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Double integrals involving Legendre functions

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1. Introduction. In this paper we prove a theorem in *operational* calculus and use it to evaluate double integrals involving Legendre functions. The results obtained are believed to be new.

We write

$$f(p) \doteqdot h(t)$$

when

$$f(p) = p\int_0^\infty e^{-pt}h(t)\,dt\,, \tag{1}$$

provided that the integral is convergent and $R(p) > 0$.

2. THEOREM. *If*

$$\varphi(p) \doteqdot h(t)$$

and

$$\psi(p) \doteqdot t^{a+\mu-1}K_\beta(bt)K_\nu(ct)h(t)\,,$$

then

$$\int_1^\infty\int_1^\infty (x^2-1)^{-\frac{1}{2}a}(y^2-1)^{-\frac{1}{2}\mu}P^a_{\beta-\frac{1}{2}}(x)P^\mu_{\nu-\frac{1}{2}}(y)(a+bx+cy)^{-1}\times \tag{2}$$
$$\times\varphi(a+bx+cy)\,dx\,dy = \frac{2(b)^{a-\frac{1}{2}}(c)^{\mu-\frac{1}{2}}}{a\pi}\psi(a)\,,$$

provided the integrals are absolutely convergent and $R(a) > 0$, $R(b) > 0$, $R(c) > 0$, $R(a) > 1$, $R(\mu) > 1$.

Proof. By definition, we have

$$\varphi(p) = p\int_0^\infty e^{-pt}h(t)\,dt\,,$$

then

$$\int_1^\infty\int_1^\infty (x^2-1)^{-\frac12\alpha}(y^2-1)^{-\frac12\mu}P^\alpha_{\beta-\frac12}(x)\,P^\mu_{\nu-\frac12}(y)\,(a+bx+cy)^{-1}\varphi(a+bx+cy)\,dx\,dy$$

$$=\int_1^\infty\int_1^\infty (x^2-1)^{-\frac12\alpha}(y^2-1)^{-\frac12\mu}P^\alpha_{\beta-\frac12}(x)\,P^\mu_{\nu-\frac12}(y)\times$$

$$\times\Big[\int_0^\infty \exp\{-(a+bx+cy)t\}\,h(t)\,dt\Big]dx\,dy$$

$$=\int_0^\infty e^{-at}h(t)\Big[\int_1^\infty (x^2-1)^{-\frac12\alpha}e^{-bxt}P^\alpha_{\beta-\frac12}(x)\,dx\int_1^\infty (y^2-1)^{-\frac12\mu}e^{-cyt}P^\mu_{\nu-\frac12}(y)\,dy\Big]dt$$

$$=\frac{2\,(b)^{\alpha-\frac12}(c)^{\mu-\frac12}}{\pi}\int_0^\infty e^{-at}t^{\alpha+\mu-1}K_\beta(bt)\,K_\nu(ct)\,h(t)\,dt$$

$$=\frac{2\,(b)^{\alpha-\frac12}(c)^{\mu-\frac12}}{a\pi}\,\psi(a)\,,$$

by (1), when changing the order of integration and evaluating the inner integrals by means of the formula ([3], p. 323; Equation (11)). The change of the order of integration can be justified by the application of de La Vallée Poussins theorem ([1], p. 504) when the integrals involved are absolutely convergent.

3. Integrals. If we take ([4], p. 342)

$$(3)\qquad h(t)=t^{\varrho-\frac12}I_\delta(t)\doteqdot\sqrt{\frac{2}{\pi}}\,p\,(p^2-1)^{-\frac12\varrho}Q^\varrho_{\delta-\frac12}(p)=\varphi(p)\,,$$

$$R(\varrho+\delta)>-\tfrac12,\qquad R(p)>1\,,$$

then from ([5], p. 86)

$$(4)\qquad t^{\alpha+\mu-1}K_\beta(bt)\,K_\nu(ct)\,h(t)$$

$$=t^{\alpha+\mu+\varrho-\frac32}K_\beta(bt)\,K_\nu(ct)\,I_\delta(t)$$

$$\doteqdot\sum_{\beta,-\beta}\sum_{\nu,-\nu}\frac{\Gamma(-\beta)\Gamma(-\nu)\Gamma(\alpha+\mu+\varrho+\nu+\beta+\delta-\frac12)c^\nu b^\beta}{\Gamma(\delta+1)p^{\alpha+\beta+\mu+\varrho+\nu+\delta-\frac32}(2)^{\beta+\nu+\delta+2}}\times$$

$$\times F_c\Big[\tfrac12(\alpha+\beta+\mu+\varrho+\nu+\delta-\tfrac12),\ \tfrac12(\alpha+\beta+\mu+\varrho+\nu+\delta-\tfrac32);$$

$$\beta+1\,,\ \nu+1\,,\ \delta+1;\ \frac{b^2}{p^2},\frac{c^2}{p^2},\frac{1}{p^2}\Big]=\psi(p)\,,$$

$$R(\alpha+\mu+\varrho+\delta\pm\beta\pm\nu-\tfrac12)>0\,,\qquad R(p+b+c)>1;$$

using (3) and (4) in (2), we have

$$(5)\quad \int_1^\infty\int_1^\infty (x^2-1)^{-\frac{1}{2}\alpha}(y^2-1)^{-\frac{1}{2}\mu}[(a+bx+cy)^2-1]^{-\frac{1}{2}\varrho}\times$$
$$\times P^{\alpha}_{\beta-\frac{1}{2}}(x)\,P^{\mu}_{\nu-\frac{1}{2}}(y)\,Q^{\varrho}_{\delta-\frac{1}{2}}(a+bx+cy)\,dx\,dy$$
$$=\sum_{\beta,-\beta}\sum_{\nu,-\nu}\frac{\Gamma(-\beta)\Gamma(-\nu)\Gamma(\alpha+\mu+\varrho+\nu+\beta+\delta-\frac{1}{2})(b)^{\alpha+\beta-\frac{1}{2}}(c)^{\mu+\nu-\frac{1}{2}}}{\Gamma(\delta+1)(a)^{\alpha+\mu+\varrho+\nu+\beta+\delta-\frac{1}{2}}(2)^{\beta+\nu+\delta+\frac{3}{2}}\sqrt{\pi}}\times$$
$$\times F_c\Big[\tfrac{1}{2}(\alpha+\beta+\mu+\varrho+\delta+\nu-\tfrac{1}{2}),\ \tfrac{1}{2}(\alpha+\beta+\mu+\varrho+\delta+\nu-\tfrac{3}{2});$$
$$\beta+1,\ \nu+1,\ \delta+1;\ \frac{b^2}{a^2},\frac{c^2}{a^2},\frac{1}{a^2}\Big],$$

valid for $R(\alpha+\mu+\varrho+\delta\pm\beta\pm\nu)>\frac{1}{2}$, $R(a)>1$, $R(a)>R(c)$, $R(a)>R(b)$. Now we take ([2], p. 198)

$$(6)\quad h(t)=t^{\varrho-\frac{1}{2}}K_\delta(t)\doteqdot\sqrt{\frac{\pi}{2}}\,\Gamma(\varrho\pm\delta+\tfrac{1}{2})\,p\,(p^2-1)^{-\frac{1}{2}\varrho}P^{-\varrho}_{\delta-\frac{1}{2}}(p)=\varphi(p),$$
$$R(\varrho\pm\delta+\tfrac{1}{2})>0,\ R(p+1)>0\,.$$

Also from ([5], p. 86)

$$(7)\quad t^{\alpha+\mu-1}K_\beta(bt)K_\nu(ct)h(t)=t^{\alpha+\mu+\varrho-\frac{3}{2}}K_\nu(ct)K_\beta(bt)K_\delta(t)$$
$$\doteqdot\sum_{\beta,-\beta}\sum_{\delta,-\delta}\sum_{\nu,-\nu}\frac{\Gamma(-\beta)\Gamma(-\delta)\Gamma(-\nu)\Gamma(\alpha+\mu+\varrho+\nu+\beta+\delta-\frac{1}{2})b^\beta c^\nu}{(2)^{\beta+\nu+\delta+2}(p)^{\alpha+\beta+\mu+\varrho+\nu+\delta-\frac{3}{2}}}\times$$
$$\times F_c\Big[\tfrac{1}{2}(\alpha+\beta+\mu+\varrho+\delta+\nu-\tfrac{1}{2}),\ \tfrac{1}{2}(\alpha+\beta+\mu+\varrho+\delta+\nu-\tfrac{3}{2});$$
$$\beta+1,\ \nu+1,\ \delta+1;\ \frac{b^2}{p^2},\frac{c^2}{p^2},\frac{1}{p^2}\Big]=\psi(p),$$
$$R(\alpha+\mu+\varrho\pm\nu\pm\beta+\delta)>\tfrac{1}{2},\ R(p+c+b+1)>0;$$

using (6) and (7) in (2), we get

$$(8)\quad \int_1^\infty\int_1^\infty (x^2-1)^{-\frac{1}{2}\alpha}(y^2-1)^{-\frac{1}{2}\mu}[(a+bx+cy)^2-1]^{-\frac{1}{2}\varrho}\times$$
$$\times P^{\alpha}_{\beta-\frac{1}{2}}(x)\,P^{\mu}_{\nu-\frac{1}{2}}(y)\,P^{-\varrho}_{\delta-\frac{1}{2}}(a+bx+cy)\,dx\,dy$$
$$=\sum_{\beta,-\beta}\sum_{\nu,-\nu}\sum_{\delta,-\delta}\frac{\Gamma(\alpha+\mu+\varrho+\nu+\beta+\delta-\frac{1}{2})\Gamma(-\beta)\Gamma(-\delta)\Gamma(-\nu)(b)^{\alpha+\beta-\frac{1}{2}}(c)^{\mu+\nu-\frac{1}{2}}}{\Gamma(\varrho\pm\delta+\frac{1}{2})(\pi)^{3/2}(2)^{\beta+\nu+\delta+\frac{1}{2}}(a)^{\alpha+\beta+\mu+\varrho+\nu+\delta-\frac{1}{2}}}\times$$
$$\times F_c\Big[\tfrac{1}{2}(\alpha+\mu+\varrho+\nu+\beta+\delta-\tfrac{1}{2}),\ \tfrac{1}{2}(\alpha+\mu+\varrho+\nu+\beta+\delta-\tfrac{3}{2});$$
$$\beta+1,\ \nu+1,\ \delta+1;\ \frac{b^2}{a^2},\frac{c^2}{a^2},\frac{1}{a^2}\Big],$$

valid for $R(\alpha+\mu+\varrho\pm\nu\pm\beta\pm\delta)>\frac{1}{2}$, $R(a)>1$, $R(a)>R(c)$, $R(a)>R(b)$.

References

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