = \beta_1\varphi$ and $\alpha_2 = \beta_2\varphi$. Then $\alpha_1(x) = \alpha_2(x)$ for $x \in W$ and $\alpha_1(a) \neq \alpha_2(a)$ contradicting 14.4.

14.6 Suppose that $\mathscr{K}$ is a category over $\mathscr{A}$ and the conditions of 14.3 are satisfied. Let us call a morphism $\alpha: B \to A$ an *injection* if it is of the form $\alpha = \varepsilon_{AC}\beta$ where $\beta: B \to C$ is an equivalence and $\varepsilon_{AC}: C \to A$ is the embedding map of an subobject C of A; then a morphism $\delta: B \to A$ is a surjection if $M_A[\delta(B)] = A$. Then $\mathscr{K}$ is a bicategory in virtue of 14.4 and 14.5.

14.7. Suppose that $\mathscr{K}$ is a category and $\mathscr{B}_1$ and $\mathscr{B}_2$ are two bicategories obtained from $\mathscr{K}$ (i.e. $\mathscr{B}_1$ and $\mathscr{B}_2$ have the same objects and morphisms but perhaps different surjections and injections). We shall write $\mathscr{B}_1 \prec \mathscr{B}_2$ if for any two objects A and B and any morphism $\alpha: A \to B$ the following condition is satisfied: If $\alpha = \beta_1\pi_1$ and $\alpha = \beta_2\pi_2$ are two surjection-injection decompositions in $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively, then there exists a bimorphism φ from the range of π_2 to the range of π_1 such that $\beta_1\varphi = \beta_2$ and $\varphi\pi_2 = \pi_1$, i.e.

$$\begin{tikzcd} & C_1 \arrow[dr, "\beta_1"] & \\ A \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & B \\ & C_2 \arrow[uu, "\varphi"] \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

It is easily seen that $\prec$ determines a partial order in the family of all bicategories obtained from $\mathscr{K}$.

14.8. *The bicategory $\mathscr{B}$ defined in* 14.6 *is smaller with respect to $\prec$ than any other bicategory obtained from $\mathscr{K}$.*

Proof. Suppose that $\alpha: A \to B$ is a morphism in $\mathscr{K}$ and $C = M_B[\alpha(A)]$; then the decomposition of α in $\mathscr{B}$ is $\alpha = \beta\pi$ where $\pi: A \to C$ is defined as $\pi(x) = \alpha(x)$ for $x \in A$ and $\beta = \varepsilon_{BC}$. Suppose that $\alpha = \beta_1\pi_1$ is a decomposition in another bicategory $\mathscr{B}_1$, where $\beta_1: C_1 \to B$ and $\pi_1: A \to C_1$. First we shall prove that $\beta_1(C_1) \subset C$. Let $x \notin C$; then, by the condition (iv) of 14.1, there exist morphisms $\delta_1: B \to D$ and $\delta_2: B \to D$ such that $\delta_1(u) = \delta_2(u)$ for $u \in C$ and $\delta_1(x) \neq \delta_2(x)$. Hence $\delta_1\alpha = \delta_2\alpha$ and $\delta_1\beta_1\pi_1 = \delta_2\beta_1\pi_1$. Since π_1 is an epimorphism, $\delta_1\beta_1 = \delta_2\beta_1$. If there existed $y \in C_1$ such that $x = \beta_1(y)$, we should have a contradiction $\delta_1(x) = \delta_1\beta_1(y) = \delta_2\beta_1(y) = \delta_2(x)$ with $\delta_1(x) \neq \delta_2(x)$. Define $\beta_2: C_1 \to C$ by the formula $\beta_2(c) = \beta_1(c)$ for $c \in C_1$. We have to show that β_2 is a bimorphism. Since $\beta_2(C_1) = \beta_1(C_1) \supset \alpha(A)$ and $C = M_B(\alpha(A))$, β_2 is an epimorphism. To show that it is a monomorphism, consider any morphisms $\gamma_1: E \to C_1$ and $\gamma_2: E \to C_1$ such that $\beta_2\gamma_1 = \beta_2\gamma_2$. Then $\beta_1\gamma_1 = \varepsilon_{BC}\beta_2\gamma_1 = \varepsilon_{BC}\beta_2\gamma_2 = \beta_1\gamma_2$, whence $\gamma_1 = \gamma_2$ because β_1 is left cancellable.

14.9. *If $\mathscr{B}_1 \prec \mathscr{B}_2$ and Q is injective in $\mathscr{B}_2$, then Q is injective in $\mathscr{B}_1$.*

Proof. Given $\alpha: C_1 \to Q$ and a $\mathscr{B}_1$-injection $\beta_1: C_1 \rightarrowtail B$, we can find a factorization $\beta_1 = \beta_2\pi_2$ in $\mathscr{B}_2$ and a bimorphism $\varphi: C_2 \to C_1$ such that $\beta_1\varphi = \beta_2$. Q being injective in $\mathscr{B}_2$, there exists $\gamma: B \to Q$ such that $\gamma\beta_2 = \alpha\varphi$, whence $\alpha\varphi = \gamma\beta_2 = \gamma\beta_1\varphi$ and $\alpha = \gamma\beta_1$ as φ is right cancellable:

$$\begin{tikzcd}
 & C_2 \arrow[r, tail, dashed, "\beta_2"] \arrow[d, dashed, "\varphi"'] \arrow[dr, dashed, "\alpha\varphi"] & B \arrow[d, dashed, "\gamma"] \\
C_1 \arrow[ur, dashed, "\pi_2"] \arrow[r, equal] & C_1 \arrow[ur, "\beta_1"] \arrow[r, "\alpha"'] & Q
\end{tikzcd}$$

§ 15. Tricategories

15.1. We shall give here an axiomatic definition of a tricategory([34]). A *tricategory* is a class $\mathscr{T}$ which is the union of four classes: $\mathscr{K}$ (class of *morphisms)*, $\mathscr{C}$ *(law of composition* of morphisms), $\mathscr{I}$ (class of *injections*) $\mathscr{S}$ (class of *surjections*) satisfying the following axioms:

(A$_0$) $\mathscr{I} \subset \mathscr{K}$, $\mathscr{S} \subset \mathscr{K}$, $\mathscr{K} \cap \mathscr{C} = \emptyset$, $\mathscr{C} \subset \mathscr{K} \times \mathscr{K} \times \mathscr{K}$.

(A$_1$) *For every $\alpha \in \mathscr{K}$ and $\beta \in \mathscr{K}$ there is at most one $\gamma \in \mathscr{K}$ such that (α, β, γ) belongs to $\mathscr{C}$; such α is designated by $\beta\alpha$, if it exists.*

(A$_2$) *If $\alpha\beta$ and $\beta\gamma$ exist, then $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ exist and are equal; if $(\alpha\beta)\gamma$ exists, so does $\beta\gamma$; if $\alpha(\beta\gamma)$ exists, so does $\alpha\beta$.*

A morphism ε is called an *identity* if it satisfies the following conditions: If $\psi \in \mathscr{K}$ and $\varepsilon\psi$ exists, then $\varepsilon\psi = \psi$; if $\varphi \in \mathscr{K}$ and $\varphi\varepsilon$ exists, then $\varphi\varepsilon = \varphi$; $\varepsilon\varepsilon$ exists.

(A$_3$) *For each $\varphi \in K$ there exist identities ε_1 and ε_2 such that $\varphi\varepsilon_1$ and $\varepsilon_2\varphi$ exist.*

The identities ε_1 and ε_2 in (A$_3$) have been shown to be unique. They will be called the *domain* and the *range* of φ and denoted by $\delta(\varphi)$ and $\varrho(\varphi)$, respectively. The terms *equivalence, epimorphism, monomorphism* and *bimorphism* are defined as in § 1.

(A$_4$) *For each pair ε_1 and ε_2 of identities the class of all φ such that $\delta(\varphi) = \varepsilon_1$ and $\varrho(\varphi) = \varepsilon_2$ is a set* ([35]).

This set will be denoted by $\mathrm{Map}(\varepsilon_1, \varepsilon_2)$.

(A$_5$) *Every surjection is an epimorphism; every injection is a monomorphism.*

(A$_6$) *The composition of two surjections is a surjection and the composition of two injections is an injection (if they exist).*

([34]) This term has been used by Isbell [2], p. 121 in his remarks on categories of topological spaces. Cf. also Sonner [1], Proposition 10.

([35]) It is possible to construct an example showing that (A$_4$) is independent of (A$_0$)-(A$_3$).

(A_7) *Every morphism φ can be uniquely written in the form $\varphi = \alpha\beta\gamma$ where γ is a surjection, β a bimorphism and α is an injection.*

(A_8) *For every identity ε, the class of all surjections φ such that $\delta(\varphi) = \varepsilon$ and that of all injections ψ such that $\varrho(\psi) = \varepsilon$ are sets* ([36]).

15.2. We may also introduce an additional axiom as follows. An *ortho-category* is the bicategory $\mathscr{B}_{\mathrm{orth}}$ obtained from $\mathscr{T}$ by defining *ortho-surjections* as compositions of surjections and bimorphisms (i.e. those maps which have an identity as the third factor in the canonical decomposition of axiom (A_7)) and *ortho-injections* as injections of $\mathscr{T}$; a *para-category* is the bicategory $\mathscr{B}_{\mathrm{para}}$ obtained from $\mathscr{T}$ by defining *para-surjections* as surjections of $\mathscr{T}$ and *para-injections* as compositions of bimorphisms and injections of $\mathscr{T}$. We may assume

(A_9) $\mathscr{B}_{\mathrm{orth}}$ *is the smallest and* $\mathscr{B}_{\mathrm{para}}$ *is the largest in the sense of* 14.7.

15.3. As a typical example let us consider the decomposition stated in 10.2; then in Column 5 of the table we have an ortho-category and in Column 6 we have a para-category. This enables us to use such terms like "the ortho-category of locally compact abelian groups" and so on.

The notions ortho-para are obviously dual, but the following example shows that this duality must be understood in "axiomatic" sense. Consider the category of Banach spaces. If $\alpha: X \to Y$ is an isomorphism (onto a closed subspace), then $\alpha^*: Y^* \to X^*$ is a quotient map. If $\beta: X \to Y$ is onto dense subset, then $\beta^*: Y^* \to X^*$ is one-one, hence "ortho" goes to "para". Now, if $\alpha: X \to Y$ is a quotient map, then $\alpha^*: Y^* \to X^*$ is an isomorphism but if $\beta: X \to Y$ is one-one, $\beta^*: Y^* \to X^*$ need not be onto a dense subset, e.g. consider $\beta(x) = x$ as a map from L_∞ to L_1.

15.4. *Let $\mathscr{T}$ be a tricategory. Then*

(i) *A morphism φ is an ortho-injection [ortho-surjection, para-injection, para-surjection] if and only if every ortho-surjection β [ortho-injection α, para-surjection β, para-injection α, respectively] such that $\varphi = \alpha\beta$, is an equivalence.*

(ii) *If $\alpha\beta$ is an equivalence, then α is a para-surjection and β is an ortho-injection.*

([36]) Axioms (A_7) and (A_8) follow MacLane's concept of a bicategory and do not agree with axiom (iv) in 1.3 that follows Isbell's treatment in which any object C equivalent to a subobject B of an object A is also considered as a subobject of A. Axioms (A_7) and (A_8) mean that this identification is abandoned. E. g. the category of all sets and all maps forms a tricategory if we define the canonical decomposition of $\varphi: A \to B$ as follows: $\varphi = \alpha\beta\gamma$ where α is the embedding $\varepsilon_{B,\varphi(A)}: \varphi(A) \to B$, γ is the quotient map of A onto $C = \{\varphi^{-1}(b) : b \in \varphi(A)\}$ and β is the natural map from C onto $\varphi(A)$.

Proof. (i) is a special case of a proposition of Isbell ([2], p. 116). In order to show (ii), let us assume that $\alpha\beta$ is an equivalence and $\alpha = \alpha_3\alpha_2\alpha_1$ and $\beta = \beta_3\beta_2\beta_1$ are the decompositions according to (A_7). Since $\alpha\beta$ is an ortho-injection and $\beta_2\beta_1$ is an ortho-surjection, (i) can be applied to $\alpha\beta = (\alpha_3\alpha_2\alpha_1\beta_3)(\beta_2\beta_1)$ and $\beta_2\beta_1$ must be an equivalence which means that β is an ortho-injection. Similar argument shows that $\alpha_3\alpha_2$ is an equivalence, too; compare ([12]).

15.5. *If an injection is an epimorphism [a surjection is a monomorphism], then it is an identity.*

Proof. Let us identify identities with objects; technically we write all identities as ε_A where A runs over a class $\mathscr{A}$ of indices ($\mathscr{A}$ may be just the class of identities), and these indices are called objects. If $\varepsilon_A = \delta(\alpha)$ and $\varepsilon_B = \varrho(\alpha)$, we write $\alpha\colon A \to B$.

Suppose that $\beta\colon A \to B$ is an injection and an epimorphism simultaneously. Then $\beta = \varepsilon_B\beta\varepsilon_A$ and $\beta = \beta\varepsilon_A\varepsilon_A$ are two decompositions which, according to (A_7), must coincide, whence $\beta = \varepsilon_B = \varepsilon_A$.

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