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INDEPENDENT SUBSETS IN SEMILATTICES

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In [1] Marczewski introduced a notion of independence for universal algebras. Concerning the class of lattices and the class of semilattices the Marczewski independence was treated in [3] and [2]. Here we continue our earlier investigations. For the definitions and notations see [3] and [4].

1. Let us consider a semilattice $(S; \cap)$. By the same way as in the proof of Theorem 1 of [2], Theorem 6 of our paper [3] can be modified as follows:

THEOREM 1. *Let T be a subset of S. Then T is dependent* (1) *in $(S; \cap)$ if and only if there exist different elements $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s$ in T such that $a_1 \cap a_2 \cap \ldots \cap a_r \leqslant b_1 \cap b_2 \cap \ldots \cap b_s$.*

Moreover, we have

THEOREM 2. *Let $(S; \cap)$ be a semilattice and T an n-element subset of S. Then T is independent if and only if the subsemilattice $\langle T \rangle$ generated by T in $(S; \cap)$ consists of $2^n - 1$ elements.*

In fact, $\langle T \rangle$ consists of all different elements of the form

$$x_{i_1} \cap x_{i_2} \cap \ldots \cap x_{i_k} \quad (x_{i_1}, x_{i_2}, \ldots, x_{i_k} \in T,\ 1 \leqslant k \leqslant n).$$

Hence T is independent if and only if any two of these $2^n - 1$ elements are different. This proves Theorem 2.

2. Clearly, any independent subset of a semilattice is totally unordered, but the converse statement does not hold in general. In this section we deal just with semilattices in which every totally unordered subset is independent.

THEOREM 3. *Let $(S; \cap)$ be a semilattice in which every totally unordered subset is independent. If S has totally unordered subset of three elements, then the length of S is infinite.*

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(1) In [3] we used the term *M-dependent*.

Proof. If $T = \{a_1, a_2, a_3\}$ is a totally unordered (hence, by our assumption, an independent) subset of S, then the elements

$$a_{ij} = a_i \cap a_j \quad (i, j = 1, 2, 3;\ i \neq j)$$

are again pairwise incomparable, for e.g. $a_{12} \leqslant a_{23}$ would imply $a_1 \cap a_2 \leqslant a_3$ which contradicts, by Theorem 1, the independence of T. Moreover, again by the independence of T, $a_{ij} \neq a_1, a_2, a_3$. Thus the set $\{a_{12}, a_{23}, a_{31}\}$ is a new independent set of three elements and $a_1 > a_{12}$. Applying again the above reasoning to that subset we get an element a_{123} $(= a_{12} \cap a_{23})$ auch that $a_1 > a_{12} > a_{123}$ and, repeating infinite times, this procedure yields an infinite descending chain of S.

Remark. In the semilattice shown in Fig. 1 the first assumption of Theorem 2 is satisfied. It is, however, of finite length, because its single totally unordered subset consists of two elements only.

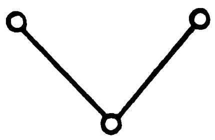

Fig. 1.

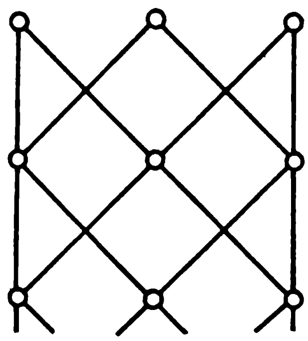

Fig. 2.

Let $w(P)$ denote the *width* of the partially ordered set P, i.e. the least upper bound of the powers of its totally unordered subsets. Fig. 2 shows a semilattice of width 3 in which every totally unordered subset is independent. There is, however, no such semilattice of width $n > 3$:

THEOREM 4. *Let $(S; \cap)$ be a semilattice in which every totally unordered subset is independent. Then either $w(S) \leqslant 3$ or $w(S)$ is infinite.*

Proof. Let $T_0 = \{a_1, a_2, \ldots, a_n\}$, where $n > 3$, be a totally unordered subset of S. Then, by the same argument as in the proof of Theorem 3, the elements

$$a_{ij} = a_i \cap a_j \quad (i, j = 1, 2, \ldots, n;\ i \neq j)$$

are pairwise incomparable. Thus we have a totally unordered subset T_1 of S whose cardinality is greater than the cardinality of T_0, because $\binom{n}{2} > n$ for $n > 3$. Hence our theorem follows by induction.

REFERENCES

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