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MODEL OF CIRCULATION AND EXCHANGE OF BANKNOTES

0. INTRODUCTION

The aim of this paper is to present a certain theoretical model of circulation of banknotes and exchange of used banknotes into the new ones. The analysis of this model might yield a solution of a problem posed by the National Bank of Poland (NBP), namaly the problem of choice of the optimal strategy of exchange of used banknotes.

The formulation of assumptions of the model will be preceded by presenting the empirical problem in intuitive terms and a short discussion of the results of papers [1] and [2] which give theoretical foundations of the concept of the degree of usage of banknotes, the central idea of this paper.

1. PRACTICAL PROBLEM

1.1. The present state of affairs. The National Bank of Poland performs a systematic exchange of overly used banknotes into the new ones. The banknotes exchanged are those which enter one of the sections of NBP and are deemed used in the degree exceeding the admissible norm (in the bank terminology they are called "destructs").

The norms of usage of banknotes are defined rather vaguely: according to the instruction, one should exchange banknotes which are torn, have distinct stains, etc., and banknotes "too much used". While the first categories are defined sufficiently unambigously for practical purposes, the basic difficulty lies in the lack of the precise definition of "degree of normal usage" beyond which the banknotes should be exchanged.

In practice, the exchange is performed by bank employees, whose basic duty is to form packages, each containing 100 banknotes; in the process of counting, they reject from the packages those banknotes which they feel ought to be exchanged, and replace them by "usable" banknotes, taken out of a special pile.

These employees are responsible materially for the correctness of the process of counting, and elimination of destructs is only their secondary task. The quality of performing the latter task is practically beyond control, and there is no motivation to do it properly (the lack of control is due mainly to the fact that it is not clear what does mean "proper elimination of destructs").

To find out the degree or arbitrariness of decision regarding the question whether a given banknote is a destruct or not, the following experiment was performed in NBP: a set of 1000 banknotes was subject to the normal procedure of counting with elimination of destructs in two branches of NBP; the same set was sorted and counted twice by the same person in a five-day period (the subjects did not know, of course, that they are being tested).

The experiments showed considerable variations in results, both for different subjects, as well as for the same subject on different days. In extreme cases, 280 and 470 banknotes were eliminated as destructs by one subject on different days, while the analogous numbers for another subject were 118 and 82.

1.2. Problems formulated by NBP. Generally, the bank would like to introduce an "optimal" policy of exchange of banknotes. The necessary conditions for formulating the criteria of optimality and the choice of policies optimal from the point of view of these criteria are the following:

(a) defining the concept of degree of usage of a banknote, and the construction of methods of evaluating the distribution of degree of usage in the population of banknotes at a given moment of time;

(b) the construction of methods of introducing in practice various policies of exchange, based on different "critical levels", i.e. levels of degree of usage beyond which the banknotes are to be exchanged;

(c) evaluation, for each critical level of usage, the expected number of banknotes which will have to be exchanged in different periods of time, and the limit distribution of degree of usage of banknotes in population, resulting from accepting the policy based on this critical level.

Solving (a), (b) and (c) is an obvious prerequisite for

(d) formulating the criteria of optimality, or equivalently, the loss function, depending at least on the costs of replacement and the distribution of the degree of usage in the population, and

(e) the choice of optimal policy of replacement, i.e. a policy (critical level) leading to the minimal loss function defined in (d).

For technical reasons, NBP cannot rely on any definitions of degree of usage of banknotes other than those based on subjective evaluations of the sorting employees. Thus, many obvious definitions, based on suitable indices (such as the amount of light absorbed, number of creases crossing a given line, or per unit of surface, etc.) are eliminated from our considerations. More precisely, the bank requires that

(f) solutions of (a)-(e) must be such that their introduction in practice cannot lead to any significant change in the present procedure, which leaves the decision concerning each banknote to subjective evaluations of bank employees.

2. DEFINITION OF DEGREE OF USAGE OF A BANKNOTE

2.1. Introductory remarks. The basic ideas which underly the definition of the degree of usage of a banknote are contained in papers [1] and [2] concerning methods of evaluation and construction of subjective classifications. We shall define the level of usage of a banknote in terms of a suitably defined classification of banknotes.

The remarks below will help in realizing the significant difference between the approach suggested here and the traditional approach.

Usually, classification is a concept secondary with respect to the characteristic which serves as a basis for it. To build a *taxonomy* (i.e. a principle of classification) of a given set of objects, one chooses a characteristic of elements of this set (possibly vector-valued, and not necessarily numerical), partitions the set of all values of this characteristic into disjoint subsets and defines a binary relation on the set of objects considered, as follows: the relation holds between two objects if and only if the values of characteristics for these two objects belong to the same set of partition. This relation is then reflexive, symmetric, and transitive, hence it is an equivalence. The classes of abstraction of this relation of equivalence are called *taxonomical categories*.

In this way, to each object of the considered set there corresponds exactly one taxonomical category (the "true" category of this object), and one may consider the problem of errors of classification, by analysing the probability that an object of a given category will be assigned to another category in the process of classification. Such an approach to problems of errors of classification is presented in [4].

In the considered case of classification of banknotes from the point of view of the degree of their usage, such an approach is not applicable, because of lack of the definition of the characteristic which would serve as a basis for classification, namely the characteristic called *degree of usage*. We shall proceed, in a sense, in the reverse direction: we shall define a certain procedure of classifying banknotes which will not be based on the definition of the characteristic according to which we want to classify, and next, we shall define the taxonomical categories of the degree of usage (the "true" categories for banknotes) using the probabilistic properties of the considered classification procedure.

Because of this "reversing" of the hierarchy of concepts, we shall start our considerations from the concepts presented in papers [1] and [2], namely those of subjective classification, methods of evaluating them, and methods of constructing them.

2.2. Subjective classifications and methods of evaluating them. As mentioned above, when considering the problem of classification, we shall not assume explicitly that to each object classified there corresponds its "true" taxonomical category. We shall build a model for the procedure of classifying objects by a given person; the criterion according to which this classification is performed is of no interest at the present moment.

Let $\mathscr{C} = \{C_1, C_2, \ldots\}$ be a non-empty finite or countable set, whose elements will be called *taxonomical categories*, or categories of classification. Let B be the set of objects classified and, finally, let S be the set of persons who perform classifications.

By a *probabilistic description of classification* or, shortly, a *classification* (of set B by persons from S according to categories from $\mathscr{C}$) we mean a family

$$\{X_s^{(i)}(b),\ s \in S,\ b \in B,\ i = 1, 2, \ldots\} \tag{1}$$

of random variables (defined on some suitable probability space), taking values in $\mathscr{C}$. We shall interpret $X_s^{(i)}(b) = C_j$ as the event "on the i-th trial, person s classified object b to category C_j".

We shall assume that

1° For $(s_1, b_1, i_1) \neq (s_2, b_2, i_2)$ the random variables $X_{s_1}^{(i_1)}(b_1)$ and $X_{s_2}^{(i_2)}(b_2)$ are independent;

2° $P\{X_s^{(i)}(b) = C_j\} = p_{s,j}(b)$ does not depend on i.

As we do not assume the existence of a true category for a given object, the evaluation of quality of classification (1) cannot be based on probabilities of erroneous classifications. Instead, we shall introduce a certain measure of quality of classification which leads only to "necessary" conditions for goodness of classification. This measure will be based on intuition according to which, whatever the principle of classifying, the classification cannot be good if with high probability there appear differences in classifications of the same object either for different classifying individuals or for the same individual classifying again the same object.

More precisely, the quality of classification (1) with respect to an object b will be expressed by the probability

$$\mathrm{P}\{X_{s_1}^{(i_1)}(b) = X_{s_2}^{(i_2)}(b)\} = u_{s_1,s_2}(b), \tag{2}$$

where $(s_1, i_1) \neq (s_2, i_2)$. By 1° and 2°, the value on the left-hand side of (2) does not depend on i_1 and i_2 and we have

$$u_{s_1,s_2}(b) = \sum_j p_{s_1,j}(b) p_{s_2,j}(b).$$

A necessary condition for goodness of classification is that the values of $u_{s_1,s_2}(b)$ should be close to 1 for $b \in B$ and $s_1, s_2 \in S$. The estimation of $u_{s_1,s_2}(b)$ for a given $b \in B$ requires of course a sufficiently large number of independent observations of classification of object b by individuals s_1 and s_2, which is usually unattainable in practice. One can, however, build an "overall" measure of quality of classification, by defining, for the set $B' = \{b_1, \ldots, b_n\} \subset B$ the parameters

$$u_{s_1,s_2}(B') = \frac{1}{n} \sum_{j=1}^{n} u_{s_1,s_2}(b_j)$$

and

$$\sigma^2_{s_1,s_2}(B') = \frac{1}{n} \sum_{j=1}^{n} [u_{s_1,s_2}(b_j) - u_{s_1,s_2}(B')]^2.$$

The paper [2] gives the construction of methods of estimating parameters $u_{s_1,s_2}(B')$ and $\sigma^2_{s_1,s_2}(B')$ based on the results of two independent observations of classifications of each object of the set B' by each of the persons s_1 and s_2 (if $s_1 = s_2$, these estimates require the results of four independent observations of classificiations of each of the elements of the set B' by the person s_1). The estimators are unbiased, and their variances are bounded from above by $1/8n$ and $3/4n$, respectively.

The use of these estimators allows then to calculate the quality of classification expressed by the values $u_{s_1,s_2}(b)$ by evaluation, for sufficiently numerous sets B', of the average of function $u_{s_1,s_2}(\cdot)$ on B' and a certain measure of spread of this function around its average on B'.

2.3. Construction of subjective classifications. In this section we present a method, suggested in [2], for the construction of a classification of banknotes with respect to the degree of their usage. This classification will not be based on any explicit definition of the trait called "degree of usage"; it will be based on the assumption that there exists a sufficiently common intuition concerning this trait. More precisely, we assume that each person from the set S is able to point out in each pair $\langle a, b \rangle$ of objects from B (banknotes) that element which he feels is "earlier" (less used). We shall not assume that these choices are consistent or that they are the same for each person; they may also vary for the same person from one occasion to another.

We shall assume that we have a family of random variables (defined on some suitable probability space)

$$\{T_s^{(i)}(a, b),\ \langle a, b \rangle \in B \times B,\ s \in S,\ i = 1, 2, \ldots\}, \tag{3}$$

where $T_s^{(i)}(a, b)$ assumes one of the values a, b. We interpret $T_s^{(i)}(a, b) = a$ as the event "on i-th presentation of the pair $\langle a, b \rangle$ individual s pointed out the element a as "earlier" in the pair $\langle a, b \rangle$".

We assume that the random variables (3) satisfy the following conditions:

1° If $(\langle a_1, b_1\rangle, s_1, i_1) \neq (\langle a_2, b_2\rangle, s_2, i_2)$, then the random variables $T_{s_1}^{(i_1)}(a_1, b_1)$ and $T_{s_2}^{(i_2)}(a_2, b_2)$ are independent;

2° the probability $\mathrm{P}\{T_s^{(i)}(a, b) = a\} = p(a, b)$ does not depend on s and i.

Roughly speaking, we shall try to find conditions for the probabilities $p(a, b)$ under which one can construct a sequence $\ldots, b_{-1}, b_0, b_1, \ldots$ of elements of B which could serve as "boundaries" between successive categories of classification. In other words, we shall try to construct a sequence $\{b_j\}$ such that the rule "put (on i-th trial) the object x into category C_j if you feel it is earlier than b_j and later than b_{j-1}" yields (with probability one) the classification of object x, i.e. assigning to it (in this trial and by this person) exactly one category C_j. We shall try to find the longest possible such sequence, hence a sequence leading to a classification into the greatest possible number of categories (a one-element sequence gives a dichotomous classification).

We assume that the probabilities $p(a, b)$ are defined for all $\langle a, b\rangle \in B\times B$ and satisfy the following conditions:

(i) $p(a, b)+p(b, a) = 1$ for all $a, b \in B$;

(ii) for all $a, b, c \in B$, if $p(a, b) \geqslant 1/2$ and $p(b, c) \geqslant 1/2$, then $p(a, c) \geqslant \max[p(a, b), p(b, c)]$;

(iii) there exists a q, $1/2 < q < 1$, such that, for all $a, b, c \in B$, the conditions $p(a, b) > q$ and $p(b, c) > q$ imply $p(a, c) = 1$.

For an arbitrary $a \in B$ and h $(1/2 < h < 1)$ write

$$A_h^+(a) = \{x \in B\colon\ p(a, x) \geqslant h\} \quad \text{and} \quad A_h^-(a) = \{x \in B\colon\ p(x, a) \geqslant h\}.$$

(iv, 1) If $A_h^+(a)$ is not empty, then there exists a $u \in A_h^+(a)$ such that $p(u, x) \geqslant 1/2$ for all $x \in A_h^+(a)$.

(iv, 2) If $A_h^-(a)$ is not empty, then there exists a $u \in A_h^-(a)$ such that $p(x, u) \geqslant 1/2$ for all $x \in A_h^-(a)$.

(v) For any $a \in B$ and h $(1/2 < h < 1)$

(v, 1) if there exists an infinite sequence $b_1, b_2, \ldots$ of elements of B such that $p(b_j, b_{j+1}) \geqslant h$ for all j, then there exists an $m = m(a)$ such that $p(a, b_m) \geqslant 1/2$;

(v, 2) if there exists an infinite sequence $b_1, b_2, \ldots$ such that $p(b_{j+1}, b_j) \geqslant h$ for all j, then there exists an $m = m(a)$ such that $p(b_m, a) \geqslant 1/2$.

We now present the construction of the classification based on this set of axioms; the proofs of the theorems may be found in [2].

First of all, note that the set Q of values of q for which axiom (iii) holds is an interval closed on the left. Let

$$q^* = \inf\{q\colon\ q \in Q\}.$$

Next, set $a \sim b$ if $p(a, b) = 1/2$, and $a \dashv b$ if $p(a, b) > 1/2$. It follows from axioms (i) and (ii) that relation $\sim$ is reflexive, symmetric and transitive in B and that relation $\dashv$ is antireflexive, antisymmetric and transitive. Moreover, for all a, b we have $a \dashv b$, $a \sim b$ or $b \dashv a$, hence B (or, to be precise, B divided by the equivalence $\sim$) is linearly ordered by $\dashv$. Note that axiom (iv) asserts that there are "first" and "last" elements in sets of a certain type, and axiom (v) corresponds to the Archimedean axiom in arithmetic.

For fixed $a \in B$ and h $(1/2 < h < 1)$ consider sequences $\ldots, b_{-1}, b_0, b_1, \ldots$ (finite or not) of elements of B such that $b_0 \sim a$ and $p(b_j, b_{j+1}) \geqslant h$ for all j.

Denote the class of all such sequences by $L(a, h)$.

Let $\{b_j\} \in L(a, h)$ for $h > q^*$. The construction of the classification scheme is based on the theorem asserting that for any $x \in B$, if $0 < p(x, b_m) < 1$ for some m, then $p(x, b_{m+k}) = p(b_{m-k}, x) = 1$ for all $k \geqslant 4$. In other words, if for some m the random variables $T_s^{(i)}(x, b_m)$ are non-degenerate (there may be differences of opinion whether x or b_m is "earlier"), then there are no differences of opinion when x is compared with elements of the sequence $\{b_j\}$ distant from b_m by four or more terms.

Thus, if $\{b_j\} \in L(a, h)$ for $h > q^*$, and $\{b_{n_j}\}$ is a subsequence of $\{b_j\}$ such that $n_{j+1} - n_j = 4$ for all j, then $\{b_{n_j}\}$ may be used as a sequence of "standards" for classification: one defines a random variable $X_s^{(i)}(b)$ with values in the set of categories of classification $\ldots, C_{-1}, C_0, C_1, \ldots$ by setting $X_s^{(i)}(b) = C_j$ if $T_s^{(i)}(b, b_{n_j}) = b$ and $T_s^{(i)}(b, b_{n_{j-1}}) = b_{n_{j-1}}$ (for finite sequences this definition requires obvious amendments for extremal categories). It follows from axiom (v) that every element of the set B will be classified into a category with a finite index; moreover, it can be shown that for every $x \in B$ the probability $p_{s,j}(x) = P\{X_s^{(i)}(x) = C_j\}$ does not depend on s and i and may be different from zero only for two successive categories j and $j+1$. In other words, for each object x with probability 1 all its classifications (for all persons s and occasions i) will fall either into one category or into two neighbouring categories. It follows that $u_{s_1,s_2}(x)$ defined in the preceding section as the probability that two independent classifications of the same object x by persons s_1 and s_2 will coincide is bounded from below by $1/2$.

Thus, an arbitrary sequence $\{b_j\} \in L(a, h)$ for any $a \in B$ and $h > q^*$ may serve, in the sense described above, as a basis for classification with the stated properties. It will be convenient to call sequences $\{b_j\}$ from class $L(a, h)$ with $h > q^*$ *primitive* and their subsequences $\{b_{n_j}\}$ with $n_{j+1} - n_j = 4$ *classification* sequences.

We present now the principle of choice leading to the best sequences in class $L(a, h)$, i.e. to the sequence which gives classification into the maximal possible number of categories.

The optimal primitive sequence $\{b_j'\}$ in the class $L(a, h)$ with $h > q^*$ is obtained as follows. We choose $b_0' \sim a$ and then construct the sequence $\{b_j'\}$ according to the inductive rule: for $k \geqslant 0$, if we already have defined b_k', consider the set $A_h^+(b_k')$. If this set is empty, then b_k' is the last term of the sequence with positive index. Otherwise, we put b_{k+1}' equivalent to the earliest element of this set; the existence of such an element follows from axiom (iv). In a similar way we extend the sequence into the negative direction.

It can be shown that the sequence so constructed is determined uniquely up to the equivalence $\sim$ and that it is the longest sequence in class $L(a, h)$ in the following sense: for every sequence $\{b_j\} \in L(a, h)$, if b_k $(k \geqslant 0)$ is defined, then b_k' is also defined and $p(b_k', b_k) \geqslant 1/2$. An analogous condition holds also for negative indices.

Thus, the classification based on the classification subsequence of the primitive sequence $\{b_j'\}$ has the maximal number of categories among all classifications based on primitive sequences from $L(a, h)$.

2.4. Definition of categories of degree of usage of banknotes. Assume that the axioms formulated in the preceding section are satisfied, and let $\{b_j\}$ be any fixed primitive sequence, i.e. a sequence from class $L(a, h)$ with $h > q^*$. Let us assign to an element x from the set B the category K_j if $p(x, b_j) \geqslant 1/2$ and $p(b_{j-1}, x) > 1/2$ (for finite sequences this definition requires an obvious modification for boundary categories). In terms of the relation $\dashv$ the element x belongs to category K_j if $b_{j-1} \dashv x \underset{\sim}{\dashv} b_j$. Such an assignment of categories to elements of the set B defines a function $B \to \{\ldots, K_{-1}, K_0, K_1, \ldots\}$, dependent on the primitive sequence $\{b_j\}$. As regards banknotes, we call the category K_j of a banknote its *true* category of usage (as measured by the sequence $\{b_j\}$). More precisely, for banknotes we introduce also an additional category K' of banknotes "used in a non-typical way", comprising banknotes which are torn, stained, etc. We assume that banknotes can be partitioned into those which belong to category K' and into the remaining ones (which are used in a normal way). At the moment we shall deal only with the latter ones.

In connection with the definition of "true" categories of usage of banknotes, there arises the question of determining the relation between the data obtained from observing the classifications based on the classification subsequence of the primitive sequence $\{b_j\}$ and the true categories defined in terms of the same sequence. From the considerations of the preceding section it follows that the scheme of classification introduced there based, let us say, on the subsequence $\{b_{4j}\}$ of the primitive sequence $\{b_j\}$ supplies us with the classification (subject to random errors) into taxonomical categories, each of them comprising four successive categories K_j; in other words, the classification is performed according to classes

$\ldots, C_0 = \{K_0, K_1, K_2, K_3\}, C_1 = \{K_4, K_5, K_6, K_7\}, \ldots$, where the "errors" of classification are such that all classifications of a given banknote fall within two neighbouring categories C_j and C_{j+1}.

In general, the problem of evaluation of the state of a given set of banknotes $B' \subset B$ may be formulated as follows: let B' be a finite set, and let N_j be the number of objects in the set B' which are in class K_j. Thus, from the point of view of the characteristic of interest, the vector $\{\ldots, N_{-1}, N_0, N_1, \ldots\}$ describes the "state" of the set B'. We want to make inference about this state on the basis of an observation of the values of a certain vector-valued random variable $\{\ldots, Z_{-1}, Z_0, Z_1, \ldots\}$, where Z_j is the joint number of objects of the set B' classified (on the given occasion by the given subject) to category $C_j = \{K_{4j}, K_{4j+1}, K_{4j+2}, K_{4j+3}\}$. It is clear that the construction of an estimate of $\{\ldots, N_{-1}, N_0, N_1, \ldots\}$ based on random variable $\{\ldots, Z_{-1}, Z_0, Z_1, \ldots\}$ is impossible for two reasons: firstly, the parameter about which we want to make inference has, roughly speaking, four times as many coordinates as the vector-valued random variable which is to serve as the basis for this inference. Secondly, the probabilities $q_j(x)$ that object x will be classified into the category C_j are not constant for objects from class C_j.

Thus, we make simplifying assumptions about the probabilities $q_j(x)$ and build a model which imposes certain restrictions on vectors $\{\ldots, N_{-1}, N_0, N_1, \ldots\}$. More precisely, we build a theoretical model of circulation and exchange of banknotes whose analysis will lead to determining the limit distribution for particular categories K_j in the population of banknotes. Knowing this distribution and making some assumptions about probabilities $q_j(x)$ we shall be able to determine the theoretical distribution of the vector $\{\ldots, Z_{-1}, Z_0, Z_1, \ldots\}$ for the classification of a large number of randomly selected banknotes; the observations of actual classifications will serve as a test for verification of the model.

As regards the assumptions concerning the probabilities $q_j(x)$, they will consist in accepting that $q_j(x)$ is constant for all x from a given class K_i.

The choice of the numerical value of $q_j(x)$ for x from class K_i will be based on the following easily proved inequalities.

Let namely $C_j = \{K_{4j}, K_{4j+1}, K_{4j+2}, K_{4j+3}\}$. Then for x from classes K_{4j} and K_{4j+1} we have $q_{j-1}(x) + q_j(x) = 1$, where in the first case we have $q_{j-1}(x) < 1/2$, and in the second case we have $q_{j-1}(x) \leqslant 1-h$. Analogous inequalities are true for elements x from classes K_{4j+2} and K_{4j+3}. In the case of the considered classification of banknotes we assume that $q_j(x)$ are equal to appropriately defined "average" values of function $q_j(x)$ on the given class K_i. We shall return to this problem in the sequel.

Finally, the last problem which requires discussion is that of errors of classifications of objects of the set B under the dichotomous classification based on one element of the primitive sequence $\{b_j\}$. To simplify the terminology, let us agree to call *destructs* all banknotes from classes $K_{j_0}, K_{j_0+1}, \ldots$ Let $\varepsilon(x)$ be the probability that banknote x will be classified as destruct, that is,

$$\varepsilon(x) = \mathrm{P}\{T_s^{(i)}(x, b_{j_0-1}) = b_{j_0-1}\}$$

(the "boundary" element b_{j_0-1} is accepted as "less used" than the element x). On the basis of axioms (i)-(v) it is easy to prove that $\varepsilon(x) = 0$ for x from classes K_{j_0-m} with $m \geqslant 3$, and $\varepsilon(x) = 1$ for x from classes K_{j_0+m} with $m \geqslant 2$. For values $\varepsilon(x)$ for x from classes $K_{j_0-2}, K_{j_0-1}, K_{j_0}$ and K_{j_0+1} we shall make certain simplifying numerical assumptions when considering the model of circulation and exchange of banknotes.

2.5. Application to classification of banknotes. Before passing to the description of the model of circulation of banknotes, we shall sketch briefly the results of experiments concerning classification of banknotes based on the theory outlined in the preceding sections. The detailed discussion of the empirical procedure may be found in [2]; here we shall merely present the results.

The experiments concerned the 20 zł banknotes, and were performed in the Institute of Mathematics of the Polish Academy of Sciences in Warsaw. It was assumed that the random variables $T_s^{(i)}(a, b)$ describing the choices of a banknote from the pair $\langle a, b\rangle$ satisfy the axioms formulated in the preceding sections. The probabilities $p(a, b) = \mathrm{P}\{T_s^{(i)}(a, b) = a\}$ were estimated from observations of choices from the pair $\langle a, b\rangle$ performed by different subjects; each of the subjects made at most two choices from the same pair $\langle a, b\rangle$.

It was assumed that axiom (iii) is satisfied for $q = 0.75$ and the attempts were made to choose a primitive sequence $\{b_j\}$ of banknotes with $p(b_j, b_{j+1}) = 0.75$. For a given banknote b_j, as b_{j+1} we took that banknote which obtained 18 choices "more used than b_j" out of 24 choices. In an analogous way the sequence was extended towards negative indices. The experimentation procedure consisted, roughly speaking, on the choice, for a given banknote b_j, of an appropriate set of banknotes $Z(b_j)$ which was subject to 24 dichotomous classifications into "less" and "more" used than b_j. Next, as b_{j+1} or b_{j-1} the banknote with an appropriate number of choices was selected. The sets $Z(b_j)$ for successive banknotes b_j were chosen by experimenters from the pile of banknotes comprising 400 banknotes borrowed from the bank for the experimental procedure.

It turned out to be possible to construct a primitive sequence consisting of 17 banknotes. Taking every fourth of them, a classification sequence of 5 banknotes was obtained, thus creating a taxonomy with six categories. Next, 100 banknotes were classified 13 times into these six categories of degree of usage. According to the a ssumptions, with probability 1 for each of the 100 banknotes their classifications should fall within one or two neighbouring classes.

The results of experiments turned out to be satisfactorily coinciding with the expectations: for 55 banknotes all 13 classifications belonged to one or two neighbouring categories; for 37 banknotes, 12 classifications had this property, and only in one instance out of 13 the classification fell into the third category (neighbouring with the two). For 6 banknotes the classifications were in three neighbouring categories, each of them represented at least twice in the 13 trials, and for 2 banknotes, their classifications fell into four neighbouring categories.

These results seem to be promising: the consistency of classifications for different subjects and for the same subject on two occasions was much higher than that in the bank in a real situation, in spite of the fact that the experiments in the Mathematical Institute concerned the much more difficult task of classifying into six, and not two categories.

The values of estimators $u_{s_1,s_2}(B')$ and $\sigma^2_{s_1,s_2}(B')$ for the considered set of 100 banknotes are given in [2].

3. MODEL OF CIRCULATION AND EXCHANGE OF BANKNOTES

3.1. Assumptions of the model. We assume that the population of banknotes is finite and consists of N banknotes (we consider banknotes of a fixed value). We assume that there exist $r+1$ categories of degree of usage of banknotes, the first r of them based on a certain primitive sequence $\{b_j\}$, and the last class, K_{r+1}, corresponding to banknotes used in a "non-typical" way. It will be convenient to number the classes starting from index 1, so that class K_1 corresponds to new (least used) banknotes. For banknotes of the value 20 zł it is possible to consider 19 classes, 18 of them based on the 17-element primitive sequence, and one class of "non-typical" banknotes.

We assume that each banknote at every moment of time is in one of the classes $K_1, K_2, \ldots, K_{r+1}$ of the degree of usage; moreover, we assume that each banknote may be in one of the three "states": in circulation, in the bank, or in the treasury. Moreover, we assume that there exists a mint, in possession of an inexhaustible number of new banknotes to replace the used ones. The banknotes in the mint are in class K_1 and are not included in the population of N banknotes considered.

We assume that during one day N_0+N_1 banknotes are in circulation, and N_2 in treasury, where $N = N_0+N_1+N_2$. Moreover, assume that $N_2 \geqslant N_1$. Each evening, N_1 banknotes which are in circulation enter the bank, where they are subjected to the procedure of classification into destructs and non-destructs; those banknotes which are classified as destructs are replaced by new banknotes from the mint. Next morning, N_1 banknotes are taken from the treasury and put into circulation, and N_1 banknotes which entered the bank the preceding evening and were classified are put into the treasury.

For banknotes in circulation we shall make the following assumptions: banknote x which on a given day is in the class K_j and does not enter the bank passes to the state K_m with probability $\pi_m(x) = \pi_{jm}$ independently of what happens to other banknotes and independently of the previous events which occurred to this banknote.

Next, we assume that the probability that banknote x from class K_j will be classified as destruct equals $\varepsilon(x) = \varepsilon_j$ independently of decisions regarding other banknotes.

Moreover, we assume that the numbers N_0+N_1 and N_2 are large in comparison with N_1, and that the number N_1 itself is sufficiently large, so that instead of random variables one can adequately describe the phenomenon under consideration by expected values of these random variables; the random variables of interest will be the numbers of banknotes of particular classes in circulation and in the treasury. Moreover, we assume that the choice of banknotes which enter the bank from circulation, and the choice of banknotes from the treasury which are put into circulation are made independently of the class of usage of these banknotes.

3.2. Analysis of the model. Let us write

$$\alpha = \frac{N_1}{N_0+N_1}, \qquad \beta = \frac{N_1}{N_2}.$$

Thus, α is the fraction of banknotes (among those which are in circulation every day) which enter the bank each evening, and β is the fraction of banknotes in the treasury which are put into circulation each morning.

Let $\big(U_1(t), U_2(t), \ldots, U_{r+1}(t), V_1(t), \ldots, V_{r+1}(t)\big)$ be the vector describing the state of the population of banknotes on day t, where $U_j(t)$ is the number of banknotes of class K_j which are in circulation, and $V_j(t)$ is the number of banknotes of this class which are in treasury. By our assumptions, we have

$$\sum_j U_j(t) = N_0+N_1, \qquad \sum_j V_j(t) = N_2$$

for every t.

Thus, the number of banknotes from class K_j which enter the bank equals $\alpha U_j a(t)$. Among them, $\varepsilon_j \alpha U_j(t)$ will be classified as destructs and replaced by new banknotes (from class K_1). The numbers of banknotes from particular classes which will be put into circulation on the next day are $\beta V_j(t)$. We have therefore

$$U_j(t+1) = (1-a)\sum_i \pi_{ij} U_i(t) + \beta V_j(t),$$

$$V_j(t+1) = (1-\beta) V_j(t) + a(1-\varepsilon_j) U_j(t) + \delta_j^1 a \sum \varepsilon_i U_i(t),$$

where $j = 1, 2, \ldots, r+1$, and δ_j^1 is the Kronecker symbol.

It can be easily checked that if $\sum U_j(0) = N_0 + N_1$ and $\sum V_j(0) = N_2$, then $\sum U_j(t) = N_0 + N_1$ and $\sum V_j(t) = N_2$ for all t. Dividing both sides by $N = N_0 + N_1 + N_2$ and writing $p_j(t) = U_j(t)/N$, $q_j(t) = V_j(t)/N$, we obtain the system of recursive equations

$$p_j(t+1) = (1-a)\sum_i \pi_{ij} p_i(t) + \beta q_j(t),$$

(4) $$q_j(t+1) = (1-\beta) q_j(t) + a(1-\varepsilon_j) p_j(t) + \delta_j^1 a \sum_i \varepsilon_i p_i(t),$$

where $\sum p_i(t) + \sum q_i(t) = 1$ for every t.

We shall first find conditions under which the limits

$$\lim_{t\to\infty} p_j(t) = p_j \quad \text{and} \quad \lim_{t\to\infty} q_j(t) = q_j$$

exist independently of the initial state for $t = 0$.

To find these conditions, let us rewrite the system (4) in the form

$$T(t+1) = T(t)A,$$

where $T(s) = \big(p_1(s), \ldots, p_{r+1}(s), q_1(s), \ldots, q_{r+1}(s)\big)$, and A is a stochastic matrix of the form

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

In this matrix, A_1 equals to the matrix $(1-a)[\pi_{ij}]$, A_2 is a matrix of the form

$$\begin{bmatrix} a\varepsilon_1 & (1-\varepsilon_1)a & 0 & \ldots & 0 \\ a\varepsilon_2 & 0 & (1-\varepsilon_2)a & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a\varepsilon_{r+1} & 0 & 0 & \ldots & (1-\varepsilon_{r+1})a \end{bmatrix},$$

A_3 is the matrix βI and A_4 is the matrix $(1-\beta)I$.

For the existence of limit distribution $\lim p_j(t) = p_j$, $\lim q_j(t) = q_j$ it suffices that A is a matrix of an ergodic Markov chain, perhaps with transient states. This condition is satisfied if: for at least one index i we have $\pi_{ii} > 0$, in the stochastic matrix $[\pi_{ij}]$ there is a passage to every state from the state 1, and $0 < a < 1$, $0 < \beta < 1$. Then each of the states corresponding to the $r+1$ first rows is recurrent and the chain is aperiodic; in this case we have, in addition, $p_j > 0$ for all $j = 1, 2, \ldots, r+1$, which corresponds to the fact that in the limit all categories of degree of usage will be represented in the banknotes in circulation (in the treasury some categories may be missing in case of perfect elimination of destructs, so that some q_j may be zero).

If these conditions for the existence of limit distribution are satisfied, this distribution may be determined from the system of equations

$$(5)\qquad \begin{cases} p_j = (1-a)\sum\limits_i \pi_{ij}p_i + \beta q_j, \\ q_j = (1-\beta)q_j + a(1-\varepsilon_j)p_j + \delta_j^1 a \sum \varepsilon_i p_i, \\ p_1 + \ldots + p_{r+1} + q_1 + \ldots + q_{r+1} = 1. \end{cases}$$

Note first that summing up with respect to j we have

$$\beta \sum q_j = a \sum (1-\varepsilon_j)p_j + a \sum \varepsilon_j p_j = a \sum p_j,$$

which gives, in view of the last equation in system (5),

$$\sum p_j = \frac{1}{1+a/\beta}.$$

We shall now find the solution of system (5) under some simplifying assumptions about matrix $[\pi_{ij}]$. Namely, we assume that only the following three types of passages are possible: remaining in the same state, passage to the state with index higher by one, and passage to the state with index $r+1$; we assume that the second of these passages has a positive probability. In other words, we assume that for $i = 1, 2, \ldots, r$ we have

$$(6)\qquad \pi_{ii} + \pi_{i,i+1} + \pi_{i,r+1} = 1, \qquad \pi_{i,i+1} > 0 \quad \text{and} \quad \pi_{r+1,r+1} = 1.$$

The conditions for the existence of an ergodic distribution are satisfied. From the second of equations (5) we obtain

$$q_j = \frac{a}{\beta}(1-\varepsilon_j)p_j + \delta_j^1 \frac{a}{\beta} \sum \varepsilon_i p_i,$$

thus it suffices to determine the unknowns p_j. Substituting to the first of equations (5), we obtain after simple transformations

$$c_j p_j = \sum_i \pi_{ij} p_i + \frac{a}{1-a} \delta_j^1 \sum \varepsilon_i p_i,$$

where

$$(7)\qquad c_j = \frac{1-a(1-\varepsilon_j)}{1-a} \geqslant 1.$$

Using assumptions (6) we have therefore

(8)
$$\begin{aligned}
c_1 p_1 &= \pi_{11} p_1 + \frac{a}{1-a} \sum \varepsilon_i p_i, \\
c_2 p_2 &= \pi_{12} p_1 + \pi_{22} p_2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
c_r p_r &= \pi_{r-1,r} p_{r-1} + \pi_{rr} p_r, \\
p_1 + \ldots + p_{r+1} &= \frac{1}{1+a/\beta}
\end{aligned}$$

(the equation for the penultimate column has been omitted here).

Writing

$$\gamma_{j-1} = c_j - \pi_{jj}, \quad j = 1, \ldots, r,$$

we have $\gamma_j > 0$, in view of (6) and (7). We can, therefore, rewrite equations (8) (except the first and the last) in the form

$$\gamma_1 p_2 = \pi_{12} p_1, \ \gamma_2 p_3 = \pi_{23} p_2, \ldots, \gamma_{r-1} p_r = \pi_{r-1,r} p_{r-1}$$

or

$$p_2 = \frac{\pi_{12}}{\gamma_1} p_1, \ p_3 = \frac{\pi_{12}\pi_{23}}{\gamma_1\gamma_2} p_1, \ldots, p_r = \frac{\pi_{12}\pi_{23}\ldots\pi_{r-1,r}}{\gamma_1\gamma_2\ldots\gamma_{r-1}} p_1.$$

Substituting to the first and last of equations (8) we obtain a system of two equations with two unknowns p_1 and p_{r+1}:

$$p_1\left\{\gamma_0 - \frac{a}{1-a}\left[\varepsilon_1 + \varepsilon_2 \frac{\pi_{12}}{\gamma_1} + \ldots + \varepsilon_r \frac{\pi_{12}\ldots\pi_{r-1,r}}{\gamma_1\ldots\gamma_{r-1}}\right]\right\} = \frac{a}{1-a} \varepsilon_{r+1} p_{r+1},$$

$$p_1\left(1 + \frac{\pi_{12}}{\gamma_1} + \ldots + \frac{\pi_{12}\ldots\pi_{r-1,r}}{\gamma_1\ldots\gamma_{r-1}}\right) + p_{r+1} = \frac{1}{1+a/\beta}.$$

Eliminating p_{r+1}, we obtain

$$p_1\left\{\gamma_0 + \frac{a}{1-a}\left[(\varepsilon_{r+1}-\varepsilon_1) + (\varepsilon_{r+1}-\varepsilon_2)\frac{\pi_{12}}{\gamma_1} + \ldots + (\varepsilon_{r+1}-\varepsilon_r)\frac{\pi_{12}\ldots\pi_{r-1,r}}{\gamma_1\ldots\gamma_{r-1}}\right]\right\}$$
$$= \frac{a}{1-a}\cdot\frac{1}{1+a/\beta}\cdot\varepsilon_{r+1}.$$

Writing

$$M = \frac{a}{1-a}\cdot\frac{1}{1+a/\beta}\cdot\varepsilon_{r+1}$$

and putting, to simplify the notation, $\pi_{01} = \gamma_0$, we can write

$$p_1\left[\gamma_0 + \frac{a}{1-a}\sum_{j=1}^{r}(\varepsilon_{r+1}-\varepsilon_j)\frac{\pi_{01}\pi_{12}\ldots\pi_{j-1,j}}{\gamma_0\gamma_1\ldots\gamma_{j-1}}\right] = M.$$

If we assume, which seems to be reasonable, that $\varepsilon_{r+1} > 0$, we have $M > 0$, and since $p_1 > 0$, we may conclude that

$$\gamma_0 + \frac{a}{1-a} \sum_{j=1}^{r} (\varepsilon_{r+1} - \varepsilon_j) \frac{\pi_{01}\pi_{12} \ldots \pi_{j-1,j}}{\gamma_0\gamma_1 \ldots \gamma_{j-1}} > 0 .$$

Finally, for the limit distribution we obtain, in case of the considered form of matrix $[\pi_{ij}]$, the expressions

$$\begin{aligned}
p_1 &= \frac{M}{D}, \\
p_2 &= \frac{\pi_{12}}{\gamma_1} \frac{M}{D}, \\
p_3 &= \frac{\pi_{12}\pi_{23}}{\gamma_1\gamma_2} \frac{M}{D}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
p_r &= \frac{\pi_{12} \ldots \pi_{r-1,r}}{\gamma_1 \ldots \gamma_{r-1}} \frac{M}{D}, \\
p_{r+1} &= \frac{1}{1+a/\beta} - \frac{M}{D} \sum_{j=1}^{r} \frac{\pi_{01}\pi_{12} \ldots \pi_{j-1,j}}{\gamma_0\gamma_1 \ldots \gamma_{j-1}}, \\
q_j &= \frac{a}{\beta}(1-\varepsilon_j)p_j + \delta_j^1 \frac{a}{\beta} \sum \varepsilon_i p_i, \qquad i = 1, 2, \ldots, r+1,
\end{aligned} \tag{9}$$

where

$$M = \frac{a}{1-a} \frac{1}{1+a/\beta} \varepsilon_{r+1},$$

$$\gamma_{j-1} = \frac{1-a(1-\varepsilon_j)}{1-a} - \pi_{jj}, \qquad \pi_{01} = \gamma_0$$

and

$$D = \gamma_0 + \frac{a}{1-a} \sum_{j=1}^{r} \frac{\pi_{01}\pi_{12} \ldots \pi_{j-1,j}}{\gamma_0\gamma_1 \ldots \gamma_{j-1}} (\varepsilon_{r+1} - \varepsilon_j).$$

4. DISCUSSION

The discussion will be concentrated on three problems: possibility of numerical application of the suggested model in practical situations, empirical verification of the model, and problems of its possible practical usefulness.

4.1. Numerical problems: estimation of parameters. Without entering, at the moment, the problem whether the model suggested is adequate and whether it may be useful in practice, and assuming that the answer to both these questions is positive, there arises the problem of estimation of numerical values of the parameters, so as to enable us the application of formulas (9) to the limit distribution of the degree of usage of banknotes in the population. We shall merely outline the possibility of such an estimation, without going into the details of the necessary experimental procedure.

The parameters of the model are α, β, the vector $\varepsilon_1, \ldots, \varepsilon_{r+1}$ and the transition matrix $[\pi_{ij}]$.

It seems that there ought to be no special problem with estimating the values α and β, which are probably known to the directors of the bank.

As regards the vector $\varepsilon_1, \ldots, \varepsilon_{r+1}$ expressing the policy of exchange of banknotes, it appears that the following assumptions may be made: for the policy according to which the destructs are banknotes from classes $K_j, j \geqslant j_0$, we have $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{j_0-3} = 0$, $\varepsilon_{j_0+2} = \varepsilon_{j_0+3} = \ldots = 1$, and $\varepsilon_{r+1} = 1$, as it has been already stated in the preceding sections. For the values ε_{j_0-2}, ε_{j_0-1}, ε_{j_0} and ε_{j_0+1} we can take the "median" values of function

$$\varepsilon(x) = \mathrm{P}\{\text{banknote } x \text{ will be classified as destruct}\},$$

namely $\varepsilon_{j_0-2} = 1/8$, $\varepsilon_{j_0-1} = 3/8$, $\varepsilon_{j_0} = 5/8$ and $\varepsilon_{j_0+1} = 7/8$ (since for the considered primitive sequence $\{b_j\}$ with $p(b_j, b_{j+1}) = 0.75$ we have $0 < \varepsilon(x) < 1/4$ for $x \in K_{j_0-2}$, $1/4 < \varepsilon(x) < 1/2$ for $x \in K_{j_0-1}$, etc.). Thus, the considered sequence $\varepsilon_1, \ldots, \varepsilon_{r+1}$ will have the form

$$0, 0, \ldots, 0, 1/8, 3/8, 5/8, 7/8, 1, 1, \ldots; 1,$$

and for most "liberal" policies, consisting of taking as destructs only the banknotes from class K_{r+1} or only the banknotes from classes K_r and K_{r+1}, this sequence will have the forms

$$0, 0, \ldots, 0, 1 \quad \text{and} \quad 0, 0, \ldots, 0, 1/8, 3/8, 5/8, 1,$$

respectively (since category K_{r+1} is distinguished as a category of banknotes which are used in a "non-typical" way).

As regards the estimation of matrix $[\pi_{ij}]$ (for classes of degree of usage defined in terms of a given primitive sequence $\{b_j\}$), the problem is somewhat complex, even under the simplifying assumptions (6). It seems that these assumptions will have to be still simplified, so as to reduce the number of unknown parameters.

If the primitive sequence $\{b_j\}$ is chosen in such a way that $p(b_j, b_{j+1}) =$ const, then (from the intuitive point of view) the "widths" of successive categories are equal one to another, and are equal to units defined in a certain way by the "threshold" of perception of the degree of usage.

In the considered model, the mean sojourn time for a given category $K_j (j = 1, \ldots, r)$, for a banknote which is constantly in circulation, is

$$s_i = \frac{1}{1-\pi_{ii}} \quad (\infty \text{ for } \pi_{ii} = 1).$$

It appears that one can assume that the probability that the banknote will become destroyed in a non-typical way (will pass to the state K_{r+1}) is constant for every day and does not depend on the class of degree of usage K_i. We have, therefore, $\pi_{i,r+1} = b$ for $i = 1, 2, \ldots, r$. The value of b and the average s_i determine, of course, the i-th row of matrix $[\pi_{ij}]$, since we have, by assumption (6), the relations $\pi_{ii} + \pi_{i,i+1} + \pi_{i,r+1} = 1$. Thus, the problem reduces to making intuitively plausible assumptions about the numbers s_i, so as to reduce still further the number of unknown parameters.

It seems that only three types of assumptions may be of interest: the sequence $s_1, s_2, \ldots$ may be constant, decreasing or increasing.

Using the well-known laws of perception, the third assumption seems to be most reasonable, that is, the assumption that the sequence s_i is increasing. Moreover, if one is willing to assume a certain simplified model, one can determine the general form of function s_i for $i = 1, \ldots, r$.

If one assumes that the perception of degree of usage of a banknote depends on characteristic such as the amount of dirt on the banknote or the density of creases and if this trait increases linearly in time (each day the same amount of dirt remains on the banknote or the same number of new creases appears), then — according to the Weber-Fechner law — we have the following regularity: out of two banknotes, one of which has the amount of dirt, say, equal to x and the second $x + \Delta x$, the second one is perceived as more used by a given fraction of individuals if $(x + \Delta x)/x \geqslant a$, where a is a certain constant. Assuming that $x = x(t)$ is a linear function of time t, one obtains therefore an exponential form of function s_i: the time needed for a banknote to become dirtier enough to pass from the class K_i to K_{i+1} is proportional to the amount of dirt characteristic for class K_i. Thus, if we know $s_1 = 1/(1-\pi_{11})$, to determine s_2 we may use the following approximate calculations: let d be the velocity of increase of the amount of dirt remaining on the banknote. Thus, the banknote passes to the class K_2 when its amount of dirt reaches $s_1 d$. It will remain in the class K_2 until

$$\frac{s_1 d + s_2 d}{s_1 d} = a,$$

hence during the time interval $s_2 = s_1(a-1)$. In a similar way we get $s_3 = s_1 a(a-1)$. By an easy induction we get

$$s_j = s_1(a^{j-1} - a^{j-2}),$$

which implies that

$$\pi_{jj} = 1 - \frac{1}{s_1 a^{j-2}(a-1)}$$

for $j = 2, \ldots, r-1$, and, of course, $\pi_{rr} = 1-b$.

Thus, the problem is now reduced to three parameters, namely s_1, a and b.

The verification of the suggested assumption may be obtained by means of studying the physical properties of primitive sequences, in particular the primitive sequence of banknotes obtained as a result of experiments in the Institute of Mathematics. One should analyse physical properties such as number of creases, etc., and see whether they increase exponentially with the number of banknotes of the sequence.

If the results of this verification should turn out to be positive, one could easily suggest experiments leading to estimation of the numerical values of parameters s_1, a and b.

The estimate of parameter b may be obtained as follows: roughly speaking, if the stationary distribution has already been attained, then the number of banknotes which pass daily to the class K_{r+1} equals to the number of banknotes which are replaced daily by new ones because they are classified as "non-typical destructs". Thus, if the daily number of such banknotes is H, and there are N_0+N_1 banknotes in circulation, then the number H_1 of banknotes in class K_{r+1} may be estimated from the proportion

$$\frac{H_1}{N_0+N_1} = \frac{H}{N_1} \quad \text{or} \quad H_1 = \frac{N_0+N_1}{N_1} H = \frac{H}{a}.$$

Thus, out of $N_0+N_1-H_1 = N_0+N_1-H/a$ banknotes in circulation which are not in class K_{r+1}, each day H banknotes pass to the state K_{r+1}. This gives an approximate estimate of b, namely

$$b \cong \frac{H}{N_0+N_1-H/a}.$$

Finally, for the estimation of s_1 and a it is necessary to perform some experiments, consisting of the possibly exact evaluation of the degree of usage of a banknote whose total time spent in circulation is known. Then s_1 and a may be estimated by means of least squares.

4.2. Empirical verification of the model. In this section we shall discuss briefly the methods of empirical verification of the assumptions of the suggested model. The methods of testing the assumptions underlying the classification of banknotes according to the degree of their usage is discussed in [2], and we shall omit them. It is only worth while to note that the final results of a classification based on a classification sequence chosen from the obtained primitive sequence seem to indicate that these assumptions are satisfied reasonably well.

To verify the assumptions of the model of circulation and exchange of banknotes, one can compare the obtained limit distribution of particular classes of usage with the theoretical distribution derived from the model. As mentioned in section 2, the observations cannot concern the frequencies of banknotes in particular classes, but in classes grouped together, four at a time.

Roughly, the method of verification would look as follows: if the assumptions of the model are met, then the limit distribution for banknotes in circulation is $p_1^*, \ldots, p_{r+1}^*$, where $p_j^* = p_j/\sum p_k = p_j(1+\alpha/\beta)$. Thus, in a large sample of M randomly selected banknotes, there ought to be approximately $M_j = Mp_j(1+\alpha/\beta)$ banknotes of the class of usage K_j.

For class K_{r+1} the problem is relatively simple, as the observation of the actual number m_{r+1} of banknotes of class K_{r+1} in the sample is possible.

For the remaining banknotes one should proceed as follows: $M-m_{r+1}$ banknotes should be classified according to the classification subsequence chosen out of the selected primitive sequence.

Let $q_j(x)$ be the probability that the banknote x will be included into class $C_j = \{K_{4j}, K_{4j+1}, K_{4j+2}, K_{4j+3}\}$. It seems reasonable to assume that for $x \epsilon K_{4j}$ we shall have approximately $q_{j-1}(x) = 3/8$, $q_j(x) = 5/8$. For $x \epsilon K_{4j+1}$ we have $q_{j-1}(x) = 1/8$, $q_j(x) = 7/8$ and similarly for $x \epsilon K_{4j+2}$ and $x \epsilon K_{4j+3}$. These assumptions are based on inequalities similar to those used in making assumptions about ε_j.

Thus, if the assumptions of the model are met, then the expected number of banknotes classified into class C_j will be

$$M(1+\alpha/\beta)\left[\frac{5}{8}p_{4j}+\frac{7}{8}p_{4j+1}+\frac{7}{8}p_{4j+2}+\frac{5}{8}p_{4j+3}+\right.$$
$$\left.+\frac{3}{8}p_{4j-1}+\frac{1}{8}p_{4j-2}+\frac{3}{8}p_{4j+4}+\frac{1}{8}p_{4j+5}\right].$$

Observing the numbers of banknotes classified to particular classes C_j and to class K_{r+1} we may be able to evaluate to what extent the predictions given by the model are adequate.

4.3. Practical usefulness of results obtained by means of analysis of the model. Assuming that the practical problems connected with the given method of classifying are solved, that the parameters of the model have been estimated, and that the results of the tests turned out to be positive, the suggested model may be used to solving the problem posed by the bank, namely the problem of choice of an optimal policy of exchange.

First of all, note that (under the limit distribution corresponding to the given policy) the expected number of exchanges for one day is equal to $(N_0+N_1)a\sum\varepsilon_i p_i$. This number, therefore, may be determined theoretically for each of the contemplated policies. Thus, the problem consists in defining a loss function, depending on the cost of exchange and on the limit distribution $p_1, \ldots, p_{r+1}, q_1, \ldots, q_{r+1}$ (or, to be more precise, only on the coordinates $p_1, \ldots, p_{r+1}$, as the remaining coordinates representing the frequencies in treasury are functions of the probabilities $p_1, \ldots, p_{r+1}$).

To define the loss function, it is necessary to define an ordering in the set of vectors $(p_1, \ldots, p_{r+1})$, that is, one has to accept a certain criterion which allows us to assert which of the vectors $p = (p_1, \ldots, p_{r+1})$ and $p' = (p_1', \ldots, p_{r+1}')$ is "more desirable" (note that all these vectors have the same sum of coordinates, equal to $(1+a/\beta)^{-1}$).

Under rather general assumptions imposed on such orderings, consisting, roughly speaking, of the requirement that the relation $p \gg p'$ (distribution p is more desirable than distribution p') is transitive and gives a linear order of all distributions, that linear combinations of "better" distributions are "better" than analogous linear combinations of "worse" distributions and that for every three distributions there exist linear combinations of "best" and "worst" of the three which are "better" and "worse" than the middle one, one can show (see [3]) that there exists a function φ defined on the set of all categories of the degree of usage, such that $p = (p_1, \ldots, p_{r+1}) \gg p' = (p_1', \ldots, p_{r+1}')$ if and only if

$$\sum \varphi(K_j)p_j \leqslant \sum \varphi(K_j)p_j'.$$

Intuitively, φ is the measure of degree of usage, and the distribution p is better than the distribution p' if and only if p leads to a smaller average degree of usage than p'.

The choice of the numerical values of function φ must, of course, be made by the bank. When this choice is made, one could assume that the loss function for the policy producing the distribution $(p_1, \ldots, p_{r+1})$ is

$$A\sum \varphi(K_j)p_j + B\sum \varepsilon_j p_j. \tag{10}$$

One will then be able to determine, by means of an analysis of the model, the policy which yields the minimum of function (10).

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O PEWNYM MODELU OBIEGU I WYMIANY BANKNOTÓW

STRESZCZENIE

W pracy przedstawiony jest model obiegu i wymiany banknotów na nowe. Analiza tego modelu dostarcza teoretycznych przesłanek wyboru optymalnej polityki wymiany, tj. polityki prowadzącej – ogólnie mówiąc – do najlepszego rozkładu stopnia zużycia banknotów w populacji przy możliwie najmniejszej liczbie banknotów wymienianych.

W pierwszym paragrafie podane jest sformułowanie zagadnienia. W drugim paragrafie omówiono zasady klasyfikacji banknotów ze względu na stopnie ich zużycia, w trzecim – podano model wymiany i obiegu banknotów i jego analizę teoretyczną, a w ostatnim – dyskusję stosowalności modelu i testów dla jego weryfikacji.