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REMARKS ON THE SIMPLE CONNECTEDNESS OF BASINS OF SINKS FOR ITERATIONS OF RATIONAL MAPS

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§ 1

Consider a polynomial $P\colon \mathbf{C}\to\mathbf{C}$. Newton's method of looking for its roots is to consider iterates of the rational function $NP(z)=z-\frac{P(z)}{P'(z)}$ on the Riemann sphere $\hat{\mathbf{C}}$. Roots of P are fixed points, sinks for NP. For every p, a root of P, the set of points whose trajectories under iteration of NP converge to p splits into components. The component containing p is called the *immediate basin* of attraction to p.

In this note we prove the following.

THEOREM A. *Immediate basins of attraction to the roots of a complex polynomial, for Newton's method, are simply connected.*

This answers a question of A. Manning [M].

We can also say something in the connection with the question of M. Rees: find examples of rational functions with non simply connected immediate basins of attraction to periodic sinks of period at least 2.

(In the case where a sink p is of period k for a rational map f, we mean by its immediate basin of attraction the immediate basin for f^k.)

Namely we prove.

THEOREM B. *Immediate basins of attraction to periodic sinks of period at least 2 for a rational map of degree 2 are simply connected.*

The methods we use are classical, going back to the famous memoirs by P. Fatou and G. Julia [F1], [F2], [J], but to my knowledge Theorems A and B have stayed unknown.

§ 2

To prove Theorem A we need a lemma.

LEMMA 1. *Let A be the immediate basin of attraction to a fixed point a sink for a rational map $f: \hat{C} \to \hat{C}$. Assume that A is not simply connected. Then there exist in $\hat{C}$ two disjoint open connected sets U_0, U_1 intersecting A, such that $V = f(U_0) = f(U_1) \supset \operatorname{cl} U_0 \cup \operatorname{cl} U_1$, $f(\partial U_j) = \partial V \subset A$ for $j = 0, 1$, $V \cup A = \hat{C}$ and V is homeomorphic to a disc.*

Proof of Lemma 1. We almost repeat a part of Fatou's proof that a non-simply connected immediate basin is of infinite connectivity [F2 pp. 74–76]. Analogous considerations can be found also in [J, §§ 32, 46].

Denote our sink by p. Let γ be a simple closed curve around p such that $f(\gamma)$ is in the interior of the disc S_0 which has boundary γ and contains p. Assume additionally that $\gamma \cap \bigcup_{n>0} f^n(\operatorname{Crit} f) = \emptyset$, where $\operatorname{Crit} f$ is the set of critical points, i.e. such that $f'(z) = 0$. Denote by S_n the component of $f^{-n}(S_0)$ containing p. Let $n = N$ be the first integer for which S_n is not simply connected. Denote two distinct components of $\hat{C} \backslash S_N$ by $\hat{U}_0$ and $\hat{U}_1$ and their boundary curves by δ_0, δ_1. Let $V = \hat{C} \backslash S_{N-1}$. Finally, define U_0, U_1 to be the components of $f^{-1}(V)$ inside $\hat{U}_0$, $\hat{U}_1$ having δ_0, δ_1 for portions of their boundaries respectively. ∎

Proof of Theorem A. Suppose that A is a non simply connected immediate basin of attraction for $f = NP$ to a root p of polynomial P. Choose $z \in V \cap A$, V given by Lemma 1, and branches $f_{v_i}^{-1}$, $i = 0, 1$ so that $w_i = f_{v_i}^{-1}(z) \in U_i \cap A$. Join z with w_i by a curve $\gamma_i^0 \subset V \cap A$. Take care additionally to have $\gamma_i^0 \cap \operatorname{cl} \bigcup_{n>0} f^n(\operatorname{Crit} f) = \emptyset$. Define by induction $\gamma_i^n = f_{v_i}^{-1}(\gamma_i^{n-1})$, where $f_{v_i}^{-1}$ is the extension of the preliminary branch along the curve $\bigcup_{j=0}^{n-1} \gamma_i^j$. Define $\gamma_i = \bigcup_{n=0}^{\infty} \gamma_i^n$. The curve γ_i converges to an f-fixed point $q_i \in U_i$.

The reason is that $f_{v_i}^{-1} \circ \ldots \circ f_{v_i}^{-1}$ n times, $n = 0, 1, \ldots$, is a normal family of functions on a neighbourhood of γ_i^0 with the set of limit functions in $\operatorname{Fr} A$ which is nowhere dense. So all limit functions are constant, hence $\lim_{n\to\infty} \operatorname{diam}(\gamma_i^n) = 0$. Therefore all limit points of the sequence of curves γ_i^n are fixed points for f. They constitute a continuum. On the other hand they must be isolated from each other. So we have actually only one limit point.

The conclusion is that $\operatorname{Fr} A$ contains two different fixed points q_0, q_1 (belonging to two different components of $\operatorname{Fr} A$). But the only fixed points for NP are the roots of P and ∞. We arrived at a contradiction. ∎

Remarks 1. The proof above of the existence of q_i is in fact a repetition of Fatou's arguments [F2, p. 81]. I developed his ideas studying the convergence of "geometric coding trees" in [*P*]. In particular "geometric branches" corresponding to periodic codes converge to periodic points. See also [Pom].

2. From Theorem A it follows that from every immediate basin of attraction for NP to a root of P the point ∞ is accessible along at least (exactly?) $\deg(NP|_A)-1$ different radii (in a Riemann map parametrization of A by the unit disc). Geometrically it seems that when we perturb the polynomial z^d-1, the preimages of an immediate basin A of a root, lying between immediate basins of other roots, join each other and A to form a number of new canals along which A touches ∞. The number of canals would be at least $\deg(NP|_A)-1$.

3. In particular we conclude that $\operatorname{Fr} A \ni \infty$. If there is only one fixed point for a rational map f which is a source (denote it by q) and all other fixed points are sinks, then all immediate basins of attraction to sinks contain q in their boundaries; this fact was explicitly stated and proved by Fatou [F2, p. 211]. For estimates of widths of some canals touching ∞ for $f = NP$ see [M].

§ 3

Theorem B will be concluded from the following

PROPOSITION. *If A is an immediate basin of attraction to a fixed point, a sink, for a rational map f on $\hat{C}$, A is not simply connected and $\hat{C}\setminus A$ is not homeomorphic to a Cantor set, then* $\deg(f|_A) \geqslant 3$.

Proof of Proposition. Let U_0, U_1, V be as in Lemma 1. Suppose that

$$(1) \qquad \deg f|_{U_0\cap A} = \deg f|_{U_1\cap A} = 1 \quad \text{and} \quad \deg(f|_A) = 2.$$

Then we can consider $g_i = (f|_{U_i\cap A})^{-1}\colon V\cap A \to U_i \cap A$ for $i = 0, 1$. Each connected component K of $\operatorname{Fr} A$ is contained in the intersection of the decreasing sequence of sets $K_n = \operatorname{cl} g_{K,n}(V)$, where $g_{K,n} = g_{i_0}\circ g_{i_1}\circ\ldots\circ g_{i_n}$ and (i_n) is a 0, 1 sequence such that $f^n(K) \subset U_{i_n}$. (No component of $\operatorname{Fr} A$ can lie in a component of $\hat{U}_i$, the set definined in the proof of Lemma 1, other than U_i. Otherwise we would have $\deg(f|_A) \geqslant 3$.). Now, $(g_{K,n})$ is a normal family of functions on $V\cap A$ with all limit functions having values in the nowhere dense set $\operatorname{Fr} A$, hence constants. We conclude that $\operatorname{diam} K_n \to 0$ hence K is a single point. This means that $\operatorname{Fr} A$ is homeomorphic to a Cantor set. So (1) contradicts our assumption that $\operatorname{Fr} A$ is not homeomorphic to a Cantor set. ∎

To prove Theorem B we also need the following

LEMMA 2. *Let A be the immediate basin of attraction to a fixed point p, a sink, for a rational map f: $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$. If A is not simply connected then* $\# \operatorname{Crit}(f|_A) \geqslant \deg(f|_A)$ *(critical points counted with their multiplicities).*

Remark. We need this fact for Theorem B but it seems to be of interest in itself. It seems to belong to a folklore knowledge but I was unable to find it in the literature except for the case where $\# \operatorname{Crit}(f|_A) = 1 \Rightarrow A$ is simply connected, see [F2, p. 77]. So I will give a proof here; it will be just a minor development of Fatou's arguments. First, however, let us show how Theorem B is hence derived.

Proof of Theorem B. Let p be a periodic sink for a rational map f, of minimal period $k \geqslant 2$. Let A be the immediate basin of attraction to p for iteration of f^k. Suppose that A is not simply connected. There must be at least one f-critical point in $\bigcup_{j=0}^{k-1} f^j(A)$. Denote it by c_1. We may assume $c_1 \in A$. There must be another f-critical point $c_2 \in \bigcup_{j=0}^{k-1} f^j(A)$ (or the multiplicity of c_1 is at least 2). Otherwise $\# \operatorname{Crit}(f^k|_A) = 1$, $\deg(f^k|_A) \geqslant 2$, which contradicts Lemma 2. Suppose there is no other f-critical point $c_3 \in \bigcup_{j=0}^{k-1} f^j(A)$ (and the multiplicities of c_1, c_2 are equal to 1 or, in the case of $c_1 = c_2$, the multiplicity of this critical point is 2). If $c_2 \in A$ then $\# \operatorname{Crit}(f^k|_A) = 2$ but $\deg(f^k|_A) \geqslant 3$ (by Proposition because $\hat{\mathbf{C}} \backslash A$ contains $f(A)$ and so cannot be a Cantor set). This contradicts Lemma 2. If $c_2 \in f^s(A)$ for $0 < s < k$ then $\# \operatorname{Crit}(f^k|_A) = 1 + \deg(f^s|_A)$. Since $\deg(f|_{f^s(A)}) \geqslant 2$, we get $\# \operatorname{Crit}(f^k|_A) \leqslant 1 + \frac{1}{2}\deg(f^k|_A) < \deg(f^k|_A)$ (since in this case we even have $\deg(f^k|_A) \geqslant 4$), contrary to Lemma 2. The conclusion is that $\# \operatorname{Crit} f = 2\deg(f) - 2 \geqslant 3$. Hence $\deg f \geqslant 3$.

Proof of Lemma 2. If some critical points of f in A are multiple perturb $f \in C^\infty$ a little in small neighbourhoods of these points by replacing z^k by a finite Blaschke product type of maps to obtain multiplicities of all critical points in A equal to 1. If necessary, perturb it further by composing with small translations supported by neighbourhoods of critical values to have $\bigcup_{n>0} f^n(c_1) \cap \bigcup_{n>0} f^n(c_2) = \emptyset$ for every critical points c_1, $c_2 \in A$, $c_1 \neq c_2$ and $p \notin \bigcup_{n>0} f^n(c_1)$.

Order the points of $f^{-1}(p) \cap A$ by writing $w_1 = p, w_2, \ldots, w_d$, where $d = \deg(f|_A)$. Consider a smooth simple closed curve γ_1 around p such that no critical value for $f|_A$ is contained in the domain S_1^1 bounded by γ and containing p and such that $\gamma_0 = f(\gamma_1) \subset S_1^1$. Consider a smooth isotopy γ_t, $t \in \langle 0, 1 \rangle$ in the annulus bounded by γ_0 and γ_1. In the construction keep care about not having a pair of two different points $f^{m_1}(c_1), f^{m_2}(c_2)$ for f-critical points c_1, c_2 in one curve γ_t. For $t \in \langle 0, 1 \rangle$ denote by S_t^1 the domain bounded by γ_t and containing p. For every $t \geqslant 0$ write

$$S_t^1 = \text{the component of } f^{-E(t)}(S_{t-E(t)}^1) \text{ containing } p$$

(here E stands for Entier).

For $1 \leqslant i \leqslant d$, $t \geqslant 1$ define

$$S_t^i = \text{the component of } f^{-1}(S_{t-1}^1) \text{ containing } w_i.$$

If $t < 2$ then S_t^i are disjoint from each other. With growing t they begin to coincide.

As in the proof of Lemma 1 consider T such that S_T^1 is not simply connected, whereas S_{T-1}^1 is. Then

$$\# \operatorname{Crit}(f|_{S_T^1}) \geqslant \deg(f|_{S_T^1}). \tag{2}$$

This is so because by extending $f|_{S_T^1}$ to $\hat{f}$ defined on the union of the family of topological discs $(D_j)_{j=1}^r$ complementary of S_T^1 in $\hat{\mathbf{C}}$ by maps of $z \to z^{s_j}$ type we obtain

$$\# \operatorname{Crit}(\hat{f}|_{\cup D_j}) = \deg \hat{f} - r.$$

This together with

$$\# \operatorname{Crit}(\hat{f}) = 2\deg \hat{f} - 2 \tag{3}$$

yields

$$\# \operatorname{Crit}(f|_{S_T^1}) = \deg f - 2 + r$$

and (2) follows. (The proof of (3) is straightforward.)

As t grows from 2, we have a sequence of parameters $t_1 < t_2 < \ldots < t_{d-1}$ indicating instants where new coincidences occur between certain pairs of sets S_t^i. Every new coincidence results from a confluence of two topological circles $\delta_{t_j}^i$, $\delta_{t_j}^{i'}$, components of $\operatorname{Fr} S_{t_j}^i$ and $\operatorname{Fr} S_{t_j}^{i'}$ respectively, at a point c_j. We can look at the sets $\operatorname{Fr} S_t^i$ as levels where a certain Morse function G is constant. If we had not needed preliminary perturbations of f and if p were critical of multiplicity d' then by placing p at ∞ we might take for G Green's function $\lim_{n\to\infty} d'^{-n} \log|f^n(z)|$. Of course, c_j is a critical point for G and its forward trajectory contains a critical point for f. We claim, however, that c_j itself is a critical point for f.

Assume the contrary. Denote $\delta_{t_j}^i \cup \delta_{t_j}^{i'}$ by δ_{t_j} and, for $\varepsilon \approx 0$, the union of components of the boundary of $S_{t_j+\varepsilon}^i \cup S_{t_j+\varepsilon}^{i'}$ close to δ_{t_j} by $\delta_{t_j+\varepsilon}$. Choose a small disc U with the origin at c_j. For $\varepsilon < 0$ the two-component set $(S^i_{t_j+\varepsilon} \cup S^{i'}_{t_j+\varepsilon}) \cap U$ has a two component image in $f(U)$, according to the assumption that c_j is not critical for f.

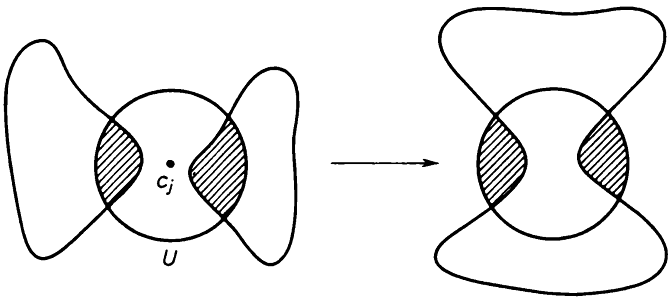

Fig. 1

Both components of the image belong however to the same connected set $S^1_{t_j-1+\varepsilon}$. The components of $\hat{C}\backslash S^1_{t_j-1+\varepsilon}$ are simply connected. Both components of $f(\delta_{t_j+\varepsilon} \cap U)$ belong to the boundary of the same component of $\hat{C}\backslash S^1_{t_j-1+\varepsilon}$; hence, they belong to the same simple closed curve ξ_ε a component of $\operatorname{Fr} S^1_{t_j-1+\varepsilon}$, see Fig. 1. (We have eliminated a possibility shown on Fig. 2 by considering subsets of the sphere.) So $f(\delta^i_{t_j+\varepsilon}) = f(\delta^{i'}_{t_j+\varepsilon}) = \xi_\varepsilon$. But now, for $\varepsilon > 0$ the set ξ_ε in $\operatorname{Fr}(S^1_{t_j-1+\varepsilon})$, close to ξ_ε for $\varepsilon < 0$ and obtained from it by surgery through $f(c_j)$, has two components and is the image of the simple closed curve $\delta_{t_j+\varepsilon}$, which is impossible.

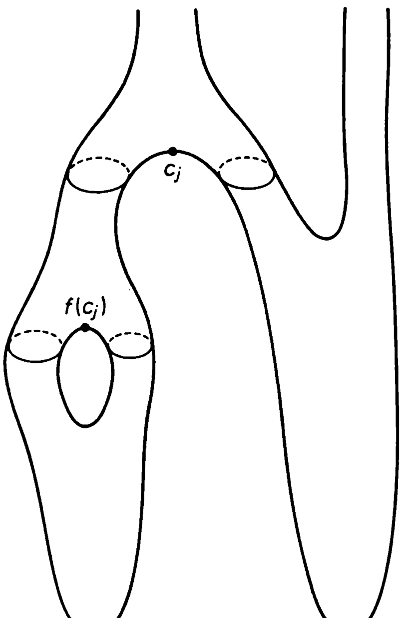

Fig. 2

The conclusion is that every parameter t_j gives rise to the absorption of a new f-critical point. By induction we obtain $\# \operatorname{Crit}(f|_{S_t^i}) \geqslant \deg(f|_{S_t^i})-1$. Starting with this inequality for every $i \geqslant 2$, $t = T$, together with the inequality (2), we can continue induction until all sets S_t^i coincide. Then we get $\# \operatorname{Crit}(f|_{S_t^i}) \geqslant \deg(f|_A)$. ∎

Added in proof. Recently M. Shishikura has informed me that he proved a theorem stronger than Theorem A: *the Julia set of NP is connected.*

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