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The recurrence relations for the moments of the discrete probability distributions

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1. The presentation of the problem

1.1. Introduction. The importance of the moments of the probability distributions is very well known. The calculation of the moments of the higher order may very often cause some trouble and even in the case of the simple distributions this has occurred recently. As given by Karl Pearson ([30], p. 157) in 1895 he published the first four moments about the mean of the binomial distribution probably for the first time (Phil. Trans. 186, p. 347). In 1899 the same author published the first five moments of the hypergeometric distribution in the Philosophical Magazine (February 1899, p. 239). In his opinion ([30], p. 157) the method of reaching those expressions needed a lot of transformations and did not give general results. Hence it was justifiable to search for the methods that lead to the expressions giving easy calculations of the moments of an arbitrary order both about the origin and the mean. Among those expressions the most remarkable are the recurrence relations because of the convenience and facility in application. In some cases they have been known for a very long time. In this paper an outline of the attainments in this field as well as the author's own results will be presented.

1.2. The presentation of the known results. The establishment of the recurrence relations for the moments of the probability distributions has until recently been a troublesome problem. In 1919 K. Pearson ([29], footnote p. 270) wrote on that subject: „A simple reduction formula for the moments of a binomial $(p+q)^n$ about its mean was sought in vain. After a good deal of energy had been spent, we believe that μ_r being the rth moment about the mean

$$(1.1) \qquad \mu_r = \left[\frac{d^r}{dt^r}(pe^{qt}+qe^{-pt})^r\right]_{t=0}, \quad 0<p<1,\ q=1-p,\ r=2,3,\ldots$$

is, perhaps, the easiest expression for reaching these moment-coeffitients by successive differentiation".

As it can be seen from the formula (1.1) the moments about the mean are being reached by the successive differentiation of the moment-generating function of the binomial distribution given in the form

$$(1.2) \qquad P(X=k) = \binom{n}{k}p^k q^{n-k}, \quad k = 0, 1, 2, \ldots, n.$$

In 1923, however, V. Romanovsky ([34], comp. also [33], p. 39–40) published a recurrence formula for the moments about the mean of the binomial distribution (1.2), convenient and easy enough in calculation, which together with some other results had been communicated to the Society of Naturalists of University of Warsaw as far back as 1915. This formula is as follows(1):

$$\mu_{r+1} = pq\left(nr\mu_{r-1} + \frac{d\mu_r}{dp}\right), \quad r = 1, 2, 3, \ldots \tag{1.3}$$

Romanovsky has obtained it by making use of the very simple method: the differentiation of the moment-generating function of the binomial distribution, appearing in the formula (1.1), with regard to p and the comparison of the result with the respective coefficients at t^r in the series expansion of that function.

In 1924 K. Pearson [30] gave the recurrence formula for the moments about the mean of the hypergeometric distribution defined by

$$P(X = k) = \binom{n}{k} \frac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}} \tag{1.4}$$

when samples of n are drawn from a population of N individuals (M individuals of A kind and $N-M$ individuals of B kind) if a sample can contain k individuals of A kind and $n-k$ individuals of B kind and $p = M/N$, $0 < p < 1$, $q = 1-p$, $\max(0, n-Nq) \leqslant k \leqslant \min(n, Np)$.

Using the moment-generating function and the differential equation of Gauss, which determines the hypergeometric function, K. Pearson obtained the following relation

$$N\mu_{r+1} = [(1+E)^r - E^r]\{\mu_2 - [Np + n(q-p)]\mu_1 + npq(N-n)\mu_0\}, \tag{1.5}$$
$$r = 1, 2, 3, \ldots$$

where

$$\mu_2 = npq\frac{N-n}{N-1}, \quad \mu_1 = 0, \quad \mu_0 = 1$$

and E denotes the operation of raising the order of a moment by unity, i.e.

$$E\mu_r = \mu_{r+1}.$$

Assuming $N \to \infty$ Pearson found from the formula (1.5) the recur-

(1) As it has been made known by R. Frisch ([8], p. 166) in professor Al. A. Tchouproff's opinion the formula (1.3) was first published by Bohlmann (Cf. Bortkiewicz, Jahresberichte der deutschen Mathematikervereinigung, Bd. XXVII (1918), p. 73).

rence relations for the moments about the mean of the binomial distribution

$$\mu_{r+1} = [(1+E)^r - E^r](npq\mu_0 - p\mu_1) \tag{1.6}$$

and consequently assuming $\lim_{n\to\infty} np = \lambda > 0$ he reached the recurrence relation for the moments about the mean of the Poisson distribution

$$\mu_{r+1} = \lambda[(1+E)^r - E^r]\mu_0. \tag{1.7}$$

With regard to $E = 1+\Delta$ the relations given here can be also dealt as the relations with finite differences. These relations are very elegant and leave nothing to desire if we evaluate the moments one after another. But it does not enable us to write down any desired moment without knowing the preceding ones.

The aim of the paper of V. Romanovsky [35] in 1925 was to provide a formula which permits to obtain the moments of the hypergeometric distribution independently. For demonstration he also used the Gauss-function $F(\alpha, \beta, \gamma, t)$ for which the following relation holds:

$$\left[\frac{d^r F}{dt^r}\right]_{t=1} = U_r.$$

In this formula U_r denotes the so-called factorial moment of rth order, i.e.

$$U_r = E[X(X-1)\dots(X-r+1)]$$

equal for the hypergeometric distribution

$$U_r = \frac{n(n-1)\dots(n-r+1)(Nq)(Nq-1)\dots(Nq-r+1)}{N(N-1)\dots(N-r+1)}. \tag{1.8}$$

Making use of the Newton interpolation formula Romanovsky has hereafter established the relation

$$\mu_r = \sum_{k=0}^{r}\sum_{l=0}^{r-k}(-1)^l C_r^l S_{k,r-l}(rq)^l U_k \tag{1.9}$$

where $S_{k,r-l}$ denotes the so-called Stirling's numbers of the second kind.

If we assume $N \to \infty$ we derive from (1.8) the formula for the factorial moments of the binomial distribution

$$U_r = n(n-1)\dots(n-r+1)q^r$$

and next, with the same meaning of U_r, the formula for the moments about the mean of the binomial distribution which is of the same form as the formula (1.9).

Also in 1925 Ragnar Frisch ([8] and [9]) being occupied with the question of the establishment of the formula for the moments about the mean of the binomial distribution showed that the Romanovsky's formula can be obtained in a quite simple way by differentiation with regard to p of the definition formula for the so-called incomplete moments about the mean

$$(1.10) \qquad \mu_r(s) = \sum_{k=s}^{n} \big(k - E(X)\big)^r \Pi(k), \qquad r = 1, 2, 3, \ldots$$

where in this case $E(X) = np$ and $\Pi(k) = P(X = k)$ is defined by (1.2). Thus the formula (1.3) is of the more general character because it holds also for the incomplete moments.

Using the formula for the partial summation known in numerical analysis

$$(1.11) \qquad \mu_r(s) = \sum_{k=s}^{n} f_k g_k = f_n \sum_{k=s}^{n} g_k - \sum_{k=s}^{n-1} \Delta f_k \sum_{i=s}^{k} g_i$$

where

$$f_k = (k - np)^{r-1}, \qquad g_k = (k - np)\Pi(k), \qquad \Delta f_k = f_{k+1} - f_k$$

Frisch obtained the further formula for the moments of the distribution (1.2) about its mean

$$(1.12) \quad \mu_r(s) = sq\Pi(s)(s - np)^{r-1} + npq \sum_{i=0}^{r-2} \binom{r-1}{i} \mu_i - p \sum_{i=0}^{r-2} \binom{r-1}{i} \mu_{i+1}$$

which at $s = 0$ gives the formula (1.6) obtained by K. Pearson but presented now in the form

$$(1.13) \qquad \mu_r = npq \sum_{i=0}^{r-2} \binom{s-1}{i} \mu_i - p \sum_{i=0}^{r-2} \binom{s-1}{i} \mu_{i+1}.$$

Thus (1.12) is the generalization of the relation (1.6) or (1.13).

Next considering general properties of the linear equations between the moments of the point binomial Frisch found the relation

$$\sum_{i=0}^{r} \left\{ \left[\binom{r-k-1}{i} + \binom{k}{i}\right] [(-1)^{i-1} q - p] + [1 + (-1)^i] \times \right.$$
$$\left. \times \left[\binom{r-k-1}{i-1} + \binom{k}{i-1}\right] npq \right\} \mu_{r-i} = 0$$

which by choosing k conveniently permits to reduce the evaluations only to some values of μ_i ([8], pp. 169–170).

In 1933 Alfred Guldberg [15] on the occasion of deducing the correlation coefficient of the binomial distribution used the difference equation of that distribution for reaching the moments. He mentioned in the paper that the method of the difference equations may be used for finding the recurrence relations for the moments of other distributions.

In the same year R. Risser and C. E. Traynard [33] used that method for deducing the recurrence relations for the moments about the origin α_r

and for the incomplete moments about the origin $a_r(s)$ of the distributions: Poisson, binomial, negativ binomial and hypergeometric. (As it is known $a_r(s)$ may be obtained from the formula (1.10) at the assumption $E(X) = 0$). They have obtained the following results for $r = 0, 1, 2, \ldots$

The binomial distribution:

$$a_{r+1}(s) = qs^{r+1}\Pi(s)+p\sum_{i=0}^{r}\left[n\binom{r}{i}-\binom{r}{i+1}\right]a_{r-i}(s), \tag{1.14}$$

$$a_{r+1} = p\sum_{i=0}^{r}\left[n\binom{r}{i}-\binom{r}{i+1}\right]a_{r-i}. \tag{1.15}$$

The Poisson distribution:

$$a_{r+1}(s) = \lambda\left[s^r\Pi(s-1)+\sum_{i=0}^{r}\binom{r}{i}a_{r-i}(s)\right], \tag{1.16}$$

$$a_{r+1} = \lambda\sum_{i=0}^{r}\binom{r}{i}a_{r-i}. \tag{1.17}$$

The hypergeometric distribution:

$$\begin{aligned} a_{r+1}(s) = {} & \left[(Nq-n)s^{r+1}+s^{r+2}\right]\Pi(s)+ \\ & + \frac{1}{N-r}\sum_{i=0}^{r}\left[Nnp\binom{r}{i}-(Np+n)\binom{r}{i+1}+\binom{r}{i+2}\right]a_{r-i}(s), \end{aligned} \tag{1.18}$$

$$a_{r+1} = \frac{1}{N-r}\sum_{i=0}^{r}\left[Nnp\binom{r}{i}-(Np+n)\binom{r}{i+1}+\binom{r}{i+2}\right]a_{r-i}. \tag{1.19}$$

The negativ binomial distribution given in the form:

$$P(X = k) = \Pi(k) = (-1)^k\binom{-n}{k}q^np^k, \quad k = 0, 1, 2, \ldots \tag{1.20}$$

$$a_{r+1}(s) = \frac{1}{q}\left\{s^{r+1}\Pi(s)+p\sum_{i=0}^{r}\left[n\binom{r}{i}+\binom{r}{i+1}\right]a_{r-i}(s)\right\}, \tag{1.21}$$

$$a_{r+1} = \frac{p}{q}\sum_{i=0}^{r}\left[n\binom{r}{i}+\binom{r}{i+1}\right]a_{r-i}. \tag{1.22}$$

In 1934 A. R. Crathorne [3] gave the simple formula for the moments about the origin of the binomial distribution

$$a_{r+1} = a_1a_r+pq\frac{da_r}{dp}, \quad r = 1, 2, 3, \ldots \tag{1.23}$$

For the demonstration of that relation he used the recurrence relation for the cumulants $\varkappa_r$ of the binomial distribution

$$\varkappa_r = pq\frac{d\varkappa_{r-1}}{dp}, \quad r = 1, 2, 3, \ldots$$

and also the formula which presents the relation between the cumulants and the moments about the origin a_r

$$(1.24) \qquad a_{r-1} - \varkappa_{r-1} = \sum_{i=1}^{r} \binom{r}{i-1} \varkappa_i a_{r+1-i}, \qquad r = 1, 2, 3, \ldots$$

both of them having been proved by R. Frisch [7].

In the same year A. T. Craig [2] deduced the formula (1.3) by the same method which was shown by R. Frisch in 1925 and moreover he also established the relation (1.23) and the formula for the moments of the Poisson distribution

$$(1.25) \qquad a_{r+1} = \lambda \left(a_r + \frac{da_r}{d\lambda} \right), \qquad r = 0, 1, 2, 3, \ldots$$

and the formula for the moments about the mean of that distribution

$$(1.26) \qquad \mu_{r+1} = \lambda \left(r\mu_{r-1} + \frac{d\mu_r}{d\lambda} \right), \qquad r = 1, 2, 3, \ldots$$

In 1934 A. A. Krishnaswami Ayyangar [20] starting from the simple transformation of the expression

$$(1.27) \qquad (k - np)^r \binom{n}{k} p^k q^{n-k}$$

obtained the recurrence relation for the incomplete moments about the mean of the binomial distribution

$$(1.28) \qquad \mu_r(s) = np[(E+q)^{r-1} \mu_0^*(s-1) - \mu_{r-1}(s)], \qquad r = 1, 2, 3, \ldots$$

where $\mu_r^*(s-1)$ denotes the incomplete moment about the mean of the distribution $(p+q)^{n-1}$ calculated from $k = s-1$ to $k = n-1$ and $E\mu_r^*(s-1) = \mu_{r+1}^*(s-1)$.

Putting $s = 0$ he obtained from (1.28) the formula for the complete moments about the mean

$$(1.29) \qquad \mu_r = np[(E+q)^{r-1} \mu_0^* - \mu_{r-1}], \qquad r = 2, 3, \ldots$$

Considering, moreover, the other possibilities of the transformation of the expression (1.27) the same author obtained a brief proof of the recurrence relation for the incomplete moments about the mean of the binomial distribution

$$(1.30) \quad \mu_r(s) = (s-np)^{r-1} \mu_1(s) + \{(1+E)^{r-1} - E^{r-1}\} [npq \mu_0(s) - p\mu_1(s)]$$

which in particular by putting $s = 0$ gives the formula (1.6) derived by K. Pearson.

It results from the consideration carried further down that the incomplete moment $\mu_r(s)$ may be re-stated in the form of the relation

$$(1.31) \qquad \mu_r(s) = d_r \mu_1(s) + \mu_r \mu_0(s).$$

Taking into account (1.31) in the formula (1.30) we obtain the recurrence relation for the coefficients d_r by the comparison of the coefficients at $\mu_1(s)$:

$$d_r = (s-np)^{r-1}+[(1+E)^{r-1}-E^{r-1}](npqd_0-pd_1) \tag{1.32}$$

where $d_0 = 0$, $d_1 = 1$.

In the next paper of the same year A. A. Krishnaswami Ayyangar [21] elaborated the problem of the incomplete moments about the mean of the hypergeometric distribution (1.4).

Going out of the identities

$$k(Np-n+k) = N(k-np)+(n-k)(Np-k), \tag{1.33}$$

$$(n-k)(Np-k) = (k-np)^2-C_1(k-np)+C_2, \tag{1.34}$$

where

$$C_1 = Np+n(p-q), \qquad C_2 = npq(N-n) \tag{1.35}$$

and

$$k(Np-n+k)\Pi(k) = [n-(k-1)][Np-(k-1)]\Pi(k-1) \tag{1.36}$$

he obtained the following recurrence relation after simple transformation

$$\begin{aligned} N[\mu_r(s)-(s-np)^{r-1}\mu_1(s)] \\ = \{(1+E)^{r-1}-E^{r-1}\}[\mu_2(s)-C_1\mu_1(s)+C_2\mu_0(s)]. \end{aligned} \tag{1.37}$$

The author of the mentioned paper showed that in the case of the hypergeometric distribution the formula (1.32) holds as well; moreover, the relation for the coefficients d_r can be found in the analogous manner as described above, the relations (1.32) and (1.37) having been taken into account:

$$N[d_r-(s-np)^{r-1}] = [(1+E)^{r-1}-E^{r-1}](d_2-C_1d_1+C_2d_0). \tag{1.38}$$

In 1935 W. J. Kirkham [19] again considered the problem of establishment of the moments of the binomial distribution. He started with the relation between the moments about the origin and the moments about the mean

$$\mu_r = \sum_{i=0}^{r}(-1)^i\binom{r}{i}a_{r-i}a_1^i, \quad r = 2, 3, \ldots$$

and from the semi-recursion formula obtained from the last one

$$\mu_r = a_r-\sum_{i=1}^{r}\binom{r}{i}\mu_{r-i}a_1^i. \tag{1.39}$$

The moments about the origin were found by Kirkham from

$$a_r = np\,\mu'_{r-1}(n-1), \quad r = 1, 2, 3, \ldots \tag{1.40}$$

where $\mu'_{r-1}(n-1)$ denotes the moment of order $r-1$ about the point -1 of the distribution $(p+q)^{n-1}$ calculated as follows:

$$\mu'_{r-1}(n-1) = \sum_{i=0}^{r} \binom{r}{i} a_{r-i}(n-1). \tag{1.41}$$

Putting (1.41) into (1.40) he obtained for the binomial distribution

$$a_r = np \sum_{i=0}^{r} \binom{r}{i} a_{r-i}(n-1). \tag{1.42}$$

In 1937 J. Riordan [32] gave the recurrence relations for the moments about the origin of the binomial, Poisson and hypergeometric distributions. He obtained those relations using the moment-generating function of each distribution and the Stirling's numbers of the second kind. As it is known the numbers which satisfy the conditions:

$$t^r = \sum_{x=1}^{r} t(t-1)(t-2)\dots(t-x+1)S_{x,r}$$

or

$$S_{x,r} = \frac{\Delta^x 0^r}{x!} \tag{1.43}$$

(comp. [26], p. 50) where $\Delta^x 0^r$ denotes the so-called difference of zero of order x of the function x^r are called the Stirling's numbers of the second kind.

We form the table of the Stirling's numbers using the relation

$$S_{x,r+1} = xS_{x,r} + S_{x-1,r} \tag{1.44}$$

(see [26], p. 50, formula (23); table II, p. 409 and [10], p. 3).

It follows from the definition that

$$S_{r,r} = S_{1,r} = 1 \tag{1.45}$$

and

$$S_{x,r} = 0 \quad \text{for} \quad x > r. \tag{1.46}$$

Putting the generating function $\sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ of each considered distribution in the form

$$\sum_{x=0}^{\infty} A_x(e^t-1)^x = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{x=0}^{k} A_x x! S_{x,r}$$

Riordan obtained the relations for the moments by the comparison of both sides of this equation at the well matched coefficient A_x. So for the binomial distribution

$$a_r = \sum_{x=0}^{r} n(n-1)\dots(n-x+1)p^x S_{x,r}\,; \tag{1.47}$$

for the Poisson distribution

$$(1.48) \qquad a_r = \sum_{x=0}^{r} \lambda^x S_{x,r};$$

for the hypergeometric distribution

$$(1.49) \quad a_r = \sum_{x=0}^{r} \frac{n(n-1)\dots(n-x+1)(Np)(Np-1)\dots(Np-x+1)}{N(N-1)\dots(N-x+1)} \cdot S_{x,r}.$$

After further transformations by using the relation (1.44) he obtained the relations (1.47) and (1.48) for the binomial and Poisson distributions respectively in the form (1.23) and (1.25) and in the case of the hypergeometric distribution the relation (1.49) in the form

$$a_{r+1} = np\, a_r(Np-1, n-1, N-1) - (N+1)\Delta_N a_r$$

where $\Delta_N a_r$ denotes the difference of the expression for a_r with regard to N.

Considering the form of the moment-generating function of the discussed distributions Riordan obtained the moments about the mean by the same method and so: for the binomial distribution

$$\mu_r = \sum_{x=0}^{r} n(n-1)\dots(n-x+1)p^x \sigma_{x,r};$$

for the Poisson distribution

$$\mu_r = \sum_{x=0}^{r} \lambda^x \sigma_{x,r};$$

for the hypergeometric distribution

$$\mu_r = \sum_{x=0}^{r} \frac{n(n-1)\dots(n-x+1)(Np)(Np-1)\dots(Np-x+1)}{N(N-1)\dots(N-x+1)} \cdot \sigma_{x,r}$$

where

$$x!\, \sigma_{x,r} = \Delta^x(-a_1)^r.$$

Then after rather toilsome calculations he obtained the relations (1.3) and (1.26) for the binomial and Poisson distribution respectively and the following relation for the hypergeometric distribution

$$\mu_{r+1} = (n+1)\Big[\mu_r - \sum_{i=0}^{r} \binom{r}{i} K_1^i \mu_{r-i}(Np, n, N+1)\Big] -$$

$$- np\Big[\mu_r - \sum_{i=0}^{r} \binom{r}{i} K_2^i \mu_{r-i}(Np, n-1, N-1)\Big]$$

where

$$K_1 = -\frac{np}{N+1}, \quad K_2 = \frac{(Np-1)(n-1)}{N-1} - np.$$

In 1939 Harold D. Larsen [24] using the method of Kirkham proved that the recursion formula for the moments about the origin of the hypergeometric distribution defined by the formula (1.4) is given by (1.40) assuming that the moment μ'_{r-1} is calculated from (1.41) for the distribution with the parameters: $N-1, n-1, Np-1$. The moments about the mean have been also calculated by Larsen who used the formula (1.39).

In 1950 Albert Noack [28] defined a large class of random variables with the discrete probability distributions which can be derived from the power series

$$f(z) = \sum_{x=0}^{\infty} a_x z^x$$

where the coefficients a_x are real either non-negative (in this case $0 < z < r$) or satisfy the condition $(-1)^x a_x \geqslant 0$ (in this case $-r < z < 0$). The probability distributions of that type are defined by the equality

$$P(X = x) = \Pi(x) = \frac{a_x z^x}{f(z)}, \quad x = 0, 1, 2, \ldots \tag{1.50}$$

because it is clear that $\sum_{x=0}^{\infty} \Pi(x) = 1$. So defined distributions have been called by A. Noack power series distributions (p.s.d.).

Using the method of deriving the moments given by Frisch and applied by Craig this author found the following recurrence relations for the moments of the discussed distributions:

The moments about the origin:

$$a_{r+1} = a_1 a_r + z\frac{da_r}{dz}, \quad r = 1, 2, 3, \ldots \tag{1.51}$$

The moments about the mean:

$$\mu_{r+1} = r\mu_2 \mu_{r-1} + z\frac{d\mu_r}{dz}, \quad r = 1, 2, 3, \ldots \tag{1.52}$$

The author has discussed a few distributions which are the particular cases of the hypergeometric series $f(z) = F(a, b, c, z)$ and which at the assumption $abc > 0$ appear to give the distributions of the type (1.50) and he derived the recurrence relations for these distributions.

The formula for the moments about the mean for the Poisson distribution quoted in the paper is identical with the formula (1.26) given by Craig.

In 1957 W. Krysicki [22] obtained the recursion formula (1.17) for the moments about the origin of the Poisson distribution using the method of the difference equations which has been discussed before now.

In 1957 R. Risser and C. E. Traynard ([33], second edition) published the recurrence formula for the moments about the origin of the Pólya distribution. The probability function of this distribution is defined as follows:

$$P(X = k) = \Pi(k) = \binom{n}{k} \frac{\prod_{i=0}^{k-1} (p+ia) \prod_{i=0}^{n-k-1} (q+ia)}{\prod_{i=0}^{n-1} (1+ia)} \tag{1.53}$$

where $a = S/N$, $0 < p = M/N < 1$, $q = 1-p$, N being the number of individuals of a population, $M = Np$ being the number of individuals of A kind in the discussed population, $N-M = Nq$ the number of individuals of B kind in the same population, $n = 0, 1, 2, \ldots$ the number of successive draws during which each individual having been sampled is returned to the same population and besides S individuals of the same kind as the individual which had been drawn before are adjoined to the population ($S = 1, 2, \ldots,$) or taken off ($S = -1, -2, \ldots$) or the population is left without any change ($S = 0$), k being the number of draws in which the individuals of A kind have been obtained.

In case $S \leqslant -1$ the evident restrictions are obligatory:

$$-kS \leqslant M \quad \text{i} \quad -(n-k)S \leqslant N-M.$$

The recurrence relation for the moments about the origin of the Pólya distribution given by Risser and Traynard is as follows:

$$a_{r+1} = \frac{1}{1+ra} \sum_{i=0}^{r} \left[np\binom{r}{i} + (na-p)\binom{r}{i+1} - a\binom{r}{i+2} \right] a_{r-i}. \tag{1.54}$$

In 1960, using the method of the finite differences, I obtained a simple formula for the moments of the Poisson distribution

$$a_r = \sum_{x=1}^{r} \Delta^x 0^r \frac{\lambda^x}{x!}, \qquad r = 1, 2, 3, \ldots \tag{1.55}$$

identical with (1.48) on the ground of (1.43).

In 1962 using the same method I found the formula for the moments about the origin of the generalized Poisson distribution defined by

$$P(X = x) = Cq^x e^{-qu} \sum_{j=x}^{\infty} \frac{u^j}{j!} \tag{1.56}$$

where

$$q > 0, \quad u > 0, \quad x = 0, 1, 2, \ldots, \quad C = \frac{1-q}{\exp[(1-q)u]-q}, \; q \neq 1$$

([11], p. 7). This formula is as follows:

(1.57)

$$a_r = C\left[\sum_{n=1}^{r} \Delta^n 0^r \frac{(qu)^n}{n!} + e^{(1-q)u} \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^m \frac{1}{(n+1)!} \binom{n}{m} q^{n-m} (n-m)^r u^n\right].$$

With $q = 1$, the formula (1.56) becomes transformed into the probability function of the so-called PEBL distribution (Poisson Exponential Binomial Limit)

$$P(X = x) = ce^{-u} \sum_{j=x}^{\infty} \frac{u^j}{j!} \tag{1.58}$$

where

$$u > 0, \quad x = 0, 1, 2, \ldots, \quad c = \frac{1}{1+u}.$$

For this distribution the analogous procedure gave

$$a_r = c\left(m_r + u \sum_{n=1}^{r} \Delta^n 0^r \frac{u^n}{(n+1)!}\right) \tag{1.59}$$

where m_r denotes the moment of order r of the Poisson distribution with the parameter $\lambda = qu > 0$.

Also in this paper I communicated another recursion formula for the moments of the generalized Poisson distribution which has been found by making use of a differential equation.

In 1963 Carl Philipson [31] gave the formula for the moments about the origin of the Poisson distribution

$$a_r = \sum_{x=1}^{r} \lambda^r \sum_{k=1}^{r} \frac{(-1)^{r-k} k^r}{(x-k)!\, k!} \tag{1.60}$$

which is identical with the formula (1.55) if we take into account that

$$\sum_{k=1}^{r} \frac{(-1)^{r-k} k^r}{(x-k)!\, k!} = \frac{\Delta^x 0^r}{x!}$$

(comp. [5], (11.6), p. 62 with the formula (2) in [10], p. 3).

In 1964 Tadeusz Śródka [36] gave the formula (1.54).

In 1965 A. R. Kamat [17], being occupied among others with the problem of the incomplete moments about the mean, published some other relations. Above all he noticed that assuming $np \to \lambda$ if $p \to 0$ we derive at once from the formula (1.12) the recursion formula for the incomplete moments about the mean of the Poisson distribution

$$\mu_r(s) = s(s-\lambda)^{r-1}\Pi(s)+\lambda\sum_{i=0}^{r-2}\binom{r-1}{i}\mu_i(s) \tag{1.61}$$

where

$$\mu_0(s) = \sum_{k=s}^{\infty}\Pi(k), \qquad \mu_1(s) = s\Pi(s).$$

Similarly, from (1.3) Kamat obtained the relation (1.26) valid also for the incomplete moments which as we have already said (p. 8) is the formula for the incomplete moments of the binomial distribution.

For the geometric distribution

$$\Pi(k) = qp^k, \qquad k = 0,1,2,\ldots \tag{1.62}$$

he derived the formula for the incomplete moments about the mean

$$\mu_r(s) = qp^{m+1}\frac{d}{dp}\left(q^{-1}p^{-m}\mu_{r-1}(s)\right) \tag{1.63}$$

directly from the definition (1.10) by the simple transformation (we put $m = p/q$ after differentiation).

Also directly from the definition (1.10) Kamat derived the recursion formula for the incomplete moments $\mu_r'(s)$ about the point m of the logarithmic distribution defined by the formula

$$P(X = l) = A\cdot\frac{p^l}{l}, \qquad l = 1,2,3,\ldots \tag{1.64}$$

where

$$A = -\frac{1}{ln q}, \qquad q = 1-p \tag{1.65}$$

(comp. [18], p. 131–132 or [13], p. 310).

He obtained namely

$$\mu_r'(s) = \frac{A}{q}\beta_{r-1}-m\mu_{r-1}'(s) \tag{1.66}$$

where β_{r-1} denotes the above discussed incomplete moment about the mean of the geometric distribution [see (1.63)].

Kamat gave also the recurrence relation for the incomplete moments about the mean of the hypergeometric distribution

$$(1.67)\qquad \mu_r(s) = (N-r+1)^{-1}\Big[(Nq-n+s)(s-np)^{r-1}s\Pi(s)+ + \sum_{i=0}^{r-3}\binom{r-1}{i}\mu_{i+2}(s)-(Np+nq-np)\sum_{i=0}^{r-2}\binom{r-1}{i}\mu_{i+1}(s)+ + (N-n)npq\sum_{i=0}^{r-2}\binom{r-1}{i}\mu_i(s)\Big].$$

The contributor remarked that this formula leads at $N\to\infty$ to the formula (1.12).

It is remarkable that the formula derived by Kamat leads at $s=0$ to the formula (1.5) already established by Pearson (by another method).

Furthermore Kamat completed the results of A. Noack extending them for the moments about an arbitrary point $m=m(z)$ of the power series distribution (1.50) but using the same method as A. Noack. He obtained namely

$$(1.68)\qquad \mu'_{r+1} = (a_1-m)\mu'_r+z\,\frac{d\mu'_r}{dz}+rz\mu'_{r-1}\,\frac{dm}{dz}.$$

Apparently, in the special case of $m=a_1$ he obtained the formula (1.52) for the moments about the mean.

In 1967 I published the relations for the moments of the distributions (1.56) and (1.58), this time, using the method of the difference equations. The relations are the following [12]:

The generalized Poisson distribution (1.56):

$$a_r = \frac{q}{1-q}\left[\sum_{i=1}^{r}\binom{r}{i}a_{r-i}-\frac{C}{\lambda}m_{r+1}\right]$$

where m_{r+1} denotes the moment about the origin of order $r+1$ of the Poisson distribution with the parameter $\lambda = qu>0$ at $q\neq 1$;

The PEBL distribution (1.58):

$$a_{r-1} = \frac{1}{r}\left[\frac{c}{u}m_{r+1}-\sum_{i=2}^{r}\binom{r}{i}a_{r-i}\right]$$

where m_{r+1} denotes here the moment about the origin of the Poisson distribution with the parameter $\lambda = u>0$.

2. The recurrence relations for the moments about the origin

2.1. The power series distributions. Now I shall present here the application of the method of the difference equations for deducing the recurrence relations for the moments of the power series distributions (1.50). Apparently, using (1.50), we can write

$$\Pi(x+1) = z\frac{a_{x+1}}{a_x}\Pi(x). \tag{2.1}$$

We multiply both sides of the equation (2.1) by $(x+1)^{r+1}$ and sum up with regard to x from $x = s$ to $x = \infty$:

$$\sum_{x=s}^{\infty}(x+1)^{r+1}\Pi(x+1) = z\sum_{x=s}^{\infty}\frac{a_{x+1}}{a_x}(x+1)^{r+1}\Pi(x).$$

Considering that

$$\sum_{x=s}^{\infty}(x+1)^{r+1}\Pi(x+1) = \sum_{x=s}^{\infty}x^{r+1}\Pi(x)-s^{r+1}\Pi(s)= a_{r+1}(s)-s^{r+1}\Pi(s)$$

we have

$$a_{r+1}(s) = s^{r+1}\Pi(s)+\sum_{x=s}^{\infty}(x+1)^{r+1}\Pi(x+1).$$

If we now use the relation (2.1) we obtain the following theorem:

THEOREM 2.1. *The incomplete moments about the origin of the power series distribution* (1.50) *are given by the following relation*:

$$a_{r+1}(s) = s^{r+1}\Pi(s)+z\sum_{x=s}^{\infty}(x+1)^{r+1}\frac{a_{x+1}}{a_x}\Pi(x) \tag{2.2}$$

and the complete moments of the distribution with the variable taking values from zero by:

$$a_{r+1} = z\sum_{x=0}^{\infty}(x+1)^{r+1}\frac{a_{x+1}}{a_x}\Pi(x), \qquad r = 0,1,2,\ldots \tag{2.3}$$

We derive the formula (2.3) from the formula (2.2) at once putting $s = 0$.

Deducing from (2.2) or (2.3) the recurrence relations for the moments of the specific cases of the formula (1.50) it is necessary only to calculate the relation a_{x+1}/a_x. I give here a few applications of the formulae (2.2) and (2.3).

(a) The binomial distribution. The substitutions:

$$f(z) = (1+z)^n, \qquad a_x = C_n^x, \qquad z = \frac{p}{1-p} = \frac{p}{q}$$

allow to derive for the random variable X the binomial distribution in the form

$$\Pi(x) = C_n^x(1+z)^{-n}z^x, \qquad x = 0, 1, 2, \ldots, n.$$

In this case

$$\frac{a_{x+1}}{a_x} = \frac{n-x}{x+1}$$

thus putting it into (2.2) we have

$$(2.4) \quad a_{r+1}(s) = s^{r+1}C_n^s(1+z)^{-n}z^s + z\sum_{x=s}^{n}(n-x)(x+1)^r\Pi(x)$$

$$= s^{r+1}C_n^s p^s q^{n-s} + \frac{p}{q}\left[n\sum_{i=0}^{r}\binom{r}{i}a_{r-i}(s) - \sum_{i=0}^{r-1}\binom{r}{i}a_{r+1-i}(s)\right].$$

The next transformation leads (2.4) to the formula (1.14) and then at $s = 0$ to the formula (1.15).

(b) The Poisson distribution. If we assume

$$f(z) = e^z \quad \text{and} \quad a_x = \frac{1}{x!}$$

the function $\Pi(x)$ takes the form of the Poisson distribution

$$\Pi(x) = e^{-z}\frac{z}{x!}, \qquad x = 0, 1, 2, \ldots$$

In this case

$$\frac{a_{x+1}}{a_x} = \frac{1}{x+1}$$

then after substituting in (2.2) we have

$$(2.5) \qquad a_{r+1}(s) = s^{r+1}e^{-z}\frac{z^s}{s!} + z\sum_{x=s}^{\infty}(x+1)^r\Pi(x)$$

$$= z\left[s^r\Pi(s-1) + \sum_{i=0}^{r}\binom{r}{i}a_{r-i}(s)\right].$$

After further transformation we obtain from (2.5) the formulae (1.16) and (1.17). It is possible to derive these relations taking into consideration the fact that the Poisson distribution is the limit distribution for the binomial distribution, assuming $n \to \infty$, $p \to 0$ so that $\lim np = z > 0$.

(c) The negative binomial distribution. Taking

$$f(z) = (1-z)^{-n}, \quad a_x = (-1)^x \binom{-n}{x}, \quad n > 0,\ 0 < z < 1$$

we obtain the so-called negative binomial distribution in the form

$$\Pi(x) = (-1)^x \binom{-n}{x} (1-z)^n z^x, \quad x = 0, 1, 2, \ldots \tag{2.6}$$

equivalent to (1.20) for $z = p$.

We find that

$$\frac{a_{x+1}}{a_x} = \frac{n+x}{x+1}.$$

Then it follows from (2.2) that

$$a_{r+1}(s) = s^{r+1}\Pi(s) + p \sum_{x=s}^{\infty} (n+x)(x+1)^r \Pi(x)$$

$$= s^{r+1}\Pi(s) + p \sum_{i=0}^{r} \left[n \binom{r}{i} a_{r-i}(s) + \binom{r}{i} a_{r+1-i}(s) \right].$$

which leads to (1.21) and (1.22).

In particular, putting $n = 1$ we get the relations for the moments of the geometric distribution (1.62). Thus we have

$$a_{r+1}(s) = \frac{1}{q} \left[s^{r+1}\Pi(s) + p \sum_{i=0}^{r} \binom{r+1}{i+1} a_{r-i}(s) \right], \tag{2.7}$$

$$a_{r+1} = \frac{p}{q} \sum_{i=0}^{r} \binom{r+1}{i+1} a_{r-i}. \tag{2.8}$$

For illustration we calculate a few first moments from the formula (2.8):

$$a_1 = \frac{p}{q}, \quad a_2 = \frac{p}{q}\left(2\frac{p}{q}+1\right), \quad a_3 = \frac{p}{q}\left[6\left(\frac{p}{q}\right) + 6\frac{p}{q} + 1\right].$$

Apparently (1.16) can be also found from (1.21) when the following limit theorem is used:

If $n \to \infty$ and $p \to 0$ so that the condition $\lim np/q = \lambda > 0$ is fulfilled, the probability function of the negative binomial distribution tends to the probability function of the Poisson distribution with the parameter $\lambda > 0$ ([6], p. 181, German ed. p. 151).

Taking $z = \eta/1+\eta$ and $n = \lambda/\eta$, $\eta > 0$, $\lambda > 0$ we get from (2.6) a form of this distribution called Pólya–Eggenberger distribution

$$(2.9) \qquad \Pi(x) = \frac{\Gamma\left(\frac{\lambda}{\eta}+x\right)}{x!\,\Gamma\left(\frac{\lambda}{\eta}\right)}\left(\frac{\eta}{1+\eta}\right)^{x}(1+\eta)^{-\frac{\lambda}{\eta}}$$

$$= \binom{\frac{\lambda}{\eta}+x-1}{x}\left(\frac{\eta}{1+\eta}\right)^{x}(1+\eta)^{-\frac{\lambda}{\eta}}, \qquad x = 0, 1, 2, 3, \ldots$$

In this case from (1.21) and (1.22) we have

$$(2.10) \quad a_{r+1}(s) = (1+\eta)\{s^{r+1}\Pi(s)+\frac{\eta}{1+\eta}\sum_{i=0}^{r}\left[\frac{\lambda}{\eta}\binom{r}{i}+\binom{r}{i+1}\right]a_{r-i}(s)\},$$

$$(2.11) \qquad a_{r+1} = \sum_{i=0}^{r}\left[\lambda\binom{r}{i}+\eta\binom{r}{i+1}\right]a_{r-i}.$$

In order to get the recurrence relations (2.10) and (2.11) for the moments of the Pólya–Eggenberger distribution we can also use the following theorem:

If $n \to \infty$, $p \to 0$, $a \to 0$ so that $np \to \lambda$, $na \to \eta$, the limit form for the probability function of the Pólya distribution given by the formula (1.59) *is the probability function of the Pólya–Eggenberger distribution* (2.9). ([5], p. 128, English ed. p. 131).

For illustration I present a few first moments about the origin calculated according to the formula (1.22) and then according to the formula (2.11):

$$a_1 = n\frac{p}{q}, \qquad a_2 = n\frac{p}{q}\left[(n+1)\frac{p}{q}+1\right],$$

$$a_3 = n\frac{p}{q}\left[(n+1)(n+2)\left(\frac{p}{q}\right)^2+3(n+1)\frac{p}{q}+1\right];$$

$$a_1 = \lambda, \qquad a_2 = \lambda(\lambda+1+\eta),$$

$$a_3 = \lambda[(\lambda+2\eta)(\lambda+1+\eta)+2\lambda+\eta+1].$$

(d) The logarithmic distribution. Consider now the logarithmic distribution defined by the formulae (1.64) and (1.65) for which

$$f(z) = -\log(1-z) \quad \text{and} \quad a_x = \frac{1}{x}.$$

In this case

$$\frac{a_{x+1}}{a_x} = \frac{x}{x+1}.$$

On the ground of (2.2) we have then

$$a_{r+1}(s) = s^{r+1}\Pi(s)+p\sum_{x=s}^{\infty} x(x+1)^r \Pi(s)$$

otherwise

$$a_{r+1}(s) = s^{r+1}\Pi(s)+p\sum_{i=0}^{r}\binom{r}{i} a_{r+1-i}(s)$$

$$= s^{r+1}\Pi(s)+p\left[a_{r+1}(s)+\sum_{i=0}^{r-1}\binom{r}{i+1} a_{r-i}(s)\right]$$

then

$$a_{r+1}(s) = \frac{1}{q}\left[s^{r+1}\,\Pi(s)+p\sum_{i=0}^{r-1}\binom{r}{i+1} a_{r-i}(s)\right]. \tag{2.12}$$

Assuming $s = 1$ we get the complete moments

$$a_{r+1} = \frac{p}{q}\left[A+\sum_{i=0}^{r-1}\binom{r}{i+1} a_{r-i}\right]. \tag{2.13}$$

The relations can also be derived in another way. It follows from the formula (1.20) that

$$\sum_{k=1}^{\infty} P(X = k) = 1-q^n.$$

If we divide both sides of this equality by $1-q^n$ we obtain

$$\frac{1}{1-q^n}\sum_{k=1}^{\infty} P(X = k) = 1.$$

As we can see the above expression may be considered as the new probability function of the random variable which we denote by Y and which takes the values $l = 1, 2, 3, \ldots$

Thus we have

$$P(Y = l) = \frac{(-1)^l}{1-q^n}\binom{-n}{l}p^l q^n, \qquad l = 1, 2, 3, \ldots \tag{2.14}$$

It is the probability function of the truncated negative binomial distribution. Taking into consideration the form of (2.14) we obtain directly the formula for the moments $\gamma_r(s)$ of the truncated negative binomial distribution

$$\gamma_{r+1}(s) = \frac{1}{1-q^n}\, a_{r+1}(s), \qquad r = 0, 1, 2, \ldots \tag{2.15}$$

where $a_{r+1}(s)$ denotes here the incomplete moment about the origin of the negative binomial distribution.

It is known that if $n \to 0$ then the probability function of the logarithmic distribution (1.64) is the limit form of (2.14); with that we have

$$A = \lim_{n\to 0} \frac{nq^n}{1-q^n}, \quad q = 1-p \tag{2.16}$$

([18], pp. 131–132 or [13], p. 310) for the coefficient A defined by (1.65).

We transform (2.15) using (1.21) as follows:

$$\gamma_{r+1}(s) = \frac{s^{r+1}}{q}\cdot\frac{\Pi(s)}{1-q^n} + n\frac{p}{q}\cdot\frac{a_0(s)}{1-q^n} + \frac{p}{q}\sum_{i=0}^{r-1}\left[n\binom{r}{i}+\binom{r}{i+1}\right]\frac{a_{r-i}(s)}{1-q^n}$$

$$= \frac{s^{r+1}}{q}\cdot\frac{\Pi(s)}{1-q^n} + n\frac{p}{q}\gamma_0(s) + \frac{p}{q}\sum_{i=0}^{r-1}\left[n\binom{r}{i}+\binom{r}{i+1}\right]\gamma_{r-i}(s).$$

At $n \to 0$ the second component and the first component in the bracketts fall off and the moments of the truncated negative binomial distribution give the moments of the logarithmic distribution. Besides, at $s = 1$ using (1.64) we get the formula (2.13).

2.2. The Pólya distribution. Let us denote the so-called factorial polynomial with the step h by $x^{[k,h]}$, i.e.

$$x^{[k,h]} = x(x-h)(x-2h)\dots[x-(k-1)h]. \tag{2.17}$$

Using this notation we can put down the probability function of the Pólya distribution (1.53) as follows:

$$\Pi(k) = \binom{n}{k}\frac{p^{[k,-a]}q^{[n-k,-a]}}{1^{[n,-a]}} \tag{2.18}$$

where $0 < p < 1$, $q = 1-p$ and the numbers k and a satisfy the conditions: $k = 0, 1, 2, \dots, n$

$$-ka \leqslant p \quad \text{and} \quad -(n-k)a \leqslant q.$$

Note that

$$a_{r+1}(s) = \sum_{k=s}^{n} k^{r+1}\Pi(k) = \sum_{k=s-1}^{n-1}(k+1)^{r+1}\Pi(k+1)$$

$$= \sum_{k=s-1}^{n-1}(k+1)^{r+1}\frac{n!}{(k+1)!(n-k-1)!}\cdot\frac{p^{[k+1,-a]}q^{[n-k-1,-a]}}{1^{[n,-a]}}$$

$$= np\sum_{k=s-1}^{n-1}(k+1)^r\binom{n-1}{k}\frac{(p+a)^{[k,-a]}q^{[n-k-1,-a]}}{(1+a)^{[n-1,-a]}}$$

$$= np\sum_{k=s-1}^{n-1}(k+1)^r\Pi^*(k)$$

where

$$(2.19)\qquad \Pi^*(k) = \binom{n-1}{k}\frac{(p+a)^{[k,-a]}q^{[n-k-1,-a]}}{(1+a)^{[n-1,-a]}}, \qquad k = 0, 1, 2, \ldots, n-1$$

denotes the conditional probability in the Pólya distribution which can be interpreted as the probability of obtaining k individuals of A kind and $n-k-1$ individuals of another kind according to the discussed in 1.2 scheme of Pólya in the result of $n-1$ draws assuming that an individual of A kind has been drawn from discussed population.

Respecting the Newton formula and denoting the moment of the distribution (2.19) by a_r^* we obtain the recurrence formula for the incomplete moments about the origin

$$(2.20)\qquad a_{r+1}(s) = np\sum_{i=0}^{r}\binom{r}{i}a_i^*(s-1)$$

and in the particular case of $s = 1$ the formula for the complete moments about the origin

$$(2.21)\qquad a_{r+1} = np\sum_{i=0}^{r}\binom{r}{i}a_i^*, \qquad r = 0, 1, 2, \ldots$$

Structurally the formula (2.21) coincides with the formula (1.42) and is identical with it in the special case when we derive the binomial distribution from the Pólya distribution.

We shall evaluate a few first moments according to (2.20):

$$a_1(s) = npa_0^*(s-1).$$

Notice that $a_1^*(s-1)$ can be obtained from the just reached formula for $a_1(s)$ replacing n by $n-1$, p by $p+a$, 1 by $1+a$ and $a_0^*(s-1)$ by $a_0^{**}(s-2)$ where $a_0^{**}(s-2)$ is evaluated for the distribution (2.18) with the parameters: $n-2, p+2a, 1+2a$, i.e.

$$a_0^{**}(s-2) = \sum_{k=s-2}^{n-2}\binom{n-2}{k}\frac{(p+2a)^{[k,-a]}q^{[n-k-2,-a]}}{(1+2a)^{[n-2,-a]}}.$$

Hence

$$\begin{aligned} a_2(s) &= np[a_0^*(s-1)+a_1^*(s-1)] \\ &= np\left[a_0^*(s-1)+\frac{(n-1)(p+a)}{1+a}a_0^{**}(s-2)\right] \\ &= np\,a_0^*(s-1)+\frac{n^{[2,1]}p^{[2,-a]}}{1^{[2,-a]}}a_0^{**}(s-2). \end{aligned}$$

In similar mood we evaluate

$$\begin{aligned}a_3(s) &= np\,[a_2^*(s-1)+2a_1^*(s-1)+a_0^*(s-1)]\\&= np\left[\frac{(n-1)(p+a)}{1+a}\,a_0^{**}(s-2)+\frac{(n-1)^{[2,1]}(p+a)^{[2,-a]}}{(1+a)^{[2,-a]}}\,a_0^{***}(s-3)+\right.\\&\qquad\left.+2\,\frac{(n-1)(p+a)}{1+a}\,a_0^{**}(s-2)+a_0^*(s-1)\right]\\&= \frac{n^{[1,1]}p^{[1,-a]}}{1^{[1,-a]}}\,a_0^*(s-1)+3\,\frac{n^{[2,1]}p^{[2,-a]}}{1^{[2,-a]}}\,a_0^{**}(s-2)+\\&\qquad+\frac{n^{[3,1]}p^{[3,-a]}}{1^{[3,-a]}}\,a_0^{***}(s-3).\end{aligned}$$

For the next moments we have

$$\begin{aligned}a_4(s) &= \frac{n^{[1,1]}p^{[1,-a]}}{1^{[1,-a]}}\,a_0^*(s-1)+7\,\frac{n^{[2,1]}p^{[2,-a]}}{1^{[2,-a]}}\,a_0^{2*}(s-2)+\\&\qquad+6\,\frac{n^{[3,1]}p^{[3,-a]}}{1^{[3,-a]}}\,a_0^{3*}(s-3)+\frac{n^{[4,1]}p^{[4,-a]}}{1^{[4,-a]}}\,a_0^{4*}(s-4),\end{aligned}$$

$$\begin{aligned}a_5(s) &= \frac{n^{[1,1]}p^{[1,-a]}}{1^{[1,-a]}}\,a_0^*(s-1)+15\,\frac{n^{[2,1]}p^{[2,-a]}}{1^{[2,-a]}}\,a_0^{2*}(s-2)+\\&\qquad+25\,\frac{n^{[3,1]}p^{[3,-a]}}{1^{[3,-a]}}\,a_0^{3*}(s-3)+10\,\frac{n^{[4,1]}p^{[4,-a]}}{1^{[4,-a]}}\,a_0^{4*}(s-4)+\\&\qquad+\frac{n^{[5,1]}p^{[5,-a]}}{1^{[5,-a]}}\,a_0^{5*}(s-5).\end{aligned}$$

In all these examples $a_0^{m*}(s-m)$ $(m = 1, 2, \ldots)$ denotes the moment of order zero of the distribution (2.18) with the respectively changed parameters, i.e.

$$a_0^{m*}(s-m) = \sum_{k=s-m}^{n-m}\binom{n-m}{k}\frac{(p+ma)^{[k,-a]}q^{[n-m-k,-a]}}{(1+ma)^{[n-m,-a]}}, \qquad m = 1, 2, \ldots$$

Let us note that the coefficients at the successive expressions of the type

$$\frac{n^{[j,1]}p^{[j,-a]}}{1^{[j,-a]}}, \qquad j = 1, 2, 3, \ldots, r$$

are the Stirling's numbers of the second kind.

We demonstrate that the Stirling's numbers satisfy the following relation

$$\sum_{l=m}^{r} \binom{r}{l} S_{m,l} = S_{m+1,r+1}, \qquad m \geqslant 1. \tag{2.22}$$

We use the method of the mathematical induction. Note that (2.22) is valid for $r = 1$. Assuming now the validity of (2.22) for r we demonstrate that it is also true for $r+1$, i.e. we demonstrate the relation

$$\sum_{l=m}^{r+1} \binom{r+1}{l} S_{m,l} = S_{m+1,r+2}.$$

We have

$$\mathrm{L} = \sum_{l=m}^{r+1} \binom{r+1}{l} S_{m,l} = \sum_{l=m}^{r+1} \left[\binom{r}{l} + \binom{r}{l-1}\right] S_{m,l}.$$

On the ground of the property (1,46) of the Stirling's numbers we have next

$$\mathrm{L} = \sum_{l=m}^{r} \binom{r}{l} S_{m,l} + \sum_{l=m}^{r+1} \binom{r}{l-1} S_{m,l}.$$

Taking now into account the induction assumption we get

$$\mathrm{L} = S_{n+1,r+1} + \sum_{l=m}^{r} \binom{r}{l-1} S_{m,l} + S_{m,r+1}.$$

Putting $l-1 = j$, $m-1 \leqslant j \leqslant r-1$ we have

$$\mathrm{L} = S_{m+1,r+1} + \sum_{j=m-1}^{r-1} \binom{r}{j} S_{m,j+1} + S_{m,r+1}$$

$$= S_{m+1,r+1} + \sum_{j=m}^{r-1} \binom{r}{j} S_{m,j+1} + \binom{r}{m-1} S_{m,m} + S_{m,r+1}.$$

On the ground of (1.44) and (1.45) we have now

$$\mathrm{L} = S_{m+1,r+1} + \sum_{j=m}^{r-1} \binom{r}{j} (mS_{m,j} + S_{m-1,j}) + \binom{r}{m-1} + S_{m,r+1}$$

$$= S_{m+1,r+1} + m\sum_{j=m}^{r-1} \binom{r}{j} S_{m,j} + \sum_{j=m}^{r-1} \binom{r}{j} S_{m-1} + \binom{r}{m-1} + S_{m,r+1}$$

$$= S_{m+1,r+1} + m\sum_{j=m}^{r} \binom{r}{j} S_{m,j} - mS_{m,r} + \sum_{j=m}^{r-1} \binom{r}{j} S_{m-1,j} + \binom{r}{m-1} + S_{m,r+1}.$$

Using the induction assumption we obtain

$$\begin{aligned}
\mathrm{L} &\equiv S_{m+1,r+1} + mS_{m+1,r+1} - mS_{m,r} + \sum_{j=m}^{r-1} \binom{r}{j} S_{m-1,j} + \binom{r}{m-1} + S_{m,r+1} \\
&= (m+1)S_{m+1,r+1} - mS_{m,r} + \sum_{j=m}^{r-1} \binom{r}{j} S_{m-1,j} + \binom{r}{m-1} + S_{m,r+1} \\
&= (m+1)S_{m+1,r+1} - mS_{m,r} + \sum_{j=m-1}^{r} \binom{r}{j} S_{m-1,j} - \binom{r}{m-1} S_{m-1,m-1} - \\
&\quad - S_{m-1,r} + \binom{r}{m-1} + S_{m,r+1}.
\end{aligned}$$

Using again the induction assumption and the relations (1.44) and (1.45) we have

$$\begin{aligned}
\mathrm{L} &\equiv (m+1)S_{m+1,r+1} + S_{m,r+1} - (mS_{m,r} + S_{m-1,r}) + S_{m,r+1} \\
&= S_{m+1,r+2} - S_{m,r+1} + S_{m,r+1} = S_{m+1,r+2} \equiv \mathrm{R}.
\end{aligned}$$

On the basis of (2.22) we shall demonstrate the following theorem:

THEOREM 2.2. *The incomplete moment about the origin of r-th order of the random variable with the Pólya distribution* (2.18) *can be expressed by*

$$a_r(s) = \sum_{m=1}^{r} \frac{n^{[m,1]} p^{[m,-a]}}{1^{[m,-a]}} S_{m,r} a_0^{m*}(s-m), \qquad r = 1, 2, \ldots \tag{2.23}$$

where $S_{m,r}$ denotes the Stirling's numbers given by (1.43)–(1.46).

We demonstrate the validity of this relation using the mathematical induction and (2.20). Apparently (2.23) is valid for $r = 1$. Assuming now the validity of that formula for r we shall demonstrate that it is also true for $r+1$. We have, namely:

$$\begin{aligned}
a_{r+1}(s) &= np \sum_{l=0}^{r} \binom{r}{l} a_l^*(s-1) \\
&= np\left[a_0^*(s-1) + \sum_{l=1}^{r} \binom{r}{l} \sum_{m=1}^{l} \frac{(n-1)^{[m,1]}(p+a)^{[m,-a]}}{(1+a)^{[m,-a]}} S_{m,l} a_0^{(m+1)*}(s-1-m)\right] \\
&= np\left[a_0^*(s-1) + \sum_{m=1}^{r} \frac{(n-1)^{[m,1]}(p+a)^{[m,-a]}}{(1+a)^{[m,-a]}} a_0^{(m+1)*}(s-1-m) \sum_{l=m}^{r} \binom{r}{l} S_{m,l}\right].
\end{aligned}$$

On the ground of (2.22) we finally obtain

$$a_{r+1}(s)$$

$$= np\left[a_0^*(s-1)+\sum_{m=1}^{r}\frac{(n-1)^{[m,1]}(p+a)^{[m,-a]}}{(1+a)^{[m,-a]}}\; S_{m+1,r+1}\,a_0^{(m+1)*}(s-1-m)\right]$$

$$= npa_0^*(s-1)+\sum_{m=1}^{r}\frac{n^{[m+1,1]}\,p^{[m+1,-a]}}{1^{[m+1,-a]}}\; S_{m+1,r+1}\,a_0^{(m+1)*}(s-1-m)$$

$$= \sum_{m=1}^{r+1}\frac{n^{[m,1]}\,p^{[m,-a]}}{1^{[m,-a]}}\; S_{m,r+1}\,a_0^{m*}(s-m).$$

Putting $s = 1$ we simplify (2.23) and obtain the formula for the moments about the origin

$$a_r = \sum_{m=1}^{r}\frac{n^{[m,1]}\,p^{[m,-a]}}{1^{[m,-a]}}\; S_{m,r}. \tag{2.24}$$

The relation (2.24) allows to derive the formula for the so-called factorial moments of the discussed distribution. As it is known ([18], p. 63) we call the expression

$$a_{[r]} = \sum_{i=r}^{\infty} x_i^{[r]} p_i, \qquad r = 1,2,3,\ldots \tag{2.25}$$

the factorial moment about the origin of rth order in the case of the discrete random variable.

Between the moments and the factorial moments the following relation holds:

$$a_r = \sum_{m=1}^{r} a_{[m]}\, S_{m,r}, \qquad r = 1,2,3,\ldots \tag{2.26}$$

(comp. [16], p. 296; [4], p. 106; [1], p. 79; [14] formula (7.3) p. 217; [27], p. 371).

It results from (2.26) that in the case of the Pólya distribution the formula for the factorial moments is as follows:

$$a_{[r]} = \frac{n^{[r,1]}\,p^{[r,-a]}}{1^{[r,-a]}}, \qquad r = 1,2,3,\ldots \tag{2.27}$$

In the particular cases mentioned below it is possible to derive the factorial moments on the same ground. As we have already written

Riordan, who found the relation of the type (2.26) for binomial, Poisson and hypergeometric distributions, did not take into account the possibility of obtaining the factorial moments in his paper. As far as those distributions are concerned we shall cite the papers in which the factorial moments have been derived through the special research made for this purpose.

The incomplete moment about the origin of rth order of the hypergeometric distribution (1.4) is expressed by

$$a_r(s) = \sum_{m=1}^{r} \frac{n^{[m,1]}(Np)^{[m,1]}}{N^{[m,1]}} S_{m,r}\, a_0^{m*}(s-m). \tag{2.28}$$

We obtain that relation from (2.23) after putting $a = -1/N$ and after some transformations. In case $s = 1$ (2.28) gives (1.49). We evaluate $a_0^{m*}(s-m)$ for the distribution (1.4) with the parameters: $N-m$, $n-m$, $Np-m$.

The factorial moment about the origin of the hypergeometric distribution is as follows:

$$a_{[r]} = \frac{n^{[r,1]}(Np)^{[r,1]}}{N^{[r,1]}} \tag{2.29}$$

([18], p. 147).

The incomplete moment about the origin of rth order of the binomial distribution (1.2) is expressed by

$$a_r(s) = \sum_{m=1}^{r} n^{[m,1]} p^m S_{m,r}\, a_0^{m*}(s-m). \tag{2.30}$$

This relation is the immediate consequence of (2.23) for $a = 0$. In case $s = 1$ it is identical with (1.47). We evaluate $a_0^{m*}(s-m)$ for the distribution (1.2) with the parameters: p, $n-m$.

The factorial moment about the origin of rth order of the binomial distribution is as follows:

$$a_{[r]} = n^{[r,1]} p^r \tag{2.31}$$

([23], p. 858; [25], p. 440; [18], pp. 90 and 122; [1], p. 82).

The incomplete moment about the origin of the Poisson distribution is expressed by

$$a_r(s) = \sum_{m=1}^{r} \lambda^m S_{m,r}\, a_0(s-m). \tag{2.32}$$

The relation (2.32) is the consequence of (2.30) if we take the condition $\lambda = \lim_{n\to\infty} np$ at which the Poisson distribution is the limit distribution for the point binomial. In case $s = 1$ (2.32) is identical with (1.48).

The factorial moment about the mean of rth order of the Poisson distribution is as follows:

$$a_{[r]} = \lambda^r \tag{2.33}$$

([1], p. 83; [14], p. 219; [18], p. 147).

The moment about the origin of rth order of the random variable with the negative binomial distribution (1.20) is expressed by

$$a_r(s) = \sum_{m=1}^{r} n^{[m,-1]}\left(\frac{p}{q}\right)^m S_{m,r}\, a_0^{m*}(s-m) \tag{2.34}$$

where $a_0^{m*}(s-m)$ is the incomplete moment of this distribution with the respectively changed parameter, i.e. written on the basis of (2.17) in the form

$$\Pi^{m*}(k) = \frac{(n+m)^{[k,-1]}}{k!}\, p^k q^{n+m}, \tag{2.35}$$

$k = 0, 1, 2, \ldots,\ m = 1, 2, 3, \ldots$

The relation (2.34) may be derived by following the procedure shown by the demonstration of the relation (2.23) for the moments of the Pólya distribution.

Taking $p = \eta/1+\eta$, $n = \lambda/\eta$ and consequently $q = 1/(1+\eta)$, $p/q = \eta$ we obtain the formula for the moments of the Pólya–Eggenberger distribution (2.9):

$$a_r(s) = \sum_{m=1}^{r} \left(\frac{\lambda}{\eta}\right)^{[m,-1]} \eta^m S_{m,r}\, a_0^{m*}(s-m). \tag{2.36}$$

If $s = 1$ the relations (2.34) and (2.36) are being reduced to the moments about the origin. We can also obtain (2.36) on the basis of the limit theorem on page 22.

For illustration we shall evaluate a few first moments about the origin from (2.34):

$$a_1 = n\frac{p}{q},$$

$$a_2 = n^{[2,-1]}\left(\frac{p}{q}\right)^2 + n\frac{p}{q},$$

$$a_3 = n^{[3,-1]}\left(\frac{p}{q}\right)^3 + 3n^{[2,-1]}\left(\frac{p}{q}\right)^2 + n\frac{p}{q},$$

$$a_4 = n^{[4,-1]}\left(\frac{p}{q}\right)^4 + 6n^{[3,-1]}\left(\frac{p}{q}\right)^3 + 7n^{[2,-1]}\left(\frac{p}{q}\right)^2 + n\frac{p}{q}.$$

(Comp. with the false formula (5.13.11) in [6], p. 179, German ed. p. 150).

Inserting $n = 1$ into (2.34) or $\lambda = \eta$ into (2.36) we obtain the relations for the moments of the geometric distribution (1.62), namely

$$(2.37) \qquad a_r(s) = \sum_{m=1}^{r} \left(\frac{p}{q}\right)^m m!\, S_{m,r} a_0^{m*}(s-m),$$

$$(2.38) \qquad a_r(s) = \sum_{m=1}^{r} \eta^m m!\, S_{m,r} a_0^{m*}(s-m).$$

From the above given relations follow the relations for the factorial moments of the discussed distributions and so:

Negative binomial distribution

$$(2.39) \qquad a_{[r]} = n^{[r,-1]} \left(\frac{p}{q}\right)^r$$

([18], p. 131);

Pólya–Eggenberger distribution

$$(2.40) \qquad a_{[r]} = \left(\frac{\lambda}{\eta}\right)^{[r,-1]} \eta^r;$$

Geometric distribution

$$(2.41) \qquad a_{[r]} = \left(\frac{p}{q}\right)^r r!$$

or

$$(2.42) \qquad a_{[r]} = \eta^r r!$$

([14], p. 219).

It follows from the remarks made in 2.1 that the formula for the incomplete moments $\gamma_r(s)$ of the truncated negative binomial distribution is as follows:

$$(2.43) \qquad \gamma_r(s) = \frac{1}{1-q^n} \sum_{m=1}^{r} n^{[m,-1]} \left(\frac{p}{q}\right)^m S_{m,r} a_0^{m*}(s-m)$$

where

$$(2.44) \qquad a_0^{m*}(s-m) = \sum_{k=s-m}^{\infty} \frac{(n+m)^{[k,-1]}}{k!} p^k q^{n+m}, \qquad m = 1, 2, \ldots$$

Inserting $s = 1$ into (2.43) we obtain the moments about the origin of the discussed distribution. If we write (2.43) in the form

$$\gamma_r(s) = \frac{nq^n}{1-q^n} \cdot \frac{1}{q^n} \sum_{m=1}^{r} (n+1)^{[m-1,-1]} \left(\frac{p}{q}\right)^m S_{m,r} a_0^{m*}(s-m)$$

and assume $n \to 0$ then according to (2.44) and to the already discussed limit theorem (2.16) we obtain the relation for the moments of the logarithmic distribution (1.64):

$$a_r(s) = A \sum_{m=1}^{r} (m-1)! \left(\frac{p}{q}\right)^m S_{m,r}\, \delta_0^{m*}(s-m)$$

where

$$\delta_0^{m*}(s-m) = q^m \sum_{k=s-m}^{\infty} \frac{m^{[k,-1]}}{k!}\, p^k.$$

Hence

$$a_r(s) = A \sum_{m=1}^{r} (m-1)!\, p^m S_{m,r} \sum_{k=s-m}^{\infty} \frac{m^{[k,-1]}}{k!}\, p^k. \tag{2.45}$$

From (2.45) we obtain the complete moments at $s = 1$:

$$\begin{aligned} a_r &= A \sum_{m=1}^{r} (m-1)!\, p^m S_{m,r} \sum_{k=0}^{\infty} \frac{m^{[k,-1]}}{k!}\, p^k \\ &= A \sum_{m=1}^{r} (m-1)! \left(\frac{p}{q}\right)^m S_{m,r} \end{aligned} \tag{2.46}$$

or on the basis of (1.43) in the form:

$$a_r = A \sum_{m=1}^{r} \left(\frac{p}{q}\right)^m \frac{\Delta^m 0^r}{m}. \tag{2.47}$$

The first moments of the discussed distribution are the following:

$$a_1 = A\,\frac{p}{q}, \quad a_2 = A\,\frac{p}{q^2}, \quad a_3 = A\,\frac{p(1+p)}{q^3}, \quad a_4 = A\,\frac{p(1+4p+p^2)}{q^4}.$$

The methods of reaching the moments for the logarithmic distribution given here seem to be much more practical than the method presented by M. G. Kendall and A. Stuart ([18], p. 133). There the moments have been found as the coefficients of the series expansion of the moment-generating function and this method does not give any general results.

The factorial moment about the origin of rth order of the logarithmic distribution is the following:

$$a_{[r]} = (r-1)! \left(\frac{p}{q}\right)^r. \tag{2.48}$$

3. The recurrence relations for the moments about the mean of the Pólya distribution

3.1. The first method. The incomplete moment about the mean of rth order of the Pólya distribution is defined by (1.10) where $E(X) = a_1 = np$ and $\Pi(k)$ is given by (2.18).

Let us note that for the Pólya distribution given by (2.18) the following identities hold:

$$k[q+(n-k)a] = k-np+(n-k)(p+ka), \tag{3.1}$$

$$(n-k)(p+ka) = -a(k-np)^2+c_1(k-np)+c_2 \tag{3.2}$$

where

$$c_1 = na(q-p)-p, \quad c_2 = npq(1+na), \tag{3.3}$$

$$k[q+(n-k)a]\Pi(k) = [n-(k-1)][p+(k-1)a]\Pi(k-1). \tag{3.4}$$

Proof. To 3.1.

$$\begin{aligned} \mathrm{L} \equiv k[q+(n-k)a] &= k-kp+nka-k^2a \\ &= k-np+np+nka-kp-k^2a \\ &= k-np+n(p+ka)-k(p+ka) \\ &= k-np+(n-k)(p+ka) \equiv \mathrm{R}. \end{aligned}$$

To 3.2.

$$\begin{aligned} \mathrm{L} &\equiv (n-k)(p+ka) = nka-k^2a+np-kp \\ &= -k^2a+nka(p+q)-kp+np(p+q) \\ &= -k^2a+nkap+nkaq-kp+np^2+npq \\ &= -ak^2+2aknp-an^2p^2-knap+an^2p^2+knaq-kp+np^2+npq \\ &= -a(k-np)^2+k(naq-nap-p)-np(naq-nap-p)+npq+n^2apq \\ &= -a(k-np)^2+(k-np)[na(q-p)-p]+npq(1+na) \equiv \mathrm{R}. \end{aligned}$$

To 3.4. At first note that the probability distribution (2.18) can be expressed as follows:

$$\Pi(k) = \binom{n}{k}\frac{p^{[k,-a]}}{(q+na)^{[k+1,a]}}\cdot\frac{q^{[n,-a]}}{1^{[n,-a]}}\cdot(q+na).$$

Thus

$$\begin{aligned} \mathrm{L} \equiv k[q+(n-k)a]\Pi(k) &= \binom{n}{k}\frac{p^{[k,-a]}}{(q+na)^{[k+1,a]}}\cdot\frac{q^{[n,-a]}}{1^{[n,-a]}}\cdot(q+na)k[q+(n-k)a] \\ &= \frac{n^{[k,1]}}{(k-1)!}\cdot\frac{p^{[k-1,-a]}}{(q+na)^{[k+1,a]}}\cdot[p+(k-1)a]\cdot\frac{q^{[n,-a]}}{1^{[n,-a]}}\cdot(q+na)[q+(n-k)a] \\ &= \binom{n}{k-1}\frac{p^{[k-1,-a]}}{(q+na)^{[k,a]}}\cdot\frac{q^{[n-(k-1),-a]}}{1^{[n,-a]}}\cdot(q+na)[n-(k-1)][p+(k-1)a] \\ &= [n-(k-1)][p+(k-1)a]\Pi(k-1). \end{aligned}$$

THEOREM 3.1. *The incomplete moment of r-th order of the Pólya distribution defined by the formula* (2.18) *may be expressed by the recurrence relation*

$$(3.5)\qquad \{(1+E)^{r-1}-E^{r-1}\}\big(-a\mu_2(s)+c_1\mu_1(s)+c_2\mu_0(s)\big) = \mu_r(s)-(s-np)^{r-1}\mu_1(s)$$

where the values of the coefficients c_1 and c_2 are defined by the relations (3.3) *and $r = 2, 3, \ldots$*

Proof. Multiply both sides of the identity (3.4) by $(k-np)^{r-1}$ and sum them up for the values k from s to n and then use the identities (3.1) and (3.2)

$$\begin{aligned} \mathrm{L} &\equiv \sum_{k=s}^{n} k[q+(n-k)a](k-np)^{r-1}\Pi(k) \\ &= \sum_{k=s}^{n} [(k-np)+(n-k)(p+ka)](k-np)^{r-1}\Pi(k) \\ &= \sum_{k=s}^{n} [(k-np)-a(k-np)^2+c_1(k-np)+c_2](k-np)^{r-1}\Pi(k), \end{aligned}$$

$$\mathrm{R} \equiv \sum_{k=s}^{n} [n-(k-1)][p+(k-1)a](k-np)^{r-1}\Pi(k-1).$$

Substituting $k-1 = k'$, $k = k'+1$ we get

$$\begin{aligned} \mathrm{R} &\equiv \sum_{k'=s-1}^{n-1} (n-k')(p+k'a)(k'-np+1)^{r-1}\Pi(k') \\ &= \sum_{k'=s}^{n} [-a(k'-np)^2+c_1(k'-np)+c_2](k'-np+1)^{r-1}\Pi(k')+ \\ &\quad +(s-np)^{r-1}[n-(s-1)][p+(s-1)a]\Pi(s-1). \end{aligned}$$

Regarding that $\mu_r(s) = E\mu_{r-1}(s)$ and $\mathrm{L} = \mathrm{R}$ we obtain

$$\begin{aligned} &\mu_r(s)+E^{r-1}\big(-a\mu_2(s)+c_1\mu_1(s)+c_2\mu_0(s)\big) \\ &= (1+E)^{r-1}\big(-a\mu_2(s)+c_1\mu_1(s)+c_2\mu_0(s)\big)+(s-np)^{r-1}s[q+(n-s)a]\Pi(s). \end{aligned}$$

After arrangement of this expression we have

$$\{(1+E)^{r-1}-E^{r-1}\}\big(-a\mu_2(s)+c_1\mu_1(s)+c_2\mu_0(s)\big) = \mu_r(s)-(s-np)^{r-1}\cdot s[q+(n-s)a]\Pi(s).$$

Putting $r = 1$ in this formula we obtain

$$(3.6)\qquad \mu_1(s) = s[q+(n-s)a]\Pi(s).$$

Taking this expression into account we reach the required formula (3.5).

If we assume $s = 0$ then $\mu_r(s) = \mu_r$ and we get the recurrence relation for the complete moments about the mean of the Pólya distribution

$$(3.7) \qquad \{(1+E)^{r-1} - E^{r-1}\}(-a\mu_2 + c_1\mu_1 + c_2\mu_0) = \mu_r .$$

We will demonstrate now that it is possible to establish the relation between the incomplete and complete moments about the mean.

THEOREM 3.2. *If the complete moment μ_r about the mean of the Pólya distribution is known then the incomplete moment $\mu_r(s)$ about the mean of r-th order of the same distribution may be obtained by the equality*

$$(3.8) \qquad \mu_r(s) = \mu_r\mu_0(s) + d_r\mu_1(s), \qquad r = 1, 2, 3, \ldots$$

where

$$(3.9) \qquad d_0 = \mu_1 = 0, \qquad d_1 = \mu_0 = 1$$

and the coefficients d_r satisfy the following recurrence relation

$$(3.10) \qquad \{(1+E)^{r-1} - E^{r-1}\}(-ad_2 + c_1 d_1 + c_2 d_0) = d_r - (s-np)^{r-1}$$

where c_1 and c_2 are keeping the meaning given by (33) *and d_0 and d_1 are defined by* (3.9).

Proof. We must fit the values of the coefficients d_r so that the relation (3.8) would be satisfied. Putting (3.8) into (3.5) we get

$$\{(1+E)^{r-1} - E^{r-1}\}[-a(\mu_2\mu_0(s) + d_2\mu_1(s)) + c_1(\mu_1\mu_0(s) + d_1\mu_1(s)) + \\ + c_2(\mu_0\mu_0(s) + d_0\mu_1(s))] = \mu_r\mu_0(s) + d_r\mu_1(s) - (s-np)^{r-1}\mu_1(s)$$

otherwise

$$\{(1+E)^{r-1} - E^{r-1}\}[\mu_1(s)(-ad_2 + c_1 d_1 + c_2 d_0) + \mu_0(s)(-a\mu_2 + c_1\mu_1 + c_2\mu_0)] \\ = \mu_1(s)[d_r - (s-np)^{r-1}] + \mu_0(s)\mu_r .$$

Comparing the coefficients at $\mu_1(s)$ we get the relation (3.10). For example putting $r = 2$ we have

$$-ad_2 + c_1 d_1 = d_2 - (s-np) \qquad \text{or} \qquad d_2(1+a) = c_1 d_1 + s - np .$$

On the ground of (3.9) we have

$$d_2 = \frac{c_1 + s - np}{1+a}$$

and taking (3.3) into account we get

$$d_2 = \frac{na(q-p) - p + s - np}{1+a} = \frac{na(q-p) + s - p(1+n)}{1+a} .$$

Making the analogous calculations at $r = 3$ we obtain

$$d_3 = \frac{d_2[2naq-2p(1+na)-a]+2npq(1+na)+na(q-p)-p+(s-np)^2}{1+2a}.$$

Putting in the discussed formulae $a = -1/N$ with N being the number of the individuals in the population we get as a particular case the relations (1.35), (1.37), (1.38).

If $a = 0$ we immediately obtain the formulae (1.30)–(1.32) for the moments of the binomial distribution.

Assuming $\lim_{n\to\infty} np = \lambda > 0$ we derive the formula for the incomplete moments of the Poisson distribution

$$\lambda\{(1+E)^{r-1}-E^{r-1}\}\mu_0(s) = \mu_r(s)-(s-\lambda)^{r-1}\mu_1(s) \tag{3.11}$$

which for $s = 0$ gives the formula (1.7) for the complete moments that had been obtained earlier by K. Pearson (but by another method). In particular, while keeping the conditions (3.9) and (3.10) we get the relation

$$\lambda\{(1+E)^{r-1}-E^{r-1}\}d_0 = d_r-(s-\lambda)^{r-1}. \tag{3.12}$$

The analogous procedure allows us to reach the recurrence relation for the incomplete moments of the negative binomial distribution

$$\frac{p}{q}\{(1+E)^{r-1}-E^{r-1}\}\left(\mu_1(s)+\frac{n}{q}\mu_0(s)\right)+\left(s-n\frac{p}{q}\right)^{r-1}\mu_1(s) = \mu_r(s) \tag{3.13}$$

and keeping the conditions (3.9) and (3.10) — the relation

$$\frac{p}{q}\{(1+E)^{r-1}-E^{r-1}\}\left(d_1+\frac{n}{q}d_0\right)+\left(s-n\frac{p}{q}\right)^{r-1} = d_r. \tag{3.14}$$

Using the substitutions already given on page 31 we obtain the relations for the moments of the Pólya–Eggenberger distribution in the form

$$\eta\{(1+E)^{r-1}-E^{r-1}\}\left[\mu_1(s)+\frac{\lambda}{\eta}(1+\eta)\mu_0(s)\right]+(s-\lambda)^{r-1}\mu_1(s) = \mu_r(s) \tag{3.15}$$

or keeping the conditions (3.9) and (3.10) — the relation

$$\eta\{(1+E)^{r-1}-E^{r-1}\}\left[d_1+\frac{\lambda}{\eta}(1+\eta)d_0\right]+(s-\lambda)^{r-1} = d_r. \tag{3.16}$$

Putting $s = 0$ we get the relations for the complete moments about the mean. For example, if we evaluate according to (3.13) for $r = 2, 3, 4$ we obtain

$$\mu_2 = n\frac{p}{q^2}, \quad \mu_3 = n\frac{p}{q^3}(1+p), \quad \mu_4 = n\frac{p}{q^2}\left[3(n+1+p)\frac{p}{q^2}+3\frac{p}{q}+1\right].$$

In the case of (3.15) the moments are given in the form

$$\mu_2 = \lambda(1+\eta),\quad \mu_3 = \lambda(1+\eta)(1+2\eta),\quad \mu_4 = \lambda(1+\eta)[1+3(1+\eta)(\lambda+2\eta)].$$

Putting $n = 1$ in the formulae (3.13), (3.14) or $\lambda = \eta$ in the formulae (3.15), (3.16) we obtain as a special case the relation for the moments about the mean of the geometric distribution, namely

$$(3.17)\quad \mu_r(s) = \frac{p}{q}\{(1+E)^{r-1}-E^{r-1}\}\left[\mu_1(s)+\frac{1}{q}\mu_0(s)\right]+\left(s-\frac{p}{q}\right)^{r-1}\mu_1(s)$$

or

$$(3.18)\quad \mu_r(s) = \eta\{(1+E)^{r-1}-E^{r-1}\}[\mu_1(s)+(1+\eta)\mu_0(s)]+(s-\eta)^{r-1}\mu_1(s).$$

Whereas keeping the conditions (3.9) and (3.10) we get the relations

$$(3.19)\qquad d_r = \frac{p}{q}\{(1+E)^{r-1}-E^{r-1}\}\left[d_1+\frac{1}{q}d_0\right]+\left(s-\frac{p}{q}\right)^{r-1},$$

$$(3.20)\qquad d_r = \frac{p}{q}\{(1+E)^{r-1}-E^{r-1}\}[d_1+(1+\eta)d_0]+(s-\eta)^{r-1}.$$

From (3.17) and (3.18) we obtain the complete moments about the mean at $s = 0$.

3.2. The second method. We will here discuss another method of reaching the moments about the mean of the Pólya distribution. For this purpose with the view to the form of the probability function (2.18) of that distribution and its expected value $a_1 = np$ we first perform the following transformations

$$\begin{aligned}(k-a_1)^r\Pi(k) &= (k-np)^r\Pi(k)\\ &= np(k-np)^{r-1}\left(1-p\frac{n}{k}\right)\binom{n-1}{k-1}\cdot\frac{(p+a)^{[k-1,-a]}q^{[n-k,-a]}}{(1+a)^{[n-1,-a]}}\\ &= np(k-np)^{r-1}[\Pi^*(k-1)-\Pi(k)]\end{aligned}$$

where $\Pi^*(k-1)$ is defined by (2.19). Denoting the expected value of the distribution $\Pi^*(k)$ by

$$a_1^* = \frac{(n-1)(p+a)}{1+a}$$

we can write

$$\begin{aligned}&\frac{1}{np}(k-a_1)^r\Pi(k)\\ &=\left[k-1-a_1^*+\frac{1+a+(n-1)(p+a)-np(1+a)}{1+a}\right]^{r-1}\Pi^*(k-1)-(k-a_1)^{r-1}\Pi(k)\\ &=\left[k-1-a_1^*+\frac{1+na(1-p)-p}{1+a}\right]^{r-1}\Pi^*(k-1)-(k-a_1)^{r-1}\Pi(k)\\ &=\left[k-1-a_1^*+q\frac{1+na}{1+a}\right]^{r-1}\Pi^*(k-1)-(k-a_1)^{r-1}\Pi(k).\end{aligned}$$

Thus

$$\frac{1}{np}\mu_r(s) = \sum_{k=s}^{n}\left[k-1-a_1^*+q\frac{1+na}{1+a}\right]^{r-1}\Pi^*(k-1)-\sum_{k=s}^{n}(k-a_1)^{r-1}\Pi(k).$$

Putting $k-1 = k'$ in the first component of the sum we get

$$\frac{1}{np}\mu_r(s) = \sum_{k'=s-1}^{n-1}\left[k'-a_1^*+q\frac{1+na}{1+a}\right]^{r-1}\Pi^*(k')-\mu_{r-1}(s)$$
$$= \left(E+q\frac{1+na}{1+a}\right)^{r-1}\mu_0^*(s-1)-\mu_{r-1}(s).$$

We have then

THEOREM 3.3. *The incomplete moment about the mean of r-th order of the Pólya distribution defined by the formula* (2.18) *is expressed by the recurrence relation*

$$\mu_r(s) = np\left[\left(E+q\frac{1+na}{1+a}\right)^{r-1}\mu_0^*(s-1)-\mu_{r-1}(s)\right], \tag{3.21}$$
$$r = 1, 2, 3, \ldots$$

where $\mu_r^*(s-1)$ *denotes the incomplete moment about the mean of the Pólya distribution defined by* (2.19), *i.e.*

$$\mu_0^*(s-1) = \sum_{k=s-1}^{n-1}\Pi^*(k), \qquad \mu_r^*(s-1) = \sum_{k=s-1}^{n-1}(k-a_1^*)^r\Pi^*(k).$$

If $s = 0$ we get the formula for the complete moments about the mean of the Pólya distribution

$$\mu_r = np\left[\left(E+q\frac{1+na}{1+a}\right)^{r-1}\mu_0^*-\mu_{r-1}\right], \qquad r = 2, 3, \ldots \tag{3.22}$$

For example we derive the second and the third moment about the mean from (3.22) by putting $r = 2$ and $r = 3$: We evaluate μ_2^* from the relation $\mu_2^* = a_2^*-a_1^{*2}$ using the formulae for a_1^* and a_2^* given on page 26. Thus

$$\mu_2 = npq\frac{1+na}{1+a},$$

$$\mu_3 = np\left[\left(E+q\frac{1+na}{1+a}\right)^2\mu_0^*-\mu_2\right] = np\left[\mu_2^*+q^2\left(\frac{1+na}{1+a}\right)^2-\mu_2\right]$$
$$= np\left\{\frac{(n-1)(p+a)}{1+a}\left[1+\frac{(n-2)(p+2a)}{1+2a}-\frac{(n-1)(p+a)}{1+a}\right]+\right.$$
$$\left.+q\frac{1+na}{1+a}\left(\frac{1+na}{1+a}q-np\right)\right\}.$$

If $a = 0$ we get the relations for the moments about the mean of the binomial distribution. In particular, the formula for the complete moments coincides with (1.29).

If $a = -1/N$ we obtain the formula for the moments about the mean of the hypergeometric distribution

$$(3.23)\qquad \mu_r(s) = np\left[\left(E+q\frac{N-n}{N-1}\right)^{r-1}\mu_0^*(s-1)-\mu_{r-1}(s)\right].$$

In the case of $\lim_{n\to\infty} np = \lambda$ we get the formula for the incomplete moments about the mean of the Poisson distribution

$$(3.24)\qquad \mu_r(s) = \lambda[(E+1)^{r-1}\mu_0(s-1)-\mu_{r-1}(s)].$$

The analogous procedure to that given above allows us to obtain the relation for the moments about the mean of the negative binomial distribution in the form

$$(3.25)\qquad \mu_r(s) = n\frac{p}{q}\left[\left(E+\frac{1}{q}\right)^{r-1}\mu_0^*(s-1)-\mu_{r-1}(s)\right]$$

where μ_r^* being the moment of the distribution (2.35) for $m = 1$.

The substitutions cited on page 31 give the formula for the moments of the Pólya–Eggenberger distribution

$$(3.26)\qquad \mu_r(s) = \lambda\{[E+(1+\eta)]^{r-1}\mu_0^*(s-1)-\mu_{r-1}(s)\}.$$

where $\mu_r^*(s)$ denotes the incomplete moment about the mean of the discussed distribution with the respectively changed parameter (evaluating $\mu_r^*(s)$ we put $\lambda+\eta$ in place of λ in $\mu_r(s)$). We reach this modified distribution from (2.35) by the mentioned substitutions.

In the case of $n = 1$ or $\lambda = \eta$ we get the relations for the moments of the geometric distribution

$$(3.27)\qquad \mu_r(s) = \frac{p}{q}\left[\left(E+\frac{1}{q}\right)^{r-1}\mu_0^*(s-1)-\mu_{r-1}(s)\right]$$

or

$$(3.28)\qquad \mu_r(s) = \eta\{[E+(1+\eta)]^{r-1}\mu_0^*(s-1)-\mu_{r-1}(s)\}.$$

The formulae (3.27) and (3.28) have a formal character. In order to reach $\mu_r^*(s-1)$ from $\mu_r(s-1)$ we must keep the rule of substitution $n = 1$ by $n = 2$ or $\lambda = \eta$ by 2η which is apparently easier to realize direct by making use of the relations (3.25) or (3.26).

If we put $s = 0$ in the above given relations we get the formulae for the complete moments about the mean of the discussed distributions.

3.3. The third method. The third method of reaching the relation for the moments about the mean of the Pólya distribution depends on the use of the formula (1.11) for the summation by parts where

$$f_k = (k-np)^{r-1}, \qquad g_k = (k-np)\Pi(k)$$

and $\Pi(k)$ is the probability function (2.18) of the discussed distribution.

The formula (3.6) allows to state that

$$\sum_{k=s}^{n} g_k = \mu_1(s) = s[q+(n-s)a]\Pi(s),$$

$$\sum_{i=s}^{k} g_i = \sum_{i=s}^{n} g_i - \sum_{i=k+1}^{n} g_i$$
$$= s[q+(n-s)a]\Pi(s)-(k+1)[q+(n-k-1)a]\Pi(k+1).$$

Then in accordance with (1.11)

$$\mu_r(s) = s[q+(n-s)a]\Pi(s)f_n - s[q+(n-s)a]\Pi(s)\sum_{k=s}^{n-1}\Delta f_k + \\ + \sum_{k=s}^{n-1}(k+1)[q+(n-k-1)a]\Pi(k+1)\Delta f_k.$$

Note that

$$f_n - \sum_{k=s}^{n-1}\Delta f_k = f_n - [(f_{s+1}-f_s)+(f_{s+2}-f_{s+1})+\ldots+(f_n-f_{n-1})]$$
$$= f_n-(f_n-f_s) = f_s$$

and

(3.29) $$f_s = (s-np)^{r-1},$$

$$(k+1)[q+(n-k-1)a]\Pi(k+1)$$
$$= (k+1)[q+(n-k-1)a]\frac{n!}{(k+1)k![n-(k+1)]!}\,\frac{p^{[k+1,-a]}q^{[n-(k+1),-a]}}{1^{[n,-a]}}$$
$$= np[q+(n-k-1)a]\binom{n-1}{k}\frac{(p+a)^{[k,-a]}q^{[n-1-k,-a]}}{(1+a)^{[n-1,-a]}}$$
$$= np[q-(k-n+1)a]\Pi^*(k)$$

$\Pi^*(k)$ being defined as formerly by (2.19). Then we have

$$\mu_r(s) = s[q+(n-s)a]\Pi(s)f_s + npq\sum_{k=s}^{n-1}\Pi^*(k)\Delta f_k - \\ - npa\sum_{k=s}^{n-1}(k-n+1)\Pi^*(k)\Delta f_k.$$

We find that

$$k-n+1 = k-a_1^*+\frac{(n-1)(p+a)}{1+a}-n+1$$

$$= k-a_1^*+q\frac{1-n}{1+a}, \quad q = 1-p;$$

$$\Delta f_k = \sum_{i=1}^{r-1}\binom{r-1}{i}(k-np)^{r-1-i};$$

$$k-np = k-a_1^*+\frac{(n-1)(p+a)}{1+a} = k-a_1^*+\frac{qna-(p+a)}{1+a}.$$

Denoting

$$b = q\frac{1-n}{1+a}, \tag{3.30}$$

$$c = \frac{qna-(p+a)}{1+a} \tag{3.31}$$

we obtain

$$\mu_r(s) = \mu_1(s)f_s+npq\sum_{k=s}^{n-1}\sum_{i=1}^{r-1}\binom{r-1}{i}(k-a_1^*+c)^{r-1-i}\Pi^*(k)-$$

$$-npa\sum_{k=s}^{n-1}\sum_{i=1}^{r-1}(k-a_1^*+b)\binom{r-1}{i}(k-a_1^*+c)^{r-1-i}\Pi^*(k)$$

$$= \mu_1(s)f_s+npq\sum_{i=1}^{r=1}\binom{r-1}{i}(E+c)^{r-1-i}\mu_0^*(s)-$$

$$-npa\sum_{i=1}^{r-1}\left[\binom{r-1}{i}(E+c)^{r-1-i}\mu_1^*(s)+b(E+c)^{r-1-i}\mu_0^*(s)\right]$$

$$= \mu_1(s)f_s+np\left[(q-ab)\sum_{i=1}^{r-1}\binom{r-1}{i}(E+c)^{r-1-i}\mu_0^*(s)-\right.$$

$$\left.-a\sum_{i=1}^{r-1}\binom{r-1}{i}(E+c)^{r-1-i}\mu_1^*(s)\right].$$

We have then the following theorem:

THEOREM 3.4. *The incomplete moment about the mean* $\mu_r(s)$ *of* r*-th order of the Pólya distribution defined by* (2.18) *is expressed by the recurrence relation*

$$\mu_r(s) = \mu_1(s)f_s+np\left\{(q-ab)\sum_{i=1}^{r-1}\binom{r-1}{i}[(E+c)^{r-1-i}\mu_0^*(s)-\right.$$

$$\left.-a(E+c)^{r-1-i}\mu_1^*(s)]\right\}, \quad r = 1,2,3,\ldots \tag{3.32}$$

where $q = 1-p$ *and the incomplete moment* $\mu_1(s)$ *being defined by* (3.6), f_s *by* (3.29), b *and* c *by* (3.30) *and* (3.31) *respectively and* $\mu_0^*(s)$ *and* $\mu_1^*(s)$ *denote the incomplete moments of the Pólya distribution* (2.19) *of order* 0 *and* 1 *respectively, i.e.*

$$\mu_0^*(s) = \sum_{k=s}^{n-1} \Pi^*(k), \qquad \mu_1^*(s) = \sum_{k=s}^{n-1} (k-\alpha_1^*)\Pi^*(k).$$

If $s = 0$ we get the relation for the complete moments about the mean of the Pólya distribution

$$(3.33) \qquad \mu_r = np\left\{(q-ab)\sum_{i=1}^{r-1}\binom{r-1}{i}\,[(E+c)^{r-1-i}\mu_0^* - a(E+c)^{r-1-i}\mu_1^*]\right\}, \qquad r = 2, 3, \ldots$$

Substituting $a = -1/N$ we get the relation for the incomplete moments about the mean of the hypergeometric distribution

$$(3.34) \qquad \mu_r(s) = \mu_1(s)f_s + np\left\{q\frac{N-n}{N-1}\sum_{i=1}^{r-1}\binom{r-1}{i}\left[(E+c_1)^{r-1-i}\mu_0^*(s) + \frac{1}{N}(E+c_1)^{r-1-i}\mu_1^*(s)\right]\right\}, \qquad r = 2, 3, \ldots$$

where

$$c_1 = \frac{1-pN-qn}{N-1}, \qquad \mu_1(s) = \frac{s}{N}(Nq+s-n)\Pi(s)$$

and $\Pi(s)$ is defined by (1.4) for $k = s$ and $\mu_0^*(s)$, $\mu_1^*(s)$ denote the incomplete moments about the mean of this distribution in which n must be replaced by $n-1$ and p by $(Np-1)/N$.

If $a = 0$ we get from (3.32) the relation for the incomplete moments of the binomial distribution

$$(3.35) \qquad \mu_r(s) = q\left[s\Pi(s)f_s + np\sum_{i=1}^{r-1}\binom{r-1}{i}(E-p)^{r-1-i}\mu_0^*(s)\right]$$

where $\Pi(s)$ is defined by (1.2) for $k = s$ and $\mu_0^*(s)$ denotes the moment of this distribution after replacing n by $n-1$.

In the case of $\lim_{n\to\infty} np = \lambda$ we get the relation for the incomplete moments about the mean of the Poisson distribution

$$(3.36) \qquad \mu_r(s) = s\Pi(s)(s-\lambda)^{r-1} + \lambda\sum_{i=1}^{r-1}\binom{r-1}{i}\mu_{r-1-i}(s).$$

The relation (3.36) is identical with (1.61).

The analogous treatment to the one described above allows us to establish the recurrence relation for the incomplete moments of the negative binomial distribution (1.20) in the form

$$\mu_r(s) = \frac{s}{q}\,\Pi(s) + \frac{np}{q^2}\sum_{i=1}^{r-1}\binom{r-1}{i}\left(E+\frac{p}{q}\right)^{r-1-i}\mu_0^*(s) \tag{3.37}$$

where $\mu_0^*(s)$ denotes the moment of the distribution (2.35) for $m = 1$.

The relation for the moments of the Pólya–Eggenberger distribution

$$\mu_r(s) = (1+\eta)\left[s\Pi(s) + \lambda\sum_{i=1}^{r-1}\binom{r-1}{i}(E+\eta)^{r-1-i}\mu_0^*(s)\right] \tag{3.38}$$

may be derived from (3.32) by using the limit theorem quoted on page 22 or from (3.37) by using the substitutions given on page 31.

Putting $n = 1$ in (3.37) or $\lambda = \eta$ in (3.38) we obtain the relations for the moments of the geometric distributions

$$\mu_r(s) = \frac{s}{q}\,\Pi(s) + \frac{p}{q^2}\sum_{i=1}^{r-1}\binom{r-1}{i}\left(E+\frac{p}{q}\right)^{r-1-i}\mu_0^*(s), \tag{3.39}$$

$$\mu_r(s) = (1+\eta)\left[s\Pi(s) + \eta\sum_{i=1}^{r-1}\binom{r-1}{i}(E+\eta)^{r-1-i}\mu_0^*(s)\right]. \tag{3.40}$$

The remarks made for the formulae (3.27) and (3.28) in 3.2 hold good for (3.39) and (3.40) as well, i.e. in practice the moments of the geometric distribution are obtained easier from (3.37) and (3.38).

It is possible to obtain the recurrence relations for the complete moments about the mean immediately from the above given relations by the substitution $s = 0$.

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