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On the probability model for solving some integral equations of second type

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Abstract. In this paper we use the probability model displayed in [10] for solving by the Monte-Carlo method some integral and linear algebraical equations and for estimating some functionals connected with the theory of the transfer equation of nuclear physics. Certain special cases of such problems have been solved with use of different probability models by other authors. Here we also compare those models with the ones defined in this paper.

1. Introduction. In order to solve the following integral equation in the space $L_\infty(\Omega)$ (1) by the Monte-Carlo method:

$$u(X)-\int_{\Omega} K(x, y)u(y)\mu(dy) = g(x) \quad (x \in \Omega), \tag{1.1}$$

(where (Ω, Σ, μ) is a measure space, $\mu(\Omega) < +\infty$), the probability model has been constructed in [10].

Suppose that there exists a set $\Omega_0 \in \Sigma$ such that $\Omega_0 \subset \Omega$, $\mu(\Omega_0) > 0$ and it fulfils the following conditions:

(A) For all $f \in L_\infty(\Omega\setminus\Omega_0)$ (1), the series $\sum_{n=0}^{\infty} T_+^n f$ converges in $L_\infty(\Omega\setminus\Omega_0)$, where the integral operator T_+ is defined by the formula:

$$[T_+f](x) = \int_{\Omega\setminus\Omega_0} |K(x, y)|f(y)\mu(dy) \quad (x \in \Omega\setminus\Omega_0); \tag{1.2}$$

(B) $K(x, y) \geqslant 0$ for $x \in \Omega/\Omega_0$ $(\mathrm{mod}\,\mu)$, $y \in \Omega_0$ $(\mathrm{mod}\,\mu)$;

(C) $K(x, y) = 0$ for $x \in \Omega_0$ $(\mathrm{mod}\,\mu)$, $y \in \Omega$ $(\mathrm{mod}\,\mu)$.

The probability model is also constructed in [10] for obtaining an estimation by the Monte-Carlo method of the value of the functional:

$$(u, \varphi) = \int_{\Omega} u(x)\ \varphi(x)\ \mu(dx), \tag{1.3}$$

(1) $L_\infty(A)$ (for $A \in \Sigma$) is the space of Σ-measurable and bounded on A $(\mathrm{mod}\,\mu)$ functions.

where $u(x)$ is the solution of equation (1.1) and $\varphi \in L_1(\Omega)$ ([2]). In this paper we shall use the probability models referred to above for solving by the Monte-Carlo method some integral equations, linear algebraic equations, difference equations and for estimating certain linear functionals connected with the theory of the transfer equation of nuclear physics.

2. The concept of the model S and model $\bar{S}$. Suppose that there exists a $\Sigma\times\Sigma$-measurable and bounded on $\Omega\times\Omega$ function $p(x,y)$ such that the following conditions are satisfied:

$$(\mathrm{P}_1)\qquad a \equiv \operatorname*{vrai\,sup}_{\mu\quad x\in\Omega\setminus\Omega_0} \Big\{\int_\Omega K(x,y)p(x,y)\mu(dy)\Big\} < 1;$$

$$(\mathrm{P}_2)\qquad K(x,y)p(x,y) \geqslant 0 \quad \text{for } x\in\Omega(\mathrm{mod}\,\mu),\ y\in\Omega(\mathrm{mod}\,\mu);$$

$$(\mathrm{P}_3)\qquad p(x,y)\neq 0 \quad \text{for } (x,y)\in\Omega\times\Omega\ (\mathrm{mod}\,\mu\times\mu).$$

Then there exists a set $A^*\in\Sigma$ such that $\mu(A^*)=0$ and it satisfies the following conditions (see [10]):

$$(\mathrm{B}^*)\quad K(x,y)\geqslant 0 \quad \text{for } x\in\Omega_A^*\setminus\Omega_0,\ y\in\Omega_0\ (\mathrm{mod}\,\mu),\ \text{where } \Omega_A^* = \Omega\setminus A^*;$$

$$(\mathrm{P}_1^*)\qquad a^* = \sup_{x\in\Omega_A^*\setminus\Omega_0}\Big\{\int_\Omega K(x,y)p(x,y)\mu(dy)\Big\} < 1;$$

$$(\mathrm{P}_2^*)\qquad K(x,y)p(x,y)\geqslant 0 \quad \text{for } x\in\Omega_A^*,\ y\in\Omega\ (\mathrm{mod}\,\mu);$$

$$(2.1)\qquad G \equiv \sup_{x\in\Omega_A^*\setminus\Omega_0}\{|g(x)|\} < +\infty.$$

In order to construct probability models solving problems (1.1), (1.3), we defined in [10] a complete measure space $(\tilde{\Omega},\tilde{\Sigma},\tilde{\mu})$, where

$$(2.2)\qquad \tilde{\Omega} = \Omega\cup\Omega^*;\quad \tilde{\Sigma} = \bar{\Sigma}\cap\Sigma^*;\quad \tilde{\mu}(\tilde{A}) = \bar{\mu}(\tilde{A}\cap\Omega)+\delta\cdot\delta_{\tilde{A}} \quad (\text{for } \tilde{A}\in\tilde{\Sigma});$$

Ω^* is a set such that $\Omega^*\neq\Phi$, $\Omega^*\cap\Omega=\Phi$; $\bar{\mu}$ is the complete measure extending the measure μ onto the σ-field $\bar{\Sigma}\supset\Sigma$; δ is a positive constant and:

$$(2.3)\qquad \Sigma^* = \{\tilde{A}\colon \tilde{A} = \bar{A}\cup\Omega^*;\ \bar{A}\in\bar{\Sigma}\};\qquad \delta_{\tilde{A}} = \begin{cases}1 & \text{if } \tilde{A}\cap\Omega^*\neq\Phi,\\ 0 & \text{if } \tilde{A}\cap\Omega^* = \Phi.\end{cases}$$

Starting from the measure space $(\tilde{\Omega},\tilde{\Sigma},\tilde{\mu})$ we constructed two homogeneous Markov processes in the broad sense in the phase space $\tilde{\Omega}$. The Markov process corresponding to the transistion probabilities $P_i(k,x,\tilde{A})$

([2]) $L_1(A)$ (for $A\in\Sigma$) is the space of μ-integrable functions on A.

($i = 1, 2$) is called the *i-th process* (see [10]), where:

$$(2.4)\qquad P_i(k, x, \tilde{A}) = \int_{\tilde{A}} P_i(k-1, y, \tilde{A})\, P_i(1, x, dy)$$

$$\text{(for } x \in \tilde{\Omega},\ \tilde{A} \in \tilde{\Sigma},\ k = 2, 3, \ldots);$$

$$(2.5)\qquad P_i(1, x, \tilde{A}) = \begin{cases} \int_{\tilde{A}} F_i(x, y)\tilde{\mu}(dy) & \text{if } x \in \Omega_A^* \setminus \Omega_0, \\ \chi_{\tilde{A}}(x) & \text{if } x \in A^* \cup \Omega_0 \cup \Omega^*; \end{cases}$$

$$(2.6)\qquad F_i(x, y)$$

$$= \begin{cases} K(x, y)\ p(x, y) & \text{if } (x, y) \in (\Omega_A^* \setminus \Omega_0) \times (\Omega \setminus \Omega_0), \\ K(x, y)p(x, y) + \dfrac{g_i(x)p(x, y)}{\mu(\Omega_0)g_i(y)} & \text{if } (x, y) \in (\Omega_A^* \setminus \Omega_0) \times \Omega_0, \\ h_i(x) & \text{if } (x, y) \in (\Omega_A^* \setminus \Omega_0) \times \Omega^*; \end{cases}$$

$$(2.7)\qquad h_i(x)$$

$$= \frac{1}{\delta}\left[1 - \int_{\Omega} K(x, y)p(x, y)\mu(dy) - \frac{g_i(x)}{\mu(\Omega_0)} \int_{\Omega_0} \frac{p(x, y)\mu(dy)}{g_i(y)}\right];$$

$$(2.8)\qquad g_1(x) = -\left[|g(x)| + \chi_{\Omega_0}(x)\left(\frac{2GM}{1-a^*} + \Delta\right)\right];$$

$$g_2(x) = g(x) - g_1(x) \quad (x \in \Omega);$$

$$(2.9)\qquad M \equiv \sup_{(x,y) \in (\Omega_A^* \setminus \Omega_0) \times \Omega_0} \{|p(x, y)|\} < +\infty;$$

Δ is a positive constant.

Write $\Omega_1 = A^* \cup \Omega^* \setminus \Omega_0$. On the space of all trajectories of the i-th process with the same initial state $x \in \Omega_A^* \setminus \Omega_0$, which have the form:

$$(2.10)\qquad x \to x_1 \to \ldots \to x_l \, (x_l \in \Omega_0 \cup \Omega_1\colon\ x_1, \ldots, x_{l-1} \in \Omega_A^* \setminus \Omega_0);$$

we define random variables $\xi^{(i)}(x)$ ($i = 1, 2$) by the formula:

$$(2.11)\qquad \xi^{(i)}(x) = f^{(i)}(x; x_1, \ldots, x_l) = \begin{cases} \dfrac{g_i(x_l)}{p(x, x_1) \ldots p(x_{l-1}, x_l)} & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega_1. \end{cases}$$

Then (see [10], p. 21), under conditions (A), (B), (C), (P_1)–(P_3), the expected values of the random variables $\xi^{(i)}(x)$ ($i = 1, 2$) exist and are finite. The solution $u(x)$ in the space $L_\infty(\Omega)$ of equation (1.1) is defined by the formula:

(2.12) $$u(x) = M\xi^{(1)}(x) + M\xi^{(2)}(x) \quad \text{for } x \in \Omega \setminus \Omega_0 \ (\mathrm{mod}\,\mu)$$ ([3]).

By virtue of (2.12), we may solve the problem (1.1) by the Monte-Carlo method.

DEFINITION (2.1). The above probability model for solving the problem (1.1) is called the *model S*. And the random variables $\xi^{(i)}(x)$ $(i = 1, 2)$ defined in the model S are called the *estimators in the model S*. Obviously (see (2.4)–(2.9)), each model S depends on a function $p(x, y)$ satisfying conditions (P_1)–(P_3) and on values of positive constants Δ, δ:

(2.13) $$S = S\{p(x, y)\Delta, \delta,\}.$$

Let $P(\cdot)$ be the probability measure defined on the σ-field $\tilde{\Sigma}$ by the formula

(2.14) $$P(\tilde{A}) = \int_{\tilde{A}} \pi(x)\tilde{\mu}(dx) \quad (\text{for } \tilde{A} \in \tilde{\Sigma}),$$

where $\pi(x)$ is a function satisfying the following conditions:

(π_1) $$0 < \pi(x) < +\infty \quad \text{for } x \in \Omega \ (\mathrm{mod}\,\mu);$$

(π_2) $$\pi(x) \geqslant 0 \quad \text{for } x \in \tilde{\Omega} \ (\mathrm{mod}\,\tilde{\mu});$$

(π_3) $$\int_{\Omega} \pi(x)\tilde{\mu}(dx) = 1.$$

On the sample space of all trajectories of the i-th process, with the initial probability distribution $P(\cdot)$ of the form

(2.15) $$x_0 \to x_1 \to \ldots \to x_l \quad (x_l \in \Omega_0 \cup \Omega_1;\ x_0, x_1, \ldots, x_{l-1} \in \Omega_\Delta^* \setminus \Omega_0),$$

the random variables $\eta^{(i)}$ $(i = 1, 2)$ are defined by the formula:

(2.16) $$\eta^{(i)} = F^{(i)}(x_0, x_1, \ldots, x_l) = \begin{cases} \dfrac{\varphi(x_0)g_i(x_l)}{\pi(x_0)p(x_0, x_1)\ldots p(x_{l-1}, x_l)} & \text{if } x_l \in \Omega_0;\ l \geqslant 1, \\ \dfrac{\varphi(x_0)g_i(x_0)}{\pi(x_0)} & \text{if } x_0 \in \Omega_0, \\ 0 & \text{if } x \in \Omega_1. \end{cases}$$

Then (see [10], p. 30), under conditions (A), (B), (C), (P_1)–(P_3), (π_1)–(π_3) and $\varphi \in L_1(\Omega)$, the random variables $\eta^{(i)}$ $(i = 1, 2)$ have finite expected

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([3]) In the case $x \in \Omega_0$ $(\mathrm{mod}\,\mu)$, from condition (C) it follows that the solution $u(x)$ of equation (1.1) is given by the formula: $u(x) = g(x)$ for $x \in \Omega_0$ $(\mathrm{mod}\,\mu)$.

values. And we have:

$$(2.17)\qquad (u, \varphi) = M\eta^{(1)} + M\eta^{(2)},$$

where (u, φ) is defined by formula (1.3).

By virtue of (2.17), we may also solve the problem (1.3) by the Monte-Carlo method.

DEFINITION (2.2). The above probability model for solving the problem (1.3) is called the *model* $\bar{S}$. And the random variables $\eta^{(i)}$ $(i = 1, 2)$ defined in this model are called the *estimators in the model* $\bar{S}$. It is easy to deduce that (see (2.4)–(2.9) and (2.14)) each model $\bar{S}$ depends on functions $p(x, y)$, $\pi(x)$ satisfying conditions (P_1)–(P_3), (π_1)–(π_3) and on values of positive constants Δ, δ:

$$(2.18)\qquad \bar{S} = \bar{S}\{p(x, y),\ \pi(x), \Delta, \delta\}.$$

In order to solve conveniently some special cases of problems (1.1), (1.3), in this paper we shall choose special forms of functions $p(x, y)$, $\pi(x)$. Then, the estimators in models S (or in models $\bar{S}$) will have simple forms.

3. The solution of an integral equation in the space of measurable and bounded functions.

We consider the integral equation:

$$(3.1)\qquad u(x) - \int_{\Omega} K(x, y)u(y)\mu(dy) = g(x) \qquad (x \in \Omega)$$

in the space $M(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a space with a finite complete measure; $M(\Omega, \Sigma, \mu)$ is the space of Σ-measuralbe functions, bounded on Ω.

Suppose that the kernel function $K(x, y)$ belongs to $M(\Omega^2, \Sigma^2, \mu\times\mu)$ and it satisfies the following conditions:

$$(A_1)\qquad a_1 \equiv \sup_{x\in\Omega\setminus\Omega_0}\left\{\int_{\Omega} |K(x, y)|\mu(dy)\right\} < 1;$$

$$(B_1)\qquad K(x, y) \geqslant 0 \quad \text{for } x\in\Omega\setminus\Omega_0,\ y\in\Omega_0\ (\mathrm{mod}\,\mu);$$

$$(C_1)\qquad K(x, y) = 0 \quad \text{for } x\in\Omega_0,\ y\in\Omega\ (\mathrm{mod}\,\mu),$$

where $\Omega_0\in\Sigma$ is a set with $\mu(\Omega_0) > 0$.

Moreover, we also estimate the value of the functional:

$$(3.2)\qquad (u, \varphi) = \int_{\Omega} u(x)\varphi(x)\mu(dx),$$

where $u(x)$ is the solution of (3.1) and $\varphi\in L_1(\Omega)$.

Let $p_1(x, y)$ be the function defined on $\Omega \times \Omega$ by the formula:

(3.3) $$p_1(x, y) = \begin{cases} 1 & \text{if } K(x, y) \geqslant 0, \\ -1 & \text{if } K(x, y) < 0. \end{cases}$$

Then it is easy to deduce:

(3.4) $$a_1 = \sup_{x \in \Omega \setminus \Omega_0} \Big\{ \int_\Omega K(x, y) p_1(x, y) \mu(dy) \Big\} < 1;$$

(3.5) $$K(x, y) p_1(x, y) \geqslant 0 \quad \text{for } (x, y) \in \Omega \times \Omega;$$

(3.6) $$p_1(x, y) \neq 0 \quad \text{for } (x, y) \in \Omega \times \Omega,$$

i. e. treating $P_1(x, y)$ as the function $p(x, y)$ in Section 2, the conditions having forms (P_1)–(P_2) are satisfied. Moreover, from (A_1)–(C_1) it follows that conditions (A)-(C) of problem (1.1) are also satisfied (if the space $L_\infty(\Omega)$ is replaced by $M(\Omega, \Sigma, \mu)$). Then (see [10], p. 124), it is not difficult to deduce that the model $S\{p_1(x, y), \Delta, \delta\}$ may be used for solving problem (3.1) ([4]). Note that in this case we choose $A^* = \Phi$ (i. e. $\Omega^*_\Delta = \Omega$). Therefore (see (P_1^*), (A_1)), the constant a^* becomes a_1. And the constants G, M become (see (2.1), (2.9)):

(3.7) $$G_1 \equiv \sup_{x \in \Omega \setminus \Omega_0} \{ |g(x)| \} < +\infty,$$

(3.8) $$M_1 \equiv \sup_{(x, y) \in (\Omega \setminus \Omega_0) \times \Omega_0} \{ |p_1(x, y)| \} = 1 \quad (\text{see } (3.3)).$$

Since μ in problem (3.1) is a complete measure, we can regard it as the measure $\bar{\mu}$ in Section 2. Hence, from (2.2), (2.3) it follows:

(3.9) $$\tilde{\Sigma} = \Sigma \cup \Sigma^*; \quad \tilde{\mu}(\tilde{A}) = \mu(\tilde{A} \cap \Omega) + \delta \cdot \delta_{\tilde{A}} \quad (\text{for } \tilde{A} \in \tilde{\Sigma});$$

(3.10) $$\Sigma^* = \{\tilde{A} : \tilde{A} = A \cup \Omega^*, \ A \in \Sigma\}; \quad \delta_{\tilde{A}} = \begin{cases} 1 & \text{if } \tilde{A} \cap A\,\Omega^* \neq \Phi, \\ 0 & \text{if } \tilde{A} \cap \Omega^* = \Phi. \end{cases}$$

From (3.3) and (B_1) we have:

(3.11) $$K(x, y) p_1(x, y) = |K(x, y)| \quad \text{for } (x, y) \in \Omega \times \Omega;$$

(3.12) $$p_1(x, y) = 1 \quad x \in \Omega \setminus \Omega_0, \ y \in \Omega_0 \ (\text{mod}\, \mu).$$

Then formulae (2.6)–(2.8) have the forms:

(3.13) $$F_i(x, y) = \begin{cases} |K(x, y)| & \text{if } (x, y) \in (\Omega \setminus \Omega_0) \times (\Omega \setminus \Omega_0), \\ K(x, y) + \dfrac{g_i(x)}{\mu(\Omega_0) g_i(y)} & \text{if } (x, y) \in (\Omega \setminus \Omega_0) \times \Omega_0, \\ h_i(x) & \text{if } (x, y) \in (\Omega \setminus \Omega_0) \times \Omega^*, \end{cases}$$

([4]) Similarly, we may also use the model $\tilde{S}\{p_1(x, y), \pi(x), \Delta, \delta\}$ for solving problem (3.2)

$$(3.14)\qquad h_i(x) = \frac{1}{\delta}\left[1 - \int_{\Omega} |K(x, y)|\mu(dy) - \frac{g_i(x)}{\mu(\Omega_0)} \int_{\Omega} \frac{\mu(dy)}{g_i(y)}\right] \quad (x \in \Omega \setminus \Omega_0),$$

$$(3.15)\qquad g_i(x) = -\left[|g(x)| + \chi_{\Omega_0}(x)\left(\frac{2G_1}{1-a_1} + \Delta\right)\right];$$

$$g_2(x) = g(x) - g_1(x).$$

By virtue of (2.4), (2.5) and (3.13)-(3.15), we can construct the i-th process in the models $S\{p_1(x, y), \Delta, \delta\}$, $\bar{S}\{p_1(x, y), \pi(x), \Delta, \delta\}$ for solving problems (3.1), (3.2). Let $N(x_0, x_1, \ldots, x_n)$ be the number of negative terms of the set $\{K(x_0, x_1); K(x_1, x_2); \ldots; K(x_{n-1}, x_n)\}$. Then, from (3.3), (3.5) it follows:

$$(3.16)\qquad \prod_{i=1}^{n} p_1(x_{i-1}, x_i) = (-1)^{N(x_0, x_1, \ldots, x_{n-1})} \quad \text{for } x_0, \ldots, x_{n-1} \in \Omega \setminus \Omega_0; \; x_n \in \Omega_0.$$

Therefore, by (2.11), we have the following theorem:

THEOREM (3.1). *Under assumptions* (A_1), (B_1), (C_1), *suppose that the function* $p_1(x, y)$ *is defined by* (3.3). *Then the estimators* $\xi_1^{(i)}(x)$ $(i = 1, 2;$ $x \in \Omega \setminus \Omega_0)$ *in the model* $S\{p_1(x, y), \Delta, \delta\}$ *solving problem* (3.1) *are defined by the formula*:

$$(3.17)\qquad \xi_1^{(i)}(x) = f_1^{(i)}(x; x_1, \ldots, x_l) \equiv \begin{cases} g_i(x_l)\cdot(-1)^{N(x, x_1, \ldots, x_{l-1})} & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*. \end{cases}$$

COROLLARY (3.1). *Under assumptions* (A_1), (C_1), *suppose that*:

$$(B_1^*)\qquad K(x, y) \geqslant 0 \quad \text{for } x \in \Omega \setminus \Omega_0,\ y \in \Omega \ (\operatorname{mod} \mu).$$

Then the estimators $\bar{\xi}_1^{(i)}(x)$ $(i = 1, 2;$ $x \in \Omega \setminus \Omega_0)$ *in the model* $S\{p_1(x, y), \Delta, \delta\}$ *have the forms*:

$$(3.18)\qquad \bar{\xi}_1^{(i)}(x) = \bar{f}_1^{(i)}(x, x_1, \ldots, x_l) \equiv \begin{cases} g_i(x_l) & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*. \end{cases}$$

Proof. From (3.3) and condition (B_1^*) it follows:

$$(3.19)\qquad N(x, x_1, \ldots, x_{l-1}) = 0 \quad \text{for } x, x_1, \ldots, x_{l-1} \in \Omega \setminus \Omega_0.$$

Therefore, using Theorem (3.1) we have (3.18). This completes the proof.

From (2.16), (3.16) it is easy to deduce the following theorem:

THEOREM (3.2). *Under assumptions* (A_1), (B_1), (C_1), (π_1)-(π_3) *and* $\varphi \in L_1(\Omega)$, *the estimators* $\eta_1^{(i)}$ *in the model* $\bar{S}\{p_1(x, y), \pi(x), \Delta, \delta\}$ *solving* *problem* (3.2) *are defined by the formula:*

$$(3.20)\quad \eta_1^{(i)} = F_1^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} \dfrac{\varphi(x_0) g_i(x_l)}{\pi(x_0)} \cdot (-1)^{N(x_0, x_1, \ldots, x_{l-1})} & \text{if } x_l \in \Omega_0), \\ 0 & \text{if } x_l \in \Omega^*, \end{cases}$$

where

$$(3.21)\qquad N(x_0) \equiv 0 \quad \text{for } x_0 \in \Omega_0.$$

COROLLARY (3.2). *Under assumptions* (A_1), (B_1^*), (C_1), (π_1)–(π_3) *and* $\varphi \in L_1(\Omega)$, *the estimators* $\bar{\eta}_1^{(i)}$ *in the model* $\bar{S}\{p_1(x, y), \pi(x), \Delta, \delta\}$ *have the form:*

$$(3.22)\qquad \bar{\eta}_1^{(i)} = \bar{F}_1^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} \dfrac{\varphi(x_0) g_i(x_l)}{\pi(x_0)} & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*. \end{cases}$$

COROLLARY (3.3) *Under assumptions* (A_1), (B_1), (C_1), *suppose that the given function* $\varphi(x)$ *satisfies the following conditions:*

$$(3.23)\qquad \int_\Omega \varphi(x)\mu(dx) = 1; \quad \varphi(x) > 0 \quad \text{for } x \in \Omega \ (\mathrm{mod}\,\mu).$$

Then the estimators $\tilde{\eta}_1^{(i)}$ $(i = 1, 2)$ *in the model* $\bar{S}\{p_1(x, y), \pi_1(x), \Delta, \delta\}$ *are defined by the formula:*

$$(3.24)\qquad \tilde{\eta}_1^{(i)} = \tilde{F}_1^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} g_i(x_l) & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*, \end{cases}$$

where the function $\pi_1(x)$ *is defined by*

$$(3.25)\qquad \pi_1(x) = \begin{cases} \varphi(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \Omega^*. \end{cases}$$

Proof. Using Theorem (3.2), from (3.19) we get (3.22). Corollary (3.2) is proved.

By (3.23) it follows that the function $\pi_1(x)$ defined by (3.25) satisfies conditions (π_1)–(π_3). Therefore, replacing the function $\pi(x)$ in Corollary (3.2) by the function $\pi_1(x)$ from (3.22), we get (3.24). Corollary (3.3) is proved.

As an example of problem (3.2), we may consider the following problem discussed in [6]:

Let Ω be the phase space of coordinates and velocity. It is known (see [8]) that one may treat a transport process as a homogeneous Markov chain, the states of which are "the position" of the particle just before the collisions in the phase space Ω.

Let $K(x, y)$ be the probability density of the transfer from the state x to a state y of this Markov chain. Then the solution of many problems in the transport theory leads to the estimation of the functional

$$J_0 = (g, u^*) = \int_\Omega g(x) u^*(x) dx, \tag{3.26}$$

where $u^*(x)$ is the solution of the integral transport equation:

$$u^*(x) - \int_\Omega K(y, x) u^*(y) dy = \varphi(x); \tag{3.27}$$

$g(x)$ is the weight function defined on Ω; $\varphi(x)$ is the frequency distribution function of the initial collisions aroused by the particles of the physical source.

In [6] problem (3.26) has been solved by the Monte-Carlo method under the following assumptions:

$$g(x) > 0 \quad \text{for } x \in \Omega \text{ and } g(x) \in M(\Omega), \tag{3.28}$$

where $M(x)$ is the space of functions bounded on Ω;

$$\int_\Omega \varphi(x) dx = 1 \quad \text{and} \quad \varphi(x) > 0 \quad \text{for } x \in \Omega; \tag{3.29}$$

$$K(x, y) \geqslant 0 \quad \text{for } (x, y) \in \Omega \times \Omega; \tag{3.30}$$

$$\sup_{x \in \Omega} \left\{ \int_\Omega K(x, y) dy \right\} < 1, \tag{3.31}$$

where $\Omega \subset R^n$ (R^n is the n-dimensional Euclidean space); and the integral are understood in the sense of the Lebesgue measure on R^n.

In order to construct a probability model for solving problem (3.26), the homogeneous Markov chain $\{\varphi(x), K(x, y)\}$ has been defined (see [3], [6]), where $\varphi(x)$ is the density of the initial distribution and $K(x, y)$ is the density of the transition probability.

Let

$$a(x) = 1 - \int_\Omega K(x, y) dy. \tag{3.32}$$

On the sample space of trajectories

$$x_0 \to x_1 \to \ldots \to x_l \tag{3.33}$$

corresponding to the value of the probability density

$$\varphi(x_0)K(x_0, x_1)\ldots K(x_{l-1}, x_l)a(x_l), \tag{3.34}$$

the random variable $\eta[l]$ is defined as follows:

$$\eta[l] = \sum_{i=0}^{l} g(x_i). \tag{3.35}$$

Then it is known (see [3]) that

$$M\eta[l] = J_0 = (g, u^*), \tag{3.36}$$

i. e. $\eta[l]$ is the unbiased estimator of J_0 (see [4]; p. 244).

To compare this unbiased estimator with the estimators in the model $\tilde{S}\{p_1(x, y), \pi_1(x), \Delta, \delta\}$ we suppose that there exists "the absorption region" $\Omega_0 \subset \Omega$. Then it is known (see [10], p. 123) that

$$K(x, y) = 0 \quad \text{for } x \in \Omega_0, y \in \Omega \ (\text{mod}\, \mathscr{L}), \tag{3.37}$$

where $\mathscr{L}$ is the Lebesgue measure on R^n.

Moreover, it is easy to deduce that functional (3.26) may be written as

$$J_0 = (u, \varphi) \equiv \int_\Omega u(x)\varphi(x)\,dx, \tag{3.38}$$

where $u(x)$ is the solution in $M(\Omega)$ of the equation

$$u(x) - \int_\Omega K(x, y)u(y)\,dy = g(x). \tag{3.39}$$

Therefore, from (3.31), (3.30), (3.37), (3.29) it follows that the assumptions of corollary (3.3) are satisfied. Hence, we can use the estimators $\tilde{\eta}_1^{(i)}$ $(i = 1, 2)$ in the model $\tilde{S}\{p_1(x, y), \pi_1(x), \Delta, \delta\}$ for solving problem (3.38) (i. e. problem (3.26)). By (3.15), (3.24) we deduce that using the estimators $\tilde{\eta}_1^{(i)}$, in each "experiment" we only have to compute at most one value of the function $g(x)$. Therefore, if the calculation of the values of $g(x)$ is difficult, then the estimators $\tilde{\eta}_1^{(i)}$ $(i = 1, 2)$ are more convenient than the unbiased estimator $\eta[l]$ (see (3.35)).

In Sections 4 and 5 we shall solve problems similar to problems (3.1), (3.2) for other classes of integral equations in the space $M(\Omega, \Sigma, \mu)$.

4. First of all, we consider the solution of the equation:

$$u(x) - \int_\Omega K(x, y)u(y)\mu(dy) = g(x) \tag{4.1}$$

in the space $M(\Omega, \Sigma, \mu)$. Suppose that there exists a set $\Omega_0 \in \Sigma$ such that $\mu(\Omega_0) > 0$ and it fulfils the following conditions:

$$(A_2) \qquad \bar{a}_2 \equiv \sup_{x \in \Omega \setminus \Omega_0} \Big\{ \int_{\Omega \setminus \Omega_0} K(x, y)\mu(dy) \Big\} < 1;$$

$$(B_2) \qquad K(x, y) \geqslant 0 \quad \text{for } x \in \Omega \setminus \Omega_0,\ y \in \Omega \ (\mathrm{mod}\,\mu);$$

$$(C_2) \qquad K(x, y) = 0 \quad \text{for } x \in \Omega_0,\ y \in \Omega \ (\mathrm{mod}\,\mu);$$

$$(D_2) \qquad a_2^* \equiv \sup_{x \in \Omega \setminus \Omega_0} \Big\{ \int_{\Omega} K(x, y)\mu(dy) \Big\} < +\infty \ ({}^5).$$

We also estimate the value of the functional:

$$(4.2) \qquad (u, \varphi) = \int_{\Omega} u(x)\varphi(x)\mu(dx),$$

where $u(x)$ is the solution of (4.1) and $\varphi \in L_1(\Omega)$.

Let $\bar{p}_2$ be a constant satisfying the conditions:

$$(4.3) \qquad 0 < \bar{p}_2 < 1; \quad \bar{p}_2(a_2^* - \bar{a}_2) < 1 - \bar{a}_2 \ ({}^6).$$

It defines a function $p_2(x, y)$ on $\Omega \times \Omega$ as follows:

$$(4.4) \qquad p_2(x, y) = \begin{cases} 1 & \text{for } (x, y) \in (\Omega \setminus \Omega_0) \times (\Omega \setminus \Omega_0), \\ \bar{p}_2 & \text{for } (x, y) \in (\Omega \setminus \Omega_0) \times \Omega_0, \\ 1 & \text{for } (x, y) \in \Omega_0 \times \Omega. \end{cases}$$

Then

$$(4.5) \qquad \int_{\Omega} K(x, y)p_2(x, y)\mu(dy) = (1 - \bar{p}_2) \int_{\Omega \setminus \Omega_0} K(x, y)\mu(dy) + \\ + \bar{p}_2 \int_{\Omega} K(x, y)\mu(dy) \qquad (\text{for } x \in \Omega \setminus \Omega_0).$$

Therefore, from (A_2), (D_2), (4.3) we deduce

$$(4.6) \qquad a_2 = \sup_{x \in \Omega \setminus \Omega_0} \Big\{ \int_{\Omega} K(x, y)p_2(x, y)\mu(dy) \Big\} \leqslant (1 - \bar{p}_2)\bar{a}_2 + \bar{p}_2 a_2^* < 1.$$

Moreover, by (B_2), (C_2), (4.3) it follows

$$(4.7) \qquad K(x, y)p_2(x, y) \geqslant 0 \quad \text{for } x \in \Omega,\ y \in \Omega \ (\mathrm{mod}\,\mu),$$

$$(4.8) \qquad p_2(x, y) \neq 0 \quad \text{for } (x, y) \in \Omega \times \Omega.$$

---

(5) It is obvious that $a_2^* \geqslant \bar{a}_2$.

(6) I. e., if $a_2^* = \bar{a}_2$ we choose $\bar{p}_2$ such that $0 < \bar{p}_2 < 1$, and if $a_2^* > \bar{a}_2$ we choose $\bar{p}_2$ with $0 < \bar{p}_2 < \min\left\{1, \dfrac{1 - \bar{a}_2}{a_2^* - \bar{a}_2}\right\}$.

From (4.6)–(4.8) it is easy to see that regarding $p_2(x, y)$ as the function $p(x, y)$ in Section 2, the conditions of the form (P_1)–(P_3) are satisfied. Besides, from (A_2)–(C_2) it follows that conditions (A)–(C) of problem (1.1) are also satisfied (if the space $L_\infty(\Omega)$ is replaced by $M(\Omega, \Sigma, \mu)$). Like in Section 3, we can use the models $S\{p_2(x, y), \Delta, \delta\}$, $\bar{S}\{p_2(x, y), \pi(x), \Delta, \delta\}$ for solving problems (4.1), (4.2). In this case the measure space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ is also defined by formulae (3.9), (3.10). The transition probabilities $P_i(k, x, \tilde{A})$ of the i-th processes are defined by (2.4) and the following formulae:

$$P_i(1, x, \tilde{A}) = \begin{cases} \int_{\tilde{A}} F_i(x, y)\tilde{\mu}(dy) & \text{for } x \in \Omega\setminus\Omega_0, \\ \chi_{\tilde{A}}(x) & \text{for } x \in \Omega_0 \cup \Omega^*, \end{cases} \tag{4.9}$$

$$F_i(x, y) = \begin{cases} K(x, y) & \text{for } (x, y) \in (\Omega\setminus\Omega_0)\times(\Omega\setminus\Omega_0), \\ \bar{p}_2 K(x, y) + \dfrac{\bar{p}_2 g_i(x)}{\mu(\Omega_0) g_i(y)} & \text{for } (x, y) \in (\Omega\setminus\Omega_0)\times\Omega_0, \\ h_i(x) & \text{for } (x, y) \in (\Omega\setminus\Omega_0)\times\Omega^*, \end{cases} \tag{4.10}$$

$$h_i(x) = \frac{1}{\delta}\Big[1 - \int_{\Omega\setminus\Omega_0} K(x, y)\mu(dy) - \bar{p}_2\int_{\Omega_0} K(x, y)\mu(dy) - \frac{\bar{p}_2 g_i(x)}{\mu(\Omega_0)}\int_{\Omega_0}\frac{\mu(dy)}{g_i(y)}\Big], \tag{4.11}$$

$$g_1(x) = -\Big[|g(x)| + \chi_{\Omega_0}(x)\Big(\frac{2G_1\bar{p}_2}{1-a_2}\Big) + \Delta\Big]; \tag{4.12}$$

$$g_2(x) = g(x) - g_1(x),$$

where the constant G_1 is defined by (3.7).

From (4.4) it follows

$$\prod_{i=0}^{p} p_2(x_{i-1}, x_i) = \bar{p}_2 \quad (\text{for } x_0, x_1, \ldots, x_{n-1} \in \Omega\setminus\Omega_0;\ x_n \in \Omega_0). \tag{4.13}$$

Therefore, by (2.11), (2.16) it is easy to deduce the following theorems:

THEOREM (4.1). *Under assumptions (A_2)–(D_2), suppose that the function $p_2(x, y)$ is defined by (4.4). Then the estimators*

$$\xi_2^{(i)}(x) \quad (i = 1, 2;\ x \in \Omega\setminus\Omega_0)$$

in the model $S\{p_2(x, y), \Delta, \delta\}$ *solving problem* (4.1) *are defined as follows:*

$$(4.14)\qquad \xi_2^{(i)}(x) = f_2^{(i)}(x; x_1, \ldots, x_l) \equiv \begin{cases} (\overline{p}_2)^{-1} g_i(x_l) & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*. \end{cases}$$

THEOREM (4.2). *Under assumptions* (A_2)–(D_2), (π_1)–(π_3) *and* $\varphi \in L_1(\Omega)$, *the estimators* $\eta_2^{(i)}$ $(i = 1, 2)$ *in the model* $\overline{S}\{p_2(x, y), \pi(x), \Delta, \delta\}$ *solving problem* (4.2) *are defined by the formula*

$$(4.15)\qquad \eta_2^{(i)} = F_2^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} \dfrac{\varphi(x_0) g_i(x_l)}{\pi(x_0)\overline{p}_2} & \text{if } x_l \in \Omega_0;\ l \geqslant 1, \\ \dfrac{\varphi(x_0) g_i(x_0)}{\pi(x_0)} & \text{if } x_0 \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*. \end{cases}$$

COROLLARY (4.1). *Under assumptions* (A_2)–(D_2), *suppose that the given function* $\varphi(x)$ *fulfils condition* (3.23) *and the given function* $g(x)$ *satisfies the condition*

$$(4.16)\qquad g(x) = 0 \quad (\text{for } x \in \Omega_0 \ (\operatorname{mod} \mu)).$$

Then the estimators $\tilde{\eta}_2^{(i)}$ $(i = 1, 2)$ *in the model* $\overline{S}\{p_2(x, y), \pi_1(x), \Delta, \delta\}$ *solving problem* (4.2) *are defined as follows:*

$$(4.17)\qquad \tilde{\eta}_2^{(i)} = \tilde{F}_2^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} a_i \cdot (\overline{p}_2)^{-1} & \text{if } x_l \in \Omega_0,\ l \geqslant 1, \\ a_i & \text{if } x_0 \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*, \end{cases}$$

where the function $\pi_1(x)$ *is defined by* (3.25) *and*

$$a_i = (-1)^i \left(\frac{2G_1 \overline{p}_2}{1 - a_2} + \Delta \right) \qquad (i = 1, 2).$$

Proof. We know that function $\pi_1(x)$ defined by (3.25) satisfies conditions (π_1)–(π_3). Moreover, from (4.16) and (4.12) it follows

$$g_i(x) \equiv (-1)^i \left(\frac{2G_1 \overline{p}_2}{1 - a_2} + \Delta \right) \quad \text{for } x \in \Omega_0 \ (\operatorname{mod} \mu).$$

Therefore, using Theorem (4.2), from (4.15) we get (4.17). This completes the proof.

As a special case of problem (4.2) we can consider the problem discussed in [8] and described below.

Problem (3.26) of the theory of the transport equation has also been solved by the Monte-Carlo method in the mentioned paper, under assumptions (3.29), (3.30), (3.37) and the following assumptions:

$$\int_{\Omega} K(x, y)\,dy \leqslant 1 \quad \text{for } x \in \Omega \setminus \Omega_0 \ (^7), \tag{4.18}$$

$$g(x) = 0 \quad \text{for } x \in \Omega_0 \text{ and } g \in M(\Omega); \tag{4.19}$$

$$\int_{\Omega \setminus \Omega_0} K(x, y)\,dy < 1-\gamma \quad \text{for } x \in \Omega \ (0 < \gamma < 1) \ (^8). \tag{4.20}$$

The probability model for solving the mentioned problem has been constructed in the following way (see [8]):

Let $\eta[\infty]$ be the random variable defined by

$$\eta[\infty] = \sum_{l=0}^{\infty} g(x_l), \tag{4.21}$$

where $x_0 \to x_1 \to \ldots \to x_l \to \ldots$ is a trajectory of the homogeneous Markov chain $\{\varphi(x), K(x, y)\}$.

It is known that

$$M\eta[\infty] = J_0 = (g, u^*), \tag{4.22}$$

i. e. $\eta[\infty]$ is the unbiased estimator of J_0 (see [4], p. 244).

In order to compare the unbiased estimator $\eta[\infty]$ with the estimators in the model $\bar{S}\{p_2(x, y), \pi_1(x), \Delta, \delta\}$, we note that functional (3.26) may be written in form (3.38). Moreover, from (4.20), (3.30), (3.37), (4.18), (3.29) and (3.38) it follows that the assumptions of Corollary (4.1) are satisfied. Hence wec an also use the estimators $\tilde{\eta}_2^{(i)}$ in the model $\bar{S}\{p_2(x, y), \pi_1(x), \Delta, \delta\}$ for estimating the value of J_0.

Obviously, in this case the estimators $\tilde{\eta}_2^{(i)}$ $(i = 1, 2)$ are more convenient than the unbiased estimator $\eta[\infty]$ (see (4.17), (4.21)).

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(7) Since, at each state x outside "the absorption region", the particle may either fly out or transfer to a state of the phase space Ω (with the probability $P(x \to \Omega) = \int_{\Omega} K(x, y)\,dy$).

(8) I. e. the probability of the absorption or of the flight at each transition is greater than γ (see [8], p. 591).

5. Now we consider the solution of the equation

$$u(x) - \int_{\Omega} K(x, y) u(y) \mu(dy) = g(x) \tag{5.1}$$

in the space $M(\Omega, \Sigma, \mu)$. Suppose that there exists a set $\Omega_0 \in \Sigma$ such that $\mu(\Omega_0) > 0$ and it fulfils the following conditions:

(A_3) For all $f \in M(\Omega, \Sigma, \mu)$ the series $\sum_{n=0}^{\infty} [T^n f](x)$ converges in $M(\Omega, \Sigma, \mu)$, where the integral operator T is defined by the formula

$$[Tf](x) = \int_{\Omega \setminus \Omega_0} K(x, y) f(y) \mu(dy);$$

(B_3) $$K(x, y) \geqslant 0 \quad \text{for } x \in \Omega \setminus \Omega_0,\ y \in \Omega \ (\mathrm{mod}\, \mu);$$

(C_3) $$K(x, y) = 0 \quad \text{for } x \in \Omega_0,\ y \in \Omega \ (\mathrm{mod}\, \mu);$$

(D_3) $$\int_{\Omega} K(x, y) \mu(dy) \equiv a_3^* \quad \text{for all } x \in \Omega \setminus \Omega_0,$$

where a_3^* is a positive constant.

Moreover, we also consider an estimation of the value of the functional

$$(u, \varphi) = \int_{\Omega} u(x) \varphi(x) \mu(dx), \tag{5.2}$$

where $u(x)$ is the solution of (5.1) and $\varphi \in L_1(\Omega)$.

Let $\bar{p}_3$ be a constant satisfying the condition

$$0 < \bar{p}_3 < (a_3^*)^{-1}; \tag{5.3}$$

we define the function $p_3(x, y)$ on $\Omega \times \Omega$ by

$$p_3(x, y) \equiv \bar{p}_3 \quad \text{for all } (x, y) \in \Omega \times \Omega. \tag{5.4}$$

Then, from (D_3), (5.3), it is easy to see that, regarding $p_3(x, y)$ as the function $p(x, y)$ in Section 2, the conditions of the form (P_1)–(P_3) are satisfied. Besides, from (A_3)–(C_3) it follows that conditions (A)–(C) of problem (1.1) are also satisfied (if the space $L_\infty(\Omega)$ is replaced by $M(\Omega, \Sigma, \mu)$. Therefore, like in Section 3, we can use the models $\bar{S}\{\bar{p}_3, \pi(x), \Delta, \delta\}$ and $S\{\bar{p}_3, \Delta, \delta\}$ for solving problems (5.2), (5.1). In this case, the measure space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ is also defined by formulae (3.9), (3.10). And the transition probabilities $P_i(k, x, \tilde{A})$ of the i-th processes are defined by (2.4) and the following formulae:

$$(5.5)\qquad P_i(1, x, \tilde{A}) = \begin{cases} \int_{\tilde{A}} F_i(x, y)\tilde{\mu}(dy) & \text{for } x \in \Omega\setminus\Omega_0, \\ \chi_{\tilde{A}}(x) & \text{for } x \in \Omega_0 \cup \Omega^*, \end{cases}$$

$$(5.6)\qquad F_i(x, y) = \begin{cases} \overline{p}_3 K(x, y) & \text{for } (x,y) \in (\Omega\setminus\Omega_0)\times(\Omega\setminus\Omega_0), \\ \overline{p}_3 K(x, y) + \dfrac{\overline{p}_3 g_i(x)}{\mu(\Omega_0) g_i(y)} & \text{for } (x, y) \in (\Omega\setminus\Omega_0)\times\Omega_0, \\ h_i(x) & \text{for } (x, y) \in (\Omega\setminus\Omega_0)\times\Omega^*, \end{cases}$$

$$(5.7)\qquad h_i(x) = \frac{1}{\delta}\left[1 - \overline{p}_3 \int_{\Omega} K(x, y)\mu(dy) - \frac{\overline{p}_3 g_i(x)}{\mu(\Omega_0)} \int_{\Omega_0} \frac{\mu(dy)}{g_i(y)}\right] \quad (x \in \Omega\setminus\Omega_0),$$

$$(5.8)\qquad \begin{aligned} g_1(x) &= -\left[|g(x)| + \chi_{\Omega_0}(x)\left(\frac{2G_1\overline{p}_3}{1-\overline{p}_3 a_3^*} + \Delta\right)\right]; \\ g_2(x) &= g(x) - g_1(x). \end{aligned}$$

Then, from (2.11), (2.16) it is easy to deduce the following theorems:

THEOREM (5.1). *Under assumptions* (A_3)–(D_3), *suppose that* $\overline{p}_3$ *is a constant satisfying condition* (5.3). *Then the estimators* $\xi_3^{(i)}(x)$ $(i = 1, 2;$ $x \in \Omega\setminus\Omega_0)$ *in the model* $S\{\overline{p}_3, \Delta, \delta\}$ *solving problem* (5.1) *are defined by the formula*

$$(5.9)\qquad \xi_3^{(i)}(x) = f_3^{(i)}(x; x_1, \ldots, x_l) \equiv \begin{cases} (\overline{p}_3)^{-l} g_i(x_l) & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*. \end{cases}$$

THEOREM (5.2). *Under assumptions* (A_3)–(D_3), (π_1)–(π_3) *and* $\varphi \in L_1(\Omega)$, *the estimators* $\eta_3^{(i)}$ $(i = 1, 2)$ *in the model* $\overline{S}\{\overline{p}_3, \pi(x), \Delta, \delta\}$ *solving problem* (5.2) *are defined by the formula*

$$(5.10)\qquad \eta_3^{(i)} = F_3^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} \dfrac{\varphi(x_0) g_i(x_l)}{\pi(x_0)(\overline{p}_3)^l} & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*. \end{cases}$$

COROLLARY (5.1). *Under assumptions* (A_3)–(D_3), *suppose that the given function* $\varphi(x)$ *satisfies condition* (3.23) *and the function* $g(x)$ *satisfies condition* (4.16).

Then the estimators $\tilde{\eta}_3^{(i)}$ $(i = 1, 2)$ *in the model* $\overline{S}\{\overline{p}_3, \pi_1(x), \Delta, \delta\}$ *solving problem* (5.2) *are defined by the formula*

$$(5.11)\qquad \tilde{\eta}_3^{(i)} = \tilde{F}_3^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} (-1)^i (\overline{p}_3)^{-l} a & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega^*, \end{cases}$$

where the function $\pi_1(x)$ *is defined by* (3.25) *and*

$$a = \frac{2G_1\bar{p}_3}{1-\bar{p}_3 a_3^*} + \Delta. \tag{5.12}$$

Proof. We know that the function $\pi_1(x)$ defined by (3.25) satisfies conditions (π_1)–(π_3). Moreover, from (4.16), (5.8) it follows

$$g_i(x) = (-1)^i \left(\frac{2G_1\bar{p}_3}{1-\bar{p}_3 a_3^*} + \Delta\right) \quad \text{for all } x \in \Omega_0 \ (\mathrm{mod}\,\mu).$$

Hence, using Theorem (5.2), from (5.10) we get (5.11). This completes the proof.

As an example of problem (5.2) we may consider the problem discussed in [9].

In that paper, problem (3.26) of the theory of the transfer equation has also been solved by the Monte-Carlo method, assuming (3.29), (3.30), (3.37), (4.19) and the following conditions are satisfied:

(5.13) The Neumann series for integral equation (3.39) converges in $M(\Omega)$;

$$\int_\Omega K(x, y)\,dy = 1 \quad \text{for all } x \in \Omega. \tag{5.14}$$

In [9] (see p. 1085–1086) the unbiased estimator $\eta[\infty]$ of J_0 (see (4.21)) is used to solve problem (3.26). In order to compare this unbiased estimator $\eta[\infty]$ with the estimators in the model $\bar{S}\{\bar{p}_3, \pi_1(x), \Delta, \delta\}$, we note that functional (3.26) may be written in the form (3.38). Moreover, from (3.29), (3.30), (3.37), (4.19), (5.13) and (5.14) it follows that the assumptions of Corollary (5.1) are satisfied. Therefore we can also use the estimators $\tilde{\eta}_3^{(i)}$ in the model $\bar{S}\{\bar{p}_3, \pi_1(x), \Delta, \delta\}$ for estimating the value of J_0. Obviously in this case the estimators $\tilde{\eta}_3^{(i)}$ are more convenient than the unbiased estimator $\eta[\infty]$ (see (4.21), (5.11)).

6. The solution of some systems of linear algebraic equations. Now we shall point out some applications of the models S for solving equations of the form (1.1), when Ω is a discrete set.

First of all, consider the following system of linear algebraic equations:

$$u - Au = b, \tag{6.1}$$

where $u = (u_1, u_2, \ldots, u_n)$ is the solution of (6.1) and $b = (b_1, b_2, \ldots, b_n)$ is a given vector. Suppose that the elements a_{ij} of the matrix $A = (a_{ij})_{n\times n}$

satisfy the following conditions:

(6.2) $$\|A\| \equiv \max_{i \in N \setminus N_0} \Big\{ \sum_{j=1}^{n} |a_{ij}| \Big\} < 1;$$

(6.3) $$a_{ij} \geqslant 0 \quad \text{for } i \in N \setminus N_0;\ j \in N_0;$$

(6.4) $$a_{ij} = 0 \quad \text{for } i \in N_0;\ j \in N,$$

where

(6.5) $$N = \{1, 2, \ldots, n\}; \qquad N_0 = \{n_1, n_2, \ldots, n_s\} \subset N \quad (n > s \geqslant 1).$$

It is clear that we may treat this problem as a special case of problem (3.1) in the following way:

Let $\Omega = N \equiv \{1, 2, \ldots, n\}$ and let Σ be the class of all subsets of Ω. Then the measure μ of a set $\{i_1, i_2, \ldots, i_m\}$ of the σ-field Σ may be defined by

(6.6) $$\mu(\{i_1, i_2, \ldots, i_m\}) = m \quad \text{and} \quad \mu(\Phi) = 0.$$

Hence, $\mu(\Omega) = \mu(N) = n < +\infty$ and μ is complete measure on Σ, i. e. (Ω, Σ, μ) is a space with a finite complete measure. Then equation (3.1) becomes a system of linear algebraic equations (6.1). From (6.2)–(6.4) it follows that conditions (A_1)–(C_1) are satisfied, where $K(i, j) = a_{ij}$, $\Omega_0 = N_0 \equiv \{n_1, n_2, \ldots, n_s\}$ and $\mu(\Omega_0) = \mu(N_0) = s > 0$.

Therefore we can use the model S of Section 3 for solving problem (6.1) by the Monte-Carlo method. In this case, one may choose the number $n+1$ as the set Ω^*, and $\delta = 1$. Then, from (2.2), (3.9), (3.10) we have

(6.7) $$\tilde{\Omega} = N \cup \{n+1\} = \{1, 2, \ldots, n+1\};$$

(6.8) $$\tilde{\mu}(\{i_1, i_2, \ldots, i_m\}) = m \quad \text{for } \{\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_m\} \in \tilde{\Sigma},$$

where $\tilde{\Sigma}$ is the class of all subsets of $\tilde{\Omega} = \{1, 2, \ldots, n+1\}$. Then the i-th processes become Markov chains consisting of $n+1$ states $1, 2, \ldots, n+1$. And these i-th processes are called the *i-th chains.*

It is easy to deduce (see (2.5), (3.13)–(3.15)) that the probability $p_{kj}^{(i)}$ of the transfer from the state k to the state j in the i-th chain is defined as follows

(6.9) $$p_{kj}^{(i)} = \begin{cases} |a_{kj}| & \text{if } k \in N \setminus N_0;\ j \in N \setminus N_0, \\ a_{kj} + \dfrac{b_k^{(i)}}{s b_j^{(i)}} & \text{if } k \in N \setminus N_0;\ j \in N_0, \\ 1 - \displaystyle\sum_{j=1}^{n} |a_{kj}| - \frac{b_k^{(i)}}{s} \sum_{j \in N_0} (b_l^{(i)})^{-1} & \text{if } k \in N \setminus N_0;\ j = n+1, \\ 1 & \text{if } k = j \in \hat{N} \equiv \{n_1, n_2, \ldots, n_s, n+1\}, \\ 0 & \text{if } k \in \hat{N};\ j \neq k, \end{cases}$$

where

$$(6.10)\qquad b_k^{(1)} = -|b_k| + \chi_{N_0}(k)\left(\frac{2B}{1-\|A\|} + \varDelta\right); \qquad b_k^{(2)} = b_k - b_k^{(1)} \qquad (k \in N),$$

$$(6.11)\qquad B = \max_{i \in N \setminus N_0} \{|b_i|\} \qquad (\text{see } (3.7)).$$

It is known (see (3.3)) that in this case the function $p_1(x, y)$ has the form

$$(6.12)\qquad p_1(i, j) = \begin{cases} 1 & \text{if } a_{ij} \geqslant 0, \\ -1 & \text{if } a_{ij} < 0. \end{cases}$$

Thus we can construct the model $S\{p_1(i, j), \varDelta, 1\}$ for solving problem (6.1) (see definition (2.1)). And from Theorem (3.1) and Corollary (3.1) it follows

THEOREM (6.1). *Under assumptions* (6.2)–(6.5), *suppose that on the space of all trajectories of the i-th chain, with the same initial state* $k \in N \setminus N_0$, *of the form:*

$$(6.13)\qquad k \to k_1 \to \ldots \to k_l \qquad (k_l \in \hat{N};\ k_1, \ldots, k_{l-1} \in N \setminus N_0),$$

we define random variables $\xi_1^{(i)}(k)$ *as follows*

$$(6.14)\qquad \xi_1^{(i)}(k) = f_1^{(i)}(k; k_1, \ldots, k_{l-1}) \equiv \begin{cases} b_{k_l}^{(i)}(-1)^{N(k, k_1, \ldots, k_{l-1})} & \text{if } k_l \in N_0, \\ 0 & \text{if } k_l = n+1, \end{cases}$$

where $N(k, k_1, \ldots, k_{l-1})$ *is the number of negative terms of the set* $\{a_{kk_1}, a_{k_1k_2}, \ldots, a_{k_{l-2}k_{l-1}}\}$.

Then the expected values of the random variables $\xi_1^{(i)}(k)$ $(i = 1, 2)$ *exist and are finite. And the solution* $u = (u_1, u_2, \ldots, u_n)$ *of* (6.1) *is given by the formula*

$$(6.15)\qquad u_k = M\xi_1^{(1)}(k) + M\xi_1^{(2)}(k) \qquad (k \in N \setminus N_0)\ (^9),$$

i. e. $\xi_1^{(i)}(k)$ $(i = 1, 2)$ *are the estimators in the model* $S\{p_1(i, j), \varDelta, 1\}$ *for solving problem* (6.1).

COROLLARY (6.1). *Under assumptions* (6.2), (6.3), (6.5), *suppose that*

$$(6.16)\qquad a_{ij} \geqslant 0 \quad \textit{for } i \in N \setminus N_0,\ j \in N.$$

Then the estimators $\bar{\xi}_1^{(i)}(k)$ $(i = 1, 2;\ k \in N \setminus N_0)$ *in the model* $S\{p_1(i, j), \varDelta, 1\}$ *solving problem* (6.1) *are defined by the formula*

$$(6.17)\qquad \bar{\xi}_1^{(i)}(k) = \bar{f}_1^{(i)}(k; k_1, \ldots, k_l) \equiv \begin{cases} b_{k_l}^{(i)} & \text{if } k_l \in N_0, \\ 0 & \text{if } k_l = n+1. \end{cases}$$

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(9) It is known (see (6.4)) that $u_k = b_k$ for $k \in N_0$.

We know that various probability models have been constructed for solving problem (6.1). If assumption (6.3) is replaced by (6.16), the following Von Neumann–Ulam model will be the most convenient one among the considered models (see [11], p. 140).

Let

$$(6.18)\qquad p_{ij} = \begin{cases} a_{ij} & \text{if } i \in N,\ j \in N, \\ 1 - \sum_{j=1}^{n} a_{ij} & \text{if } i \in N,\ j = n+1, \\ 1 & \text{if } i = j = n+1, \\ 0 & \text{if } i = n+1,\ j \neq i, \end{cases}$$

be the probability of the transfer from the state i to the j of the Markov chain consisting of $n+1$ states $1, 2, \ldots, n+1$. On the space of all trajectories of this Markov chain, with the same initial state k, of the form

$$(6.19)\qquad k \to k_1 \to \ldots \to k_l = n+1 \quad (k, k_1, \ldots, k_{l-1} \in N),$$

the random variable $\hat{\xi}_1(k)$ is defined as

$$(6.20)\qquad \hat{\xi}_1(k) = \hat{f}_1(k; k_1, \ldots, k_l) \equiv b_{k_{l-1}}\Big(1 - \sum_{j=1}^{n} a_{k_{l-1}j}\Big)^{-1}.$$

Then (see [11], p. 141–142]) the solution $u = (u_1, u_2, \ldots, u_n)$ of (6.1) may also be defined by the formula

$$(6.21)\qquad u_k = M\hat{\xi}_1(k) \quad \text{for } k \in N,$$

i.e. $\hat{\xi}_1(k)$ is the unbiased estimator of u_k. Obviously, the estimators $\hat{\xi}_1^{(i)}(k)$ in the model $S\{p_1(i,j), \Delta, 1\}$ are more convenient than the unbiased estimator $\hat{\xi}_1(k)$ (see (6.17), (6.20)).

Note that we may find many difference equations as examples of problem (6.1) (see [2], [5], [11]). Among these examples we pay specially attention to the following ones:

EXAMPLE (6.1). Consider the boundary problem for the elliptic equation

$$(6.22)\qquad a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + c\frac{\partial u}{\partial x} + d\frac{\partial u}{\partial y} + eu = f \quad \text{for } (x, y) \in G,$$

$$(6.23)\qquad u = \varphi \quad \text{for } (x, y) \in \Gamma,$$

where Γ is the boundary of the domain $G \subset R^2$; the given functions a, b, c, d, e, f (or φ) are continuous on $\bar{G} = G \cup \Gamma$ (or on Γ) and

$$(6.24)\qquad a(x, y) > 0; \quad b(x, y) > 0; \quad e(x, y) < 0 \quad \text{for } (x, y) \in \bar{G}.$$

We have to find a solution u, continuous on $\bar{G}$, of equation (6.22) that assumes specified values φ on Γ. It is known that under some simple additional assumptions (see [1], p. 45), we can use the method of finite differences to solve this problem. Then the solution of (6.22), (6.23) leads to a system of suitable difference equations

$$(6.25)\qquad u_i^{(k)} = \frac{A_i^{(k)}}{E^{(k)}} u_{i+1}^{(k)} + \frac{B_i^{(k)}}{E_i^{(k)}} u_{i-1}^{(k)} + \frac{C_i^{(k)}}{E_i^{(k)}} u_i^{(k+1)} + \frac{D_i^{(k)}}{E_i^{(k)}} u_i^{(k-1)} - f_i^{(k)} \quad \text{for } (i, k) \in G^*,$$

$$(6.26)\qquad u_r^{(s)} = \varphi_r^{(s)} \quad \text{for } (r, s) \in \Gamma^*,$$

where G^* (or Γ^*) is the set of the *inner points* (or of the *boundary points*) of the net, and corresponds to G (or Γ); the values of $A_i^{(k)}$, $B_i^{(k)}$, $C_i^{(k)}$, $D_i^{(k)}$, $E_i^{(k)}$, $f_i^{(k)}$, $\varphi_r^{(s)}$ are fully determined by the given functions a, b, c, d, e, f, φ and the values h, l of the steps of the net. If the values h, l are sufficiently small, then the following conditions are satisfied (see [1], p. 447)

$$(6.27)\qquad \frac{A_i^{(k)}}{E_i^{(k)}} > 0; \quad \frac{B_i^{(k)}}{E_i^{(k)}} > 0; \quad \frac{C_i^{(k)}}{E_i^{(k)}} > 0; \quad \frac{D_i^{(k)}}{E_i^{(k)}} > 0;$$

$$(6.28)\qquad \frac{A_i^{(k)}}{E_i^{(k)}} + \frac{B_i^{(k)}}{E_i^{(k)}} + \frac{C_i^{(k)}}{E_i^{(k)}} + \frac{D_i^{(k)}}{E_i^{(k)}} < 1.$$

It is clear that equations (6.25), (6.26) are of the form (6.1).

From (6.26)–(6.28) it follows that the assumptions of Corollary (6.1) are satisfied. Then, by virtue of this corollary, we can solve difference equations (6.25), (6.26) by the Monte-Carlo method.

Example (6.2). It is known (see [5]) that the solution of some other boundary problems leads to the following difference equations:

$$(6.29)\qquad A_i u_{i-1} - 2B_i u_i + C_i u_{i-1} = g_i \quad (i = 1, 2, \ldots, n-1),$$

$$(6.30)\qquad u_0 = g_0; \quad u_n = g_n,$$

where A_i, B_i, C_i, g_i are given coeffcients and fulfil the following conditions:

$$(6.31)\qquad A_i > 0; \quad C_i > 0; \quad B_i > \tfrac{1}{2}(A_i + C_i) + \beta,$$

where β is a positive constant. Obviously, equations (6.29), (6.30) may be written in the form (6.1), and from (6.30), (6.31) it is easy to see that the assumptions of Corollary (6.1) are satisfied, i.e. we can also use the model $S\{p_1(i, j), \Delta, 1\}$ to solve the system of difference equations (6.29), (6.30).

Previously we considered system (6.1) of linear algebraic equations as the special case of problem (3.1), when Ω is the discrete set $\Omega = \{1, 2, \ldots, n\}$. Analogously, we may consider other systems of linear algebraic equations, corresponding to problems (4.1), (5.1). For example, using the results of Section 4, one can solve the following system of linear algebraic equations using a model $\dot{S}$

$$u - Au = b, \tag{6.32}$$

where the elements a_{ij} of the matrix $A = (a_{ij})_{n\times n}$ satisfy the following conditions:

$$\max_{i\in N\setminus N_0} \Big\{ \sum_{j\in N\setminus N_0} a_{ij} \Big\} < 1; \tag{6.33}$$

$$a_{ij} \geqslant 0 \quad \text{for } i\in N\setminus N_0,\ j\in N; \tag{6.34}$$

$$a_{ij} = 0 \quad \text{for } i\in N_0,\ j\in N. \tag{6.35}$$

Here the sets N, N_0 are defined by (6.5).

It is clear that this problem is a special case of problem (4.1), where $\Omega = N$. Therefore Theorem (4.1) applies to solve problem (6.32).

7. The solution of an integral equation in $L_\infty(\Omega)$. Now we consider the integral equation

$$u(x) - \int_\Omega K(x, y)u(y)\mu(dy) = g(x) \qquad (x\in\Omega) \tag{7.1}$$

in the space $L_\infty(\Omega)$, where $\mu(\Omega) < +\infty$. Suppose that conditions (A)–(C) are satisfied and that, moreover,

$$\text{(D)} \qquad a_4^* \equiv \operatorname*{vrai\,sup}_{\mu\ \ x\in\Omega\setminus\Omega_0} \Big\{ \int_\Omega |K(x, y)|\mu(dy) \Big\} < +\infty \ ({}^{10}).$$

Together with the solution of equation (7.1) we also estimate the value of the functional

$$(u, \varphi) = \int_\Omega u(x)\varphi(x)\mu(dx), \tag{7.2}$$

where $u(x)$ is the solution of (7.1) and $\varphi\in L_1(\Omega)$.

Since problems (7.1), (7.2) are special cases of (1.1) and (1.3), therefore models S, $\bar{S}$ can be used for solving these problems. In order to get simple estimators, we may consider the following models S, $\bar{S}$:

Let $\bar{p}_4$ be a constant satisfying the condition

$$0 < \bar{p}_4 < \frac{1}{a_4^*}. \tag{7.3}$$

([10]) $a_4^* > 0$.

We define the function $p_4(x, y)$ on $\Omega \times \Omega$ by

$$(7.4) \qquad p_4(x, y) \equiv \begin{cases} \bar{p}_4 & \text{if } K(x, y) \geqslant 0, \\ -\bar{p}_4 & \text{if } K(x, y) < 0. \end{cases}$$

Then, from (D) and (7.3) it follows:

$$(7.5) \qquad \operatorname*{vrai\,sup}_{\mu \quad x \in \Omega \setminus \Omega_0} \Big\{ \int_\Omega K(x, y) p_4(x, y) \mu(dy) \Big\} < 1;$$

$$(7.6) \qquad K(x, y) p_4(x, y) \geqslant 0 \quad \text{for } x \in \Omega \ (\operatorname{mod} \mu), \ y \in \Omega \ (\operatorname{mod} \mu),$$

$$(7.7) \qquad p_4(x, y) \neq 0 \quad \text{for } (x, y) \in \Omega \times \Omega \ (\operatorname{mod} \mu \times \mu);$$

i.e. regarding $p_4(x, y)$ as the function $p(x, y)$ in Section 2, conditions of the form (P_1)–(P_3) are satisfied. Hence, from (2.11), (2.16) it is easy to deduce the following theorems:

THEOREM (7.1). *Under assumptions* (A), (B), (C), (D), *suppose that the function* $p_4(x, y)$ *is defined by* (7.4). *Then the estimators* $\xi_4^{(i)}(x)$ $(i = 1, 2;$ $x \in \Omega_\Delta^* \setminus \Omega_0)$ *in the model* $S\{p_4(x, y), \Delta, \delta\}$ *solving problem* (7.1) *are defined by the formula*

$$(7.8) \qquad \xi_4^{(i)}(x) = f_4^{(i)}(x; x_1, \ldots, x_l) \equiv \begin{cases} \dfrac{g_i(x_l)}{(\bar{p}_4)^l} \cdot (-1)^{N(x, x_1, \ldots, x_{l-1})} & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega_1. \end{cases}$$

THEOREM (7.2). *Under assumptions* (A), (B), (C), (D), (π_1)–(π_3) *and* $\varphi \in L_1(\Omega)$, *the estimators* $\eta_4^{(i)}$ $(i = 1, 2)$ *in the model* $\bar{S}\{p_4(x, y), \pi(x), \Delta, \delta\}$ *solving problem* (7.2) *are defined by the formula*

$$(7.9) \qquad \eta_4^{(i)} = F_4^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} \dfrac{\varphi(x_0) g_i(x_l)}{\pi(x_0)(\bar{p}_4)^l} \cdot (-1)^{N(x_0, x_1, \ldots, x_{l-1})} & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega_1. \end{cases}$$

It is known that in [7] various probability models have been constructed for solving problem (7.2) under a special case of assumptions (A), (D), where $\Omega \subset R^n$, $\mu = \mathscr{L}$ and

$$(7.10) \qquad \operatorname*{vrai\,sup}_{x \in \Omega} \Big\{ \int_\Omega |K(x, y)| \, dy \Big\} \leqslant 1,$$

$$(7.11) \qquad \operatorname*{vrai\,sup}_{x \in \Omega} \Big\{ \underbrace{\int_\Omega \ldots \int_\Omega}_{(m)} |K(x, x_1) \ldots K(x_{m-1}, x_m)| \, dx_2 \ldots dx_m \Big\} < 1 \ (^{11}).$$

From (7.9) it is easy to see that the estimators $\eta_4^{(i)}$ $(i = 1, 2)$ in the model $\bar{S}\{p_4(x, y), \pi(x), \Delta, \delta\}$ are more convenient than the known unbiased estimators of (u, φ) (see [7], [4], p. 244–245).

(11) Note that in [7] one considered problem (7.2) without assumptions (B), (C).

Previously we considered some special models S, $\bar{S}$. By virtue of these models, one can construct the computational schemes S_ε, $\bar{S}_\varepsilon$ for solving the correspondent problems with the error less than ε. In [10] these schemes are called the *ε-schemes*.

Note that from the assumptions of the theorems in this paper, we can estimate the *mean times* T_ε needed to solve the problems by the correspondent ε-schemes (see [10]).

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