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Geometry of numbers in adele spaces

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Part I

1. Introduction

The geometry of numbers as originated by Minkowski applies to Euclidean space $\boldsymbol{R}^n$, but several writers have since extended many of the ideas and results of the classical geometry of numbers to other spaces. Mahler [10] replaced $\boldsymbol{R}^n$ by a non-archimedean space, while Chabauty [4] studied compactness for lattices in general topological groups; Chabauty's work was later continued by Macbeath and Swierczkowski (see [8], [9] and [14]). Rogers and Swinnerton-Dyer [13] considered an analogue of the geometry of numbers over algebraic number fields, and Cantor [2] investigated linear diophantine approximations for T-adeles of algebraic number fields.

In this paper I describe an analogue of the geometry of numbers for spaces of adeles of a number field; the situation is thus closely related to that of Cantor [2]. The relevant geometrical ideas can be introduced quite naturally into adele spaces, and lead to analogues of classical theorems which in turn yield arithmetic results about number fields.

In section 2 I introduce notation and various local preliminaries, and then, in section 3, discuss the basic adele space $\mathscr{A}$, its Haar measure and certain groups of linear transformations of $\mathscr{A}$. Section 4 deals with lattices and convex bodies, while section 5 centres round an analogue of Minkowski's convex body theorem. Section 6 contains the principal results of the paper, namely, analogues of Minkowski's estimates for the product of successive minima and applications of these results to algebraic number theory. The final section shows how the results of the earlier sections can be adapted to T-adeles. By specializing the T-adele theorems I obtain the corresponding theorems of the classical geometry of numbers and also some theorems of Rogers and Swinnerton-Dyer [13]. The proofs of the main results on successive minima in section 6 do not assume any results from any analogue of the geometry of numbers; however, in section 5 I use Macbeath's version of Blichfeldt's theorem, and in one of the applications in section 6 I use a result of Cassels on bases of lattices.

In several places where the discussion closely resembles the classical case or the corresponding section of an earlier analogue I have omitted some details, but these can usually be found in my thesis [12]. Also, since part II is devoted to compactness for lattices in adele spaces, the discussion of topological questions is kept at a minimum in the present paper.

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2. Preliminaries

2.1. Notation. As usual, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ denote the fields of rational, real and complex numbers respectively. $\boldsymbol{Z}$ denotes the rational integers, and $\boldsymbol{Z}^+$ the natural numbers. The symbol p denotes a generic prime divisor of $\boldsymbol{Q}$, and ∞ denotes the infinite prime divisor of $\boldsymbol{Q}$. The ordinary absolute value on $\boldsymbol{R}$ or $\boldsymbol{C}$ is written $|\ |$.

K is a fixed algebraic number field of degree m over $\boldsymbol{Q}$, with generic prime divisor v. For each v, $|\ |_v$ denotes the normalized valuation belonging to v, and K_v denotes a (fixed) completion of K at v. Each K_v is a locally compact topological field under the topology induced by $|\ |_v$. If v is finite (= non-archimedean) then $\mathcal{O}_v$ denotes the ring of local integers at v:

$$\mathcal{O}_v = \{a \in K_v \mid |a|_v \leqslant 1\};$$

$\mathcal{O}_v$ is open and compact in K_v.

Let S_∞ denote the collection of all infinite (= archimedean) prime divisors of K, and let Ω denote the collection of all prime divisors of K. We define Σ as the collection of all finite subsets S of Ω such that $S \supset S_\infty$. As is customary, r denotes the number of real prime divisors of K and s the number of complex ones, so that $r+2s = m$.

We write d_K for the absolute discriminant of K, and e_j for the n-vector with 1 in position j and 0 elsewhere.

2.2. Local preliminaries. Fix a prime divisor v of K. For $x = (\xi_1, \ldots, \xi_n)$, $y = (\eta_1, \ldots, \eta_n)$ belonging to the vector space K_v^n put

$$|x|_v = \max_{1 \leqslant i \leqslant n} |\xi_i|_v,$$

and define a metric d_v on K_v^n by

$$(1)\qquad \begin{aligned} d_v(x, y) &= |x-y|_v^{1/2} \quad (v \text{ complex}),\\ d_v(x, y) &= |x-y|_v \quad (v \text{ not complex}). \end{aligned}$$

With this metric K_v^n is a locally compact additive group.

The rings $\mathscr{L}_v$ and $\mathscr{I}_v$. Let $\mathscr{L}_v$ be the ring of all $n \times n$ matrices over K_v. The isomorphism of $K_v^{n^2}$ onto $\mathscr{L}_v$ (as additive groups) induces a topology on $\mathscr{L}_v$ which makes $\mathscr{L}_v$ into a locally compact topological ring. From now on the symbol $\mathscr{L}_v$ denotes this topological ring.

For $A = [a_{ij}] \in \mathscr{L}_v$ define

$$\|A\|_v = n \max_{i,j} |a_{ij}|_v \quad (v \text{ real}),$$
(2)
$$\|A\|_v = n \max_{i,j} |a_{ij}|_v^{1/2} \quad (v \text{ complex}),$$
$$\|A\|_v = \max_{i,j} |a_{ij}|_v \quad (v \notin S_\infty).$$

Then

$$\|A+B\|_v \leqslant \|A\|_v + \|B\|_v, \qquad \|AB\|_v \leqslant \|A\|_v \|B\|_v$$

for all $A, B \in \mathscr{L}_v$, and hence the mapping D_v': $\mathscr{L}_v \times \mathscr{L}_v \to R$ defined by

$$D_v'(A, B) = \|A-B\|_v \quad (A, B \in \mathscr{L}_v) \tag{3}$$

is a metric for $\mathscr{L}_v$. The metric D_v' is invariant under additive translations:

$$D_v'(A+C, B+C) = D_v'(C+A, C+B) = D_v'(A, B)$$

for all $A, B, C \in \mathscr{L}_v$.

If v is finite we denote by $\mathscr{I}_v$ the subring of $\mathscr{L}_v$ containing all those matrices with elements in $\mathcal{O}_v$. Clearly $\mathscr{I}_v$ is open and compact in $\mathscr{L}_v$. $\mathscr{I}_v$ is not defined if v is infinite.

The groups $\mathscr{G}_v$ and $\mathscr{U}_v$. With the relative topology induced from $\mathscr{L}_v$, the group of units of $\mathscr{L}_v$ is a multiplicative topological group; we denote this topological group by $\mathscr{G}_v$.

For $X, Y \in \mathscr{G}_v$ define

$$D_v(X, Y) = \log(1+\|XY^{-1}-I\|_v) + \log(1+\|YX^{-1}-I\|_v). \tag{4}$$

LEMMA 1. *D_v is a metric for the topological group $\mathscr{G}_v$. D_v is invariant under right multiplicative translations*:

$$D_v(XM, YM) = D_v(X, Y)$$

for all $X, Y, M \in \mathscr{G}_v$.

Proof. The proof that D_v is a metric is straightforward; the details are given in 3.2 of [12].

A fundamental system of neighbourhoods of the identity I of for the D_v-topology is the collection $\{\mathscr{M}(\varepsilon) \mid \varepsilon > 0\}$, where $\mathscr{M}(\varepsilon)$ is the open sphere

$$\mathscr{M}(\varepsilon) = \{A \in \mathscr{G}_v \mid D_v(A, I) < \varepsilon\}.$$

Also, because of the D_v'-continuity of the mapping $A \to A^{-1}$, the collection $\{\mathscr{N}(\varepsilon) \mid \varepsilon > 0\}$, where

$$\mathscr{N}(\varepsilon) = \{A \in \mathscr{G}_v \mid D_v'(A, I) < \varepsilon, D_v'(A^{-1}, I) < \varepsilon\},$$

is a fundamental system of neighbourhoods of I for the topology induced on $\mathscr{G}_v$ from $\mathscr{L}_v$. It is easily checked that for each $\varepsilon > 0$ we have

$$\mathscr{N}(e^{\varepsilon}-1) \subset \mathscr{M}(2\varepsilon),$$
$$\mathscr{M}\big(\log(1+\varepsilon)\big) \subset \mathscr{N}(\varepsilon).$$

Hence the two topologies are equal.

The translation invariance is immediate from the definition of D_v. The lemma is now proved.

If v is finite, we define

$$U_v = \{A \in \mathscr{I}_v \mid |\det A|_v = 1\};$$

clearly U_v is a subgroup of $\mathscr{G}_v$. U_v is not defined if v is infinite.

The space K_∞^n. Suppose that $S_\infty = \{v_1, \ldots, v_{r+s}\}$ with $v_1, \ldots, v_r$ real and $v_{r+1}, \ldots, v_{r+s}$ complex. Then we can order the $\boldsymbol{Q}$-isomorphisms $\sigma_1, \ldots, \sigma_m$ of K into $\boldsymbol{C}$ so that σ_i is associated with v_i for $1 \leqslant i \leqslant r+s$ and $\sigma_{r+s+j} = \bar{\sigma}_{r+j}$ is associated with v_{r+j} for $1 \leqslant j \leqslant s$. For each real v_i, K_{v_i} is topologically isomorphic to $\boldsymbol{R}$, and for each complex v_i, K_{v_i} is topologically isomorphic to $\boldsymbol{C}$. Hence if we define

$$K_\infty = \prod_{v \in S_\infty} K_v = K_{v_1} \times \ldots \times K_{v_{r+s}},$$

then we have a topological isomorphism

$$K_\infty \cong \boldsymbol{R}^r \times \boldsymbol{C}^s.$$

The field K is naturally embedded in $\boldsymbol{R}^r \times \boldsymbol{C}^s$ via the mapping

$$a \to (\sigma_1 a, \ldots, \sigma_{r+s} a). \tag{5}$$

Moreover, the weak approximation theorem shows that the image of K is dense in $\boldsymbol{R}^r \times \boldsymbol{C}^s$. If we write $\boldsymbol{C} = \boldsymbol{R} \times \boldsymbol{R}$, then K_∞ can be viewed as $\boldsymbol{R}^{r+2s} = \boldsymbol{R}^m$, both algebraically and topologically, and K is embedded (additively) as a dense subset of $\boldsymbol{R}^m$ by using (5). We note also that under this identification each ideal of K is a lattice (in the classical sense) in $\boldsymbol{R}^m$.

For any $n \in \boldsymbol{Z}^+$ we write

$$K_\infty^n = \prod_{v \in S_\infty} K_v^n.$$

Then K_∞^n may be viewed as $\boldsymbol{R}^{mn}$, both algebraically and topologically, and in this sense we speak of points of K_∞^n "viewed in $\boldsymbol{R}^{mn}$".

Haar measure. Let μ_v be a Haar measure on K_v. We normalize μ_v as follows.

If v is real, then K_v is $\boldsymbol{R}$ (up to isomorphism) and we take μ_v as Lebesgue measure on the real line.

If v is complex, then K_v is $\boldsymbol{C}$ (up to isomorphism) and we take μ_v as Lebesgue measure on the plane.

If v is finite we choose μ_v so that $\mu_v(\mathcal{O}_v) = 1$.

We now take the product measure on K_v^n and denote it by μ_v also. Then for each $W \in \mathcal{G}_v$ and each measurable subset T of K_v^n we have

$$(6) \qquad \mu_v\big(W(T)\big) = |\det W|_v \mu_v(T)$$

(c. f. lemma 1.2 of Lutz [7]).

Let μ_∞ denote the product measure on K_∞^n. Viewing K_∞^n as $\boldsymbol{R}^{mn}$, we see at once that μ_∞ coincides with Lebesgue measure on $\boldsymbol{R}^{mn}$.

3. The space $\mathscr{A}$

3.1. Generalities. Let $\mathscr{A}$ be the restricted direct product of the topological groups K_v^n with respect to the subgroups $\mathcal{O}_v^n$, so that $\mathscr{A}$ is a locally compact Hausdorff additive topological group. We call $\mathscr{A}$ the *space of n-dimensional adeles of K*. In our analogue $\mathscr{A}$ will correspond to $\boldsymbol{R}^n$ in the classical geometry of numbers. If it is necessary to emphasize the dimension we shall write $\mathscr{A}_n$ instead of $\mathscr{A}$. Clearly $\mathscr{A} = \mathscr{A}_n$ is just the Cartesian product of n copies of the standard adele space of K (as defined in, for example, 5-2 of Weiss [16]).

A typical element of $\mathscr{A}$ is denoted by $\boldsymbol{x} = (x_v)$, where $x_v \in K_v^n$ for all v and $x_v \in \mathcal{O}_v^n$ for almost v; the vectors $(x_v)_{v \in S_\infty}$ and $(x_v)_{v \notin S_\infty}$ are called the *infinite part* and the *finite part* of $\boldsymbol{x}$ respectively. The infinite part of a subset T of $\mathscr{A}$ is the collection of all infinite parts of points in T, and similarly for the finite part of T. In this terminology, K_∞^n is the infinite part of $\mathscr{A}$.

DEFINITION 1. Let $S \in \Sigma$ and for each $v \in S$ let a_v be a positive real number. The *open box* $B(a_v; S)$ is the set of all $\boldsymbol{x} \in \mathscr{A}$ with

$$|x_v|_v < a_v \quad (v \in S),$$

$$|x_v|_v \leqslant 1 \quad (v \notin S).$$

If $a_v = a$ for all $v \in S$ we write $B(a; S)$ instead of $B(a_v; S)$. The *closed box* $\bar{B}(a_v; S)$ is the closure of $B(a_v; S)$ in $\mathscr{A}$.

Clearly to every closed box $\bar{B}(a_v; S)$ there corresponds an idele $\boldsymbol{i} = (i_v)$ such that

$$\bar{B}(a_v; S) = \{\boldsymbol{x} \in \mathscr{A} \,|\, |x_v|_v \leqslant |i_v|_v \text{ (all } v)\};$$

we say that $\bar{B}(a_v; S)$ *is determined by* $\boldsymbol{i}$.

It is easily checked that the collection

$$\{B(\varepsilon; S) \,|\, \varepsilon > 0, \ S \in \Sigma\}$$

is a fundamental system of neighbourhoods of the identity $\boldsymbol{o}$ in $\mathscr{A}$.

A metric for $\mathscr{A}$. We now introduce a metric for $\mathscr{A}$, the use of which simplifies some of our proofs.

For each $\mathbf{x} = (x_v)$, $\mathbf{y} = (y_v) \in \mathscr{A}$ define

$$d(\mathbf{x}, \mathbf{y}) = \sup_v (Nv)^{-1/2} d_v(x_v, y_v), \tag{7}$$

where d_v is the local metric given by (1), and

$$N_v = 1 \quad (v \in S_\infty),$$

$$N_v = \text{absolute norm of } v \quad (v \notin S_\infty).$$

Lemma 2. *The function $d\colon \mathscr{A} \times \mathscr{A} \to \mathbf{R}$ is a metric on $\mathscr{A}$. It is invariant under additive translations:*

$$d(\mathbf{x}+\mathbf{a}, \mathbf{y}+\mathbf{a}) = d(\mathbf{a}+\mathbf{x}, \mathbf{a}+\mathbf{y}) = d(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{a} \in \mathscr{A}$. The d-topology on $\mathscr{A}$ coincides with the restricted direct product topology.

Proof. The proof that d is a translation invariant metric on $\mathscr{A}$ is straightforward, and we omit the details.

The equality of topologies is easily proved if we note that for each $\varepsilon > 0$ we have

$$1 \leqslant \varepsilon (Nv)^{1/2} < Nv$$

for almost all $v \notin S_\infty$, so that if $x \in K_v^n$ then the equivalence

$$|x|_v < \varepsilon (Nv)^{1/2} \Leftrightarrow |x|_v \leqslant 1$$

holds for almost all $v \notin S_\infty$. The details are given in [12].

In the d-topology on $\mathscr{A}$ every sphere with centre $\mathbf{o}$ is a box. Conversely, every box is contained in some d-sphere with centre $\mathbf{o}$. Using lemma 3.1.2 of Tate [15] we see that $G \subset \mathscr{A}$ is compact if and only if G is closed and bounded (in terms of d).

3.2. Linear transformations of $\mathscr{A}$. Let $\mathscr{L}$ be the restricted direct product of the topological rings $\mathscr{L}_v$ with respect to the subrings $\mathscr{J}_v$. Then $\mathscr{L}$ is a locally compact topological ring. We call an element of $\mathscr{L}$ a *linear transformation of $\mathscr{A}$*, since elements of $\mathscr{L}$ correspond to linear transformations of $\mathbf{R}^n$ in the classical geometry of numbers. Each $A \in \mathscr{L}$ is of the form $A = (A_v)$, where $A_v \in \mathscr{L}_v$ for all v and $A_v \in \mathscr{J}_v$ for almost all v. Each such $A \in \mathscr{L}$ can be regarded as a function on $\mathscr{A}$ by defining $A\mathbf{x} = (A_v x_v)$; clearly A is an endomorphism of $\mathscr{A}$. Moreover, the inequality

$$d(A\mathbf{x}, 0) \leqslant d(\mathbf{x}, 0) \sup_v \|A_v\|_v,$$

valid for all $A \in \mathscr{L}$ and all $\mathbf{x} \in \mathscr{A}$, shows that every linear transformation of $\mathscr{A}$ is continuous.

For all $A, B \in \mathscr{L}$ we define

$$D'(A, B) = \mathrm{up}\, \frac{D'_v(A_v, B_v)}{(N_v)^{1/2}}, \tag{8}$$

where D'_v is the local metric given by (3). Then D' is a metric for $\mathscr{L}$ and is invariant under additive translations.

Now let $\mathscr{G}$ be the (algebraic) restricted direct product of the groups $\mathscr{G}_v$ with respect to the subgroups U_v. Then $\mathscr{G}$ is just the group of units of $\mathscr{L}$. The elements of $\mathscr{G}$ are called *lattice transformations of* $\mathscr{A}$. Each lattice transformation is a topological K-linear automorphism of $\mathscr{A}$, and is thus an obvious analogue of a non-singular linear transformation of $\boldsymbol{R}^n$ in the classical case. If $n = 1$, $\mathscr{G}$ is just the idele group of K.

The group of non-singular $n \times n$ matrices over K is naturally embedded along the diagonal of $\mathscr{G}$ as a subgroup $\mathscr{P}$ of $\mathscr{G}$. We call an element P of $\mathscr{P}$ a *principal lattice transformation of* $\mathscr{A}$, and we use the same symbol P regardless of whether P is viewed as an $n \times n$ matrix over K or as an element of $\mathscr{G}$.

For each $A \in \mathscr{G}$ we define $\|A\|$ by

$$\|A\| = \prod_v |\det A_v|_v .$$

Then $\| \ \|$ is a homomorphism of $\mathscr{G}$ into the multiplicative group of positive real numbers, and $\|P\| = 1$ whenever $P \in \mathscr{P}$. In the one-dimensional case $\|A\|$ is just the volume of the idele A.

3.3. Global measure. Since $\mathscr{A}$ is a locally compact Hausdorff group there exists a Haar measure μ on $\mathscr{A}$. We normalize μ so that

$$\mu\big(\bar{B}(1; S_\infty)\big) = (2^r \pi^s)^n .$$

From the uniqueness properties of Haar measure it follows that if $S \in \Sigma$ and

$$\mathscr{A}_S = \prod_{v \in S} K_v^n \times \prod_{v \notin S} \mathscr{O}_v^n ,$$

then the restriction of μ to $\mathscr{A}_S$ coincides with the product measure of the local measures μ_v on $\mathscr{A}_S$. Using (6), we have immediately

LEMMA 3. *Let $A \in \mathscr{G}$ and suppose that $T \subset \mathscr{A}$ is measurable. Then $A(T)$ is measurable and*

$$\mu\big(A(T)\big) = \|A\| \mu(T).$$

4. Lattices and convex bodies

4.1. Lattices. It is well-known (see, for example, 5—5—1 of Weiss [16]) that the image $\mathscr{D}_1$ of the field K along the diagonal of $\mathscr{A}_1$ is a discrete subgroup of $\mathscr{A}_1$ with compact quotient $\mathscr{A}_1/\mathscr{D}_1$. Hence the image $\mathscr{D} = \mathscr{D}_n$

of the vector space K^n along the diagonal of $\mathscr{A} = \mathscr{A}_n$ is a discrete n-dimensional vector space over K with compact quotient $\mathscr{A}/\mathscr{D}$. If $d \in K^n$ then we write $\boldsymbol{d}$ for the corresponding element of $\mathscr{D}$.

The discrete subgroup $\mathscr{D}$ of $\mathscr{A}$ is a natural analogue of the integral lattice $\boldsymbol{Z}^n$ in $\boldsymbol{R}^n$. Since the lattice transformations of $\mathscr{A}$ are the "non-singular" linear transformations, we make the following definition.

DEFINITION 2. A *lattice* in $\mathscr{A}$ is a set of the form $A\mathscr{D} = A(\mathscr{D})$, where $A \in \mathscr{G}$.

Since lattice transformations of $\mathscr{A}$ are topological K-linear automorphisms of $\mathscr{A}$ it follows from the corresponding properties of $\mathscr{D}$ that each lattice Λ is an n-dimensional vector space over K, is discrete in $\mathscr{A}$, and has compact quotient $\mathscr{A}/\Lambda$. Conversely, it can be shown that any n-dimensional vector space over K which is discrete in $\mathscr{A}$ is a lattice (see the comments following the corollary to theorem 4). By a *basis* for a lattice Λ we always mean a basis for Λ as a vector space over K.

Let G be any discrete subgroup of $\mathscr{A}$. We recall that a *G-packing* is a measurable set T of $\mathscr{A}$ such that

$$T \cap (\boldsymbol{x} + T) = \emptyset$$

for every $\boldsymbol{x} \in G$, $\boldsymbol{x} \neq \boldsymbol{o}$, and that a *$G$-covering* is a measurable set W such that $G + W = \mathscr{A}$. A *fundamental domain for G* is a measurable set $\mathscr{F}$ which is both a G-packing and a G-covering. Any two fundamental domains for G have the same measure; we denote this common measure by $\Delta(G)$.

Let $\{\omega_1, \ldots, \omega_m\}$ be an integral basis for K over $\boldsymbol{Q}$. We view each ω_i in $\boldsymbol{R}^m$, and define

$$H_0 = \left\{ \sum_{i=1}^{n} t_i \omega_i \,\middle|\, 0 \leqslant t_i < 1 \right\}.$$

Then (see theorem 3, chapter 6 of Lang [5]) a fundamental domain for the lattice $\mathscr{D}_1$ in $\mathscr{A}_1$ is

$$\mathscr{F}_0 = H_0 \times \prod_{v \notin S_\infty} \mathcal{O}_v^n. \tag{9}$$

Hence $\mathscr{F}_0^n$ is a fundamental domain for $\mathscr{D}$ in $\mathscr{A}$. With our choice of local measures we have

$$\Delta(\mathscr{D}) = \mu(\mathscr{F}_0^n) = 2^{-ns} d_K^{n/2}$$

(see 5—4—4 of Weiss [16]). For an arbitrary lattice $\Lambda = A\mathscr{D}$ it is clear that $A(\mathscr{F}_0^n)$ is a fundamental domain, and hence

$$\Delta(\Lambda) = \|A\| \, 2^{-ns} d_K^{n/2}.$$

4.2. Convex bodies. Let v be a discrete prime divisor of K. Following Mahler [10] we say that $f\colon K_v^n \to \boldsymbol{R}$ is a *convex distance function* (*at* v) if the following conditions are satisfied for all $x, y \in K_v^n$, $a \in K_v$.

(i) $f(x) \geqslant 0$.

(ii) $f(ax) = |a|_v f(x)$.

(iii) $f(x+y) \leqslant \max\big(f(x), f(y)\big)$.

A *convex body at* v is a set of the form

$$\{x \in K_v^n \mid f(x) \leqslant 1\},$$

where f is a convex distance function at v. Mahler has shown that every bounded convex body B at v is a parallelotope, that is, has the shape $B = A(\mathcal{O}_v^n)$ for some $A \in \mathcal{G}_v$, and that an unbounded convex body at v is a cylinder on a parallelotopic base. In particular, every convex body at v is an additive subgroup of K_v^n. In view of this and the isomorphism between K_∞^n and $\boldsymbol{R}^{mn}$, we make the following definitions.

DEFINITION 3. Let $\boldsymbol{x} \in \mathscr{A}$, $t \in \boldsymbol{R}$. The operation

$$* : \boldsymbol{R} \times \mathscr{A} \to \mathscr{A}$$

is defined by $t * \boldsymbol{x} = \boldsymbol{y}$, where

$$\begin{aligned} (y_v)_{v \in S_\infty} &= t(x_v)_{v \in S_\infty} && (\text{as points in } R^{mn}), \\ y_v &= x_v && (v \notin S_\infty). \end{aligned}$$

DEFINITION 4. Let $B \subset \mathscr{A}$. Then B is a *convex body* means

(i) B is measurable,

(ii) for every $\boldsymbol{x}, \boldsymbol{y} \in B$, $t \in \boldsymbol{R}$ with $0 \leqslant t \leqslant 1$, we have

$$t * \boldsymbol{x} + (1-t) * \boldsymbol{y} \in B.$$

It follows easily from condition (ii) of definition 4 that a symmetric convex body $B \subset \mathscr{A}$ is the Cartesian product of its infinite part B_∞ and its finite part B':

$$B = B_\infty \times B';$$

here B_∞ is a symmetric convex body in $\boldsymbol{R}^{mn}$ in the usual sense, and B' is an additive subgroup of the finite part of $\mathscr{A}$.

An example of a convex body is any set B of the shape

$$B = B_\infty \times \prod_{v \notin S_\infty} B_v, \tag{10}$$

where B_∞ is a convex subset of $\boldsymbol{R}^{mn}$ and, for each $v \notin S_\infty$, B_v is a convex body at v such that $B_v = \mathcal{O}_v^n$ for almost all v; also B is open and symmetric if and only if B_∞ is.

In general, not every open symmetric convex body in $\mathscr{A}$ is of the shape (10) (see the remark preceding the proof of theorem 7). But if $K = \boldsymbol{Q}$ we have

THEOREM 1. *Let B be a bounded, open, symmetric convex body in $\mathscr{A} = \mathscr{A}_n^{\boldsymbol{Q}}$, the n-dimensional adele space of $\boldsymbol{Q}$. Then there exists a principal lattice transformation A of $\mathscr{A}$ such that*

$$B = A\Big(B'_\infty \times \prod_{p\neq\infty} \mathcal{O}_p^n\Big)$$

for some bounded, open, symmetric convex body B'_∞ in $\boldsymbol{R}^n$.

Proof. For any $G \subset \mathscr{A}$ we denote by G' the finite part of G, and for any idele $\boldsymbol{i} = (i_p)$ of $\boldsymbol{Q}$ we denote by $B(\boldsymbol{i})$ the closed box determined by $\boldsymbol{i}$.

Since B is bounded and open in $\mathscr{A}$, there exist ideles $\boldsymbol{i}$, $\boldsymbol{j}$ of $\boldsymbol{Q}$ such that

$$B(\boldsymbol{i}) \subset B \subset B(\boldsymbol{j}).$$

By taking finite parts, intersecting with $\mathscr{D}'$, and interpreting the results in $\boldsymbol{Q}^n$, we obtain

$$I^n \subset B' \cap \mathscr{D}' \subset J^n$$

where

$$I = \{a \in K|\ |a|_v \leqslant |i_v|_v\ (v \notin S_\infty)\},$$

$$J = \{\beta \in K|\ |\beta|_v \leqslant |j_v|_v\ (v \in S_\infty)\}.$$

I and J are fractional ideals of $\boldsymbol{Q}$, and so each of I^n, J^n is a free $\boldsymbol{Z}$-module of rank n. Hence the $\boldsymbol{Z}$-module $B' \cap \mathscr{D}'$ is also free of rank n, that is, $B' \cap \mathscr{D}' = A(\boldsymbol{Z}^n)$ for some nonsingular $n\times n$ matrix A over $\boldsymbol{Q}$. Now view $B' \cap \mathscr{D}'$ in $\mathscr{A}'$, and denote closure in $\mathscr{A}'$ by a bar. Since B' is an open (and therefore closed) subgroup of $\mathscr{A}'$ and $\mathscr{D}'$ is dense in $\mathscr{A}'$, it follows that $B' = \overline{B' \cap \mathscr{D}'}$. But

$$\overline{B' \cap \mathscr{D}'} = \overline{A(\boldsymbol{Z}^n)} = A\Big(\prod_{p\neq\infty} \mathcal{O}_p^n\Big),$$

and so

$$B' = A\Big(\prod_{p\neq\infty} \mathcal{O}_p^n\Big),$$

where $A \in \mathscr{P}$.

The infinite part B_∞ of B is bounded, open, symmetric and convex in $\boldsymbol{R}^n$, and so

$$B = B_\infty \times B' = A\Big(B'_\infty \times \prod_{p\neq\infty} \mathcal{O}_p^n\Big),$$

where $B'_\infty = A^{-1}(B_\infty)$ is bounded, open, symmetric and convex in $\boldsymbol{R}^n$. This proves the theorem.

5. An analogue of Minkowski's convex body theorem

5.1. Convex body theorem. We shall now prove an analogue of Minkowski's classical convex body theorem. Our proof will be based on the following analogue of Blichfeldt's theorem, which is a straightforward application to $\mathscr{A}$ of the corollary to theorem 4 of Macbeath [8].

LEMMA 4. *Let Λ be a lattice, $k \in \boldsymbol{Z}^+$, and B a measurable subset of $\mathscr{A}$ with*

$$\mu(B) > k\,\Delta(\Lambda).$$

Then there exist $k+1$ distinct points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{k+1}$ of B such that $\boldsymbol{x}_i - \boldsymbol{x}_j \in \Lambda$ for $1 \leqslant i, j \leqslant k+1$.

We also need the following lemma, which is easily proved by viewing the points $\boldsymbol{d}_i$ in $\boldsymbol{R}^{mn}$ and ordering them according to their first nonzero component.

LEMMA 5. *Let $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{k+1}$ be $k+1$ distinct points of $\mathscr{D}$, and set*

$$Y = \{\boldsymbol{d}_i - \boldsymbol{d}_j \mid i \neq j, 1 \leqslant i, j \leqslant k+1\}.$$

Then there exist k points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k \in Y$ such that all the points $\boldsymbol{o}$, $\pm \boldsymbol{y}_1, \ldots, \pm \boldsymbol{y}_k$ are distinct.

Our analogue of Minkowski's theorem is

THEOREM 2. *Let B be a symmetric convex body in $\mathscr{A}$ and Λ a lattice. Suppose that*

$$\mu(B) > k \cdot 2^{mn} \Delta(\Lambda)$$

for some $k \in \boldsymbol{Z}^+$. Then B contains at least k distinct pairs $\pm \boldsymbol{x}_i$ $(1 \leqslant i \leqslant k)$ of non-zero points of Λ. Moreover, the conclusion still holds if B is compact and

$$\mu(B) = k \cdot 2^{mn} \Delta(\Lambda).$$

Proof. We have

$$\mu(\tfrac{1}{2} * B) = 2^{-mn} \mu(B) > k\Delta(\Lambda),$$

and so by lemma 4 there exist $k+1$ distinct points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{k+1}$ of $\frac{1}{2} * B$ such that $\boldsymbol{a}_i - \boldsymbol{a}_h \in \Lambda$ for $1 \leqslant i, h \leqslant k+1$. Since $2 * \boldsymbol{a}_i \in B$ for each i, it follows from the symmetry and convexity of B that

$$\boldsymbol{a}_i - \boldsymbol{a}_h = \tfrac{1}{2} * (2 * \boldsymbol{a}_i) - \tfrac{1}{2} * (2 * \boldsymbol{a}_h) \in \Lambda \cap B$$

for all i and h.

For each $j = 1, 2, \ldots, k+1$ define

$$\boldsymbol{b}_j = \boldsymbol{a}_j - \boldsymbol{a}_1.$$

Then the $\boldsymbol{b}_j$ are distinct points of Λ with $\boldsymbol{b}_i - \boldsymbol{b}_h = \boldsymbol{a}_i - \boldsymbol{a}_h$ for all i and h. Suppose that $\Lambda = A\mathscr{D}$ with $A \in \mathscr{G}$. Then we may write

$$\boldsymbol{b}_j = A\boldsymbol{d}_j \quad (1 \leqslant j \leqslant k+1)$$

for some distinct points $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{k+1}$ of $\mathscr{D}$. By lemma 5 there exist $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k \in Y$ (with the notation of lemma 5) such that all of $\boldsymbol{o}, \pm\boldsymbol{y}_1, \ldots, \pm\boldsymbol{y}_k$ are distinct. Then all the points $A(\boldsymbol{o}) = \boldsymbol{o}, \pm A\boldsymbol{y}_1, \ldots, \pm A\boldsymbol{y}_k$ are distinct in Λ. But each $A\boldsymbol{y}_j$ is of the form $\boldsymbol{a}_\alpha - \boldsymbol{a}_\beta$ for some α, β with $1 \leqslant \alpha, \beta \leqslant k+1$. Thus we have the k distinct pairs $\pm\boldsymbol{x}_i$ of non-zero points of $\Lambda \cap B$ as required.

The sharpened form of the theorem when B is compact follows easily from the first part by considering the sequence of sets

$$B_t = \left(1+\frac{1}{t}\right) * B \qquad (t = 1, 2, \ldots)$$

and using the discreteness of Λ.

5.2. Applications of theorem 2. We treat briefly two applications of theorem 2; a fuller discussion of these is given in [12]. The first application gives a quantitative form of a density theorem in algebraic number theory.

THEOREM 3. *Let* $\boldsymbol{i}$ *be an idele of* K *and denote by* $\Pi(\boldsymbol{i})$ *the closed one-dimensional box determined by* $\boldsymbol{i}$. *Define* $k \in \boldsymbol{Z}$ *by*

$$k = \left[\frac{\|\boldsymbol{i}\|\pi^s}{2^s |d_K|^{1/2}}\right],$$

where the square brackets denote the integral part. Then $\Pi(\boldsymbol{i})$ *contains at least* $2k+1$ *field elements (including* 0).

Proof. The box $\Pi(\boldsymbol{i})$ is a compact symmetric convex body and

$$\mu\big(\Pi(\boldsymbol{i})\big) = 2^r \pi^s \|\boldsymbol{i}\|.$$

Hence we have

$$\mu\big(\Pi(\boldsymbol{i})\big) \geqslant k \cdot 2^{r+s} |d_K|^{1/2} = k \cdot 2^m \Delta(\mathscr{D}).$$

The result now follows from theorem 2.

(Mahler [11] derives a lower bound for the number of field elements in $\Pi(\boldsymbol{i})$ from a theorem on ideal bases; see the remarks following the corollary to theorem 7.)

Our second application of theorem 2 concerns convex distance functions and linear forms. If v is a real prime divisor of K then $K_v^n \cong \boldsymbol{R}^n$, while if v is complex then $K_v^n \cong \boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$; in either case we define convex distance functions at v via the classical definitions in $\boldsymbol{R}^n$ or $\boldsymbol{R}^{2n}$. For finite prime divisors v, convex distance functions were defined in 4.2.

THEOREM 4. *For each* v *let* f_v *be a symmetric convex distance function and let* B_v *be the corresponding convex body at* v:

$$B_v = \{x \in K_v^n \mid f_v(x) \leqslant 1\}.$$

Suppose that $B_v = \mathcal{O}_v^n$ *for almost all* v, *and let* Λ *be a lattice. If*

$$\mu\Bigl(\prod_v B_v\Bigr) = \prod_v \mu_v(B_v) \geqslant k\cdot 2^{mn}\Delta(\Lambda)$$

for some $k \in \mathbf{Z}^+$, *then the inequalities*

$$f_v(x_v) \leqslant 1 \quad (\textit{all } v) \tag{11}$$

are satisfied for at least k *distinct pairs* $\pm \boldsymbol{x}$ *of non-zero elements of* Λ.

Proof. The body $\prod_v B_v$ is closed and convex in $\mathscr{A}$, so the result follows immediately from theorem 2.

By taking $\Lambda = \mathfrak{D}$ in theorem 4 we obtain at once a result about the number of solutions in n-tuples of field elements to the inequalities (11); if we also take

$$f_v(x) = |x|_v \quad (x \in K_v^n)$$

for all $v \notin S_\infty$, then we obtain a result about integral solutions to the inequalities

$$f_v(x_v) \leqslant 1 \quad (v \in S_\infty).$$

For the special case of linear forms we have the following corollary, which is an extension of theorem 2.1 of Cantor [2].

COROLLARY. *Let* $A \in \mathscr{G}$ *and suppose that* $\boldsymbol{i}$ *is an idele and* Λ *a lattice. If*

$$\|\boldsymbol{i}\|^n \geqslant \frac{k\cdot 2^{mn}\Delta(\Lambda)\|A\|}{(2^r\pi^s)^n}$$

for some $k \in \mathbf{Z}^+$, *then the inequalities*

$$|A_v x_v|_v \leqslant |i_v|_v \quad (\textit{all } v) \tag{12}$$

are satisfied for at least k *distinct pairs* $\pm \boldsymbol{x}$ *of non-zero elements of* Λ.

Proof. Apply theorem 4 with

$$f_v(x) = |i_v|_v^{-1}|A_v x|_v$$

for each v.

Remarks. 1. Note that if the lattice transformation of the corollary belongs to $\mathscr{P}$ (the case of n linear forms over K) then $\|A\| = 1$, and hence the result is independent of A.

2. It is easily deduced from theorem 4 that if $A \in \mathscr{L}$ but $A \notin \mathscr{G}$ and $\boldsymbol{i}$ is an arbitrary idele of K then there is a non-zero $\boldsymbol{x} \in \mathfrak{D}$ for which (12) holds. It follows that if $A \in \mathscr{L}$ but $A \notin \mathscr{G}$ then $A(\mathfrak{D})$ is not discrete in $\mathscr{A}$.

Suppose now that $\Lambda \subset \mathscr{A}$ is an n-dimensional vector space over K. Then $\Lambda = A\mathfrak{D}$ for some $A \in \mathscr{L}$, and, by the above, Λ is discrete in $\mathscr{A}$ if and only if $A \in \mathscr{G}$. Thus we have shown that an n-dimensional vector space over K which is discrete in $\mathscr{A}$ is a lattice in $\mathscr{A}$.

6. Successive minima

6.1. Preliminaries. We now have at our disposal the necessary tools for developing a theory of successive minima for lattices in $\mathscr{A}$. Essentially the problem is to determine how much a convex body must be "blown up" so as to include a certain number of linearly independent points of a lattice. More precisely, we have the following definition.

Definition 5. Let $B \subset \mathscr{A}$, and let Λ be a lattice. For $1 \leqslant i \leqslant n$, the *$i$-th minimum $\lambda_i = \lambda_i(B, \Lambda)$ of Λ with respect to B* is defined by

$$\lambda_i = \inf\{t > 0 \mid (t*B) \cap \Lambda \text{ contains } i \text{ points which are linearly independent over } K\}.$$

If there are no such t, define $\lambda_i = \infty$.

We shall concentrate on finding estimates for the product $\lambda_1\lambda_2 \ldots \lambda_n$ in terms of $\mu(B)$, and later discuss some applications of our results to algebraic number theory. It is clear from definition 5 that

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n.$$

Also it is easily checked that if B is open and $\boldsymbol{o} \in B$ then each of $\lambda_1, \ldots, \lambda_n$ is finite, and that if B is bounded then each of $\lambda_1, \ldots, \lambda_n$ is positive.

The following lemma is needed for the subsequent discussion.

Lemma 6. *Let B be a bounded, open, symmetric convex body in $\mathscr{A}$, with infinite part B_∞ and finite part B', and let Λ be a lattice. Then there exists a basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ for Λ such that for each $j = 1, 2, \ldots, n$ we have*

(i) $\boldsymbol{b}_j \in \lambda_j*(\bar{B}_\infty \times B')$;

(ii) *if* $\boldsymbol{a} \in \Lambda \cap (t*B)$ *with* $t \leqslant \lambda_j$, *then* $\boldsymbol{a} = a_1\boldsymbol{b}_1 + \ldots + a_{j-1}\boldsymbol{b}_{j-1}$ *for some* $a_1, \ldots, a_{j-1} \in K$;

(iii) *the point* $t_1\lambda_1^{-1}*\boldsymbol{b}_1 + \ldots + t_n\lambda_n^{-1}*\boldsymbol{b}_n$ *lies in* B *whenever*

$$\sum_{j=1}^{n} |t_j| < 1.$$

Proof. Let

$$\mathscr{S} = \Lambda \cap (\lambda_n + 1)*(\bar{B}_\infty \times B').$$

Then $\mathscr{S}$ is finite, since the body $(\lambda_n+1)*(\bar{B}_\infty \times B')$ is bounded. Also, by the definition of λ_n, $\mathscr{S}$ contains at least one n-tuple of points which are linearly independent over K.

For each non-zero $\boldsymbol{x} \in \mathscr{S}$, there is a unique positive number $t = t(\boldsymbol{x})$ such that $\boldsymbol{x} \in t*(\bar{B}_\infty \times B')$, but $\boldsymbol{x} \notin t'*(\bar{B}_\infty \times B')$ if $t' < t$. Since $\mathscr{S}$ is finite, we can choose $\boldsymbol{b}_1 \in \mathscr{S}$, $\boldsymbol{b}_1 \neq \boldsymbol{o}$, with $t(\boldsymbol{b}_1) \leqslant t(\boldsymbol{c})$ for all non-zero $\boldsymbol{c} \in \mathscr{S}$. Then, for $j = 2, 3, \ldots, n$, having chosen $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{j-1}$, we can choose $b_j \in \mathscr{S}$ with $\boldsymbol{b}_j$ linearly independent of $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{j-1}$ and such that $t(\boldsymbol{b}_j) \leqslant t(\boldsymbol{c})$

for all $c \in \mathscr{S}$ which are linearly independent of $b_1, \dots, b_{j-1}$. It is now easily checked, using the definition of $\lambda_1, \dots, \lambda_n$, that $t(b_j) = \lambda_j$ for all j. Hence $\{b_1, \dots, b_n\}$ is a basis for Λ which satisfies (i).

Using the openness of B it is straightforward to show that $\lambda_j * B$ contains at most $j-1$ linearly independent points of Λ; (ii) then follows easily.

Part (iii) of the lemma follows immediately from (i) and the fact that B is convex and symmetric.

6.2. The product of successive minima; an upper bound. We now prove

Theorem 5. *Let B be a bounded, open, symmetric convex body in $\mathscr{A}$, Λ a lattice, and write $\lambda_i = \lambda_i(B, \Lambda)$ for $1 \leqslant i \leqslant n$. Then*

$$(\lambda_1 \lambda_2 \dots \lambda_n)^m \mu(B) \leqslant 2^{mn} \Delta(\Lambda). \tag{13}$$

Theorem 5 is an analogue of Minkowski's second inequality in the geometry of numbers, and our proof of theorem 5 is modelled on the first of the proofs of this inequality given in Bambah, Woods and Zassenhaus [1].

Reduction of the proof of theorem 5. Let $A \in \mathscr{G}$ and let B be as in theorem 5. Then

(i) $A(B)$ is a bounded, open, symmetric convex body,

(ii) $\mu(A(B)) = \|A\| \mu(B)$,

(iii) $\Delta(A(\Lambda)) = \|A\| \Delta(\Lambda)$,

(iv) $\lambda_j(A(B), A(\Lambda)) = \lambda_j(B, \Lambda)$ for $1 \leqslant j \leqslant n$;

(iv) follows since $A(t*x) = t*Ax$ for every real t and every $x \in \mathscr{A}$, and A is one-to-one and preserves linear independence over K. It follows that (13) is invariant under any lattice transformation of $\mathscr{A}$, and so we may assume that $\Lambda = \mathscr{D}$ and that the basis $\{b_1, \dots, b_n\}$ of lemma 6 is just $\{e_1, \dots, e_n\}$.

Notation for the proof of theorem 5. Each $x \in \mathscr{A}$ can be regarded as an n-tuple of one-dimensional adeles; we shall write

$$x = (\hat{x}_1, \dots, \hat{x}_n)$$

to indicate that $\hat{x}_1, \dots, \hat{x}_n$ are the n one-dimensional adeles making up x. As usual, $\mathscr{A}_i$ denotes the space of i-dimensional adeles of K and $\mathscr{A} = \mathscr{A}_n$.

We can now begin properly the proof of theorem 5, subject to the reduction explained above.

The numbers $D(T, \mathscr{D}_i')$. Let $T \subset \mathscr{A}_n$ be bounded and measurable and $G \subset \mathscr{A}_n$ be discrete. For $k \in Z^+$, $g \in G$, define

$$\Phi_k = \{x \in T + g \mid x \in \text{exactly } k-1 \text{ sets } T + t \text{ with } t \in G \sim \{g\}\}.$$

Then each Φ_k is an intersection of measurable sets and is therefore measurable. Also $\Phi_k = \emptyset$ for large k (as G is discrete and T is bounded); let

$$\nu = \max\{k \in \boldsymbol{Z}^+ \mid \Phi_k \neq \emptyset\}$$

and define

$$D(T, G, g) = \mu(\Phi_1) + \frac{1}{2}\mu(\Phi_2) + \ldots + \frac{1}{\nu}\mu(\Phi_\nu). \tag{14}$$

Since μ is translation invariant, we have

$$D(T+\boldsymbol{x}, G+\boldsymbol{y}, g+\boldsymbol{y}) = D(T, G, \boldsymbol{g}) \tag{15}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{A}_n$. If $A \in \mathscr{G}$, then it is clear from lemma 3 that

$$D(AT, AG, A\boldsymbol{g}) = \|A\| D(T, G, \boldsymbol{g}). \tag{16}$$

For $i < n$ let $\mathscr{A}'_i$ be the subset of $\mathscr{A}_n$ consisting of those adeles $(\hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_n)$ with $\hat{\boldsymbol{x}}_j$ equal to the zero (one-dimensional) adele for $i+1 \leqslant j \leqslant n$. We give $\mathscr{A}'_i$ the relative topology. Then clearly $\mathscr{A}'_i$ is algebraically and topologically isomorphic to $\mathscr{A}_i$.

Let $T[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n]$ be the subset of T consisting of those adeles $\boldsymbol{y} = (\hat{\boldsymbol{y}}_1, \ldots, \hat{\boldsymbol{y}}_n)$ of T with $\hat{\boldsymbol{y}}_j = \hat{\boldsymbol{x}}_j$ for $i+1 \leqslant j \leqslant n$ (i. e., a section of T). Any section $S = T[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n]$ of T is a Cartesian product

$$S_i \times \{(\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n)\}$$

for some $S_i \subsetneq \mathscr{A}_i$. When we talk of S "regarded as a subset of $\mathscr{A}_i$", we shall mean the subset S_i of $\mathscr{A}_i$.

Now suppose $G \subset \mathscr{A}'_i$ and $\boldsymbol{g} \in G$. We define

$$D_i(T[\hat{\boldsymbol{x}}_{i+1}, \ldots, \boldsymbol{x}_n], G, \boldsymbol{g})$$

by regarding each of $T[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n]$ and G as subsets of $\mathscr{A}_i$, and then using a definition exactly similar to (14), but applied to $\mathscr{A}_i$ instead of $\mathscr{A}_n$. Now if $G \subset \mathscr{A}'_i$ and $\boldsymbol{g} \in G$, then

$$(T+\boldsymbol{g})[\hat{\boldsymbol{x}}_{i+1}, \ldots \hat{\boldsymbol{x}}_n] = T[\boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n] + \boldsymbol{g}.$$

Using this fact, together with Fubini's theorem, we have

$$G \subset \mathscr{A}'_i \Rightarrow D(T, G, \boldsymbol{g}) = \int\limits_{\mathscr{A}_{n-i}} D_i(T[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n], G, \boldsymbol{g})\, d\mu_{n-i}. \tag{17}$$

Now let

$$\mathscr{D}'_i = \boldsymbol{K}\boldsymbol{e}_1 \oplus \ldots \oplus \boldsymbol{K}\boldsymbol{e}_i.$$

Then $\mathscr{D}'_i$ regarded as a subset of $\mathscr{A}_i$ is just $\mathscr{D}_i$. Since $\mathscr{D}'_i + \boldsymbol{d} = \mathscr{D}'_i$ for every $\boldsymbol{d} \in \mathscr{D}'_i$ it follows from (15) that $D(T, \mathscr{D}'_i, d)$ is independent of $\boldsymbol{d} \in \mathscr{D}'_i$; thus we write $D(T, \mathscr{D}'_i)$ instead.

Let $\mathscr{F}'_i$ be a fundamental domain for $\mathscr{D}_i$ in $\mathscr{A}_i$. Then $\mathscr{F}'_i = \mathscr{F}_i \times \mathscr{A}_{n-i}$

is a fundamental domain for $\mathscr{D}_i'$ in $\mathscr{A}_n$. Denote by $T/\mathscr{D}_i'$ the set of all points of T reduced modulo $\mathscr{D}_i'$ so as to lie inside $\mathscr{F}_i$. Then

$$D(T, \mathscr{D}_i') = \mu(T/\mathscr{D}_i'). \tag{18}$$

Proof of (18). Each point of T is congruent modulo $\mathscr{D}_i'$ to at most a finite number of other points of T ($\leqslant \nu - 1$ in number), since T is bounded and $\mathscr{D}_i'$ is discrete. But k points of T are congruent modulo $\mathscr{D}_i'$ and congruent to no other points of T if and only if each of the k points lies in the part of T which is covered by exactly $k-1$ sets $T+\boldsymbol{d}$ with $\boldsymbol{d} \in \mathscr{D}_i'$, $\boldsymbol{d} \neq \boldsymbol{o}$. Thus each such point is counted k times in $T/\mathscr{D}_i'$; the factor $1/k$ for this part of T in $D(T, \mathscr{D}_i')$ gives the result.

From (18) it follows that

$$D(T_1, \mathscr{D}_i') \leqslant D(T_2, \mathscr{D}_i') \tag{19}$$

whenever T_1, T_2 are bounded measurable subsets of $\mathscr{A}$ with $T_1 \subset T_2$.

The transformations W_i. Since B is convex and symmetric we have

$$B \subset t{*}B + (\boldsymbol{x} - t{*}\boldsymbol{x}) \tag{20}$$

for every $\boldsymbol{x} \in B$ and every $t \geqslant 1$. For given $t \geqslant 1$ we define the mapping W_i: $\mathscr{A}_n \to \mathscr{A}_n$ by

$$W_i(\boldsymbol{x}) = \boldsymbol{y},$$

where

$$\hat{\boldsymbol{y}}_j = t{*}\hat{\boldsymbol{x}}_j \qquad (1 \leqslant j \leqslant i),$$
$$\hat{\boldsymbol{y}}_j = \hat{\boldsymbol{x}}_j \qquad (i+1 \leqslant j \leqslant n).$$

Clearly $W_i \in \mathscr{G}$ and

$$W_i(B[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n]) = (W_i B)[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n],$$

and so, using (20), the convex body $(W_i B)[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n]$ can be translated into a position in the same section so as to contain $B[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n]$. By (15) and (19) we therefore have

$$D_i(B[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n], \mathscr{D}_i') \leqslant D_i\big((W_i B)[\hat{\boldsymbol{x}}_{i+1}, \ldots, \hat{\boldsymbol{x}}_n], \mathscr{D}_i'\big),$$

and so it follows from (17) that

$$D(B, \mathscr{D}_i') \leqslant D(W_i B, \mathscr{D}_i'). \tag{21}$$

LEMMA 7. *If* $t \geqslant 1$ *then*

$$D(t{*}B, \mathscr{D}_i') \geqslant t^{m(n-i)} D(B, \mathscr{D}_i').$$

Proof. Define a lattice transformation H_{n-i} of $\mathscr{A}_n$ by

$$H_{n-i}(\boldsymbol{x}) = \boldsymbol{y},$$

where

$$\hat{\boldsymbol{y}}_j = \hat{\boldsymbol{x}} \qquad (1 \leqslant j \leqslant i),$$
$$\hat{\boldsymbol{y}}_j = t{*}\boldsymbol{x}_j \qquad (i+1 \leqslant j \leqslant n).$$

Then $\|H_{n-i}\| = t^{m(n-i)}$ and $t*B = H_{n-i}W_iB$. From (16) and (17) it follows that

$$\begin{aligned} D(t*B, \mathscr{D}_i') &= D(H_{n-i}W_iB, H_{n-i}\mathscr{D}_i') \quad \text{(since} \quad H_{n-i}\mathscr{D}_i' = \mathscr{D}_i') \\ &= \|H_{n-i}\|D(W_iB, \mathscr{D}_i') \\ &\geqslant \|H_{n-i}\|D(B, \mathscr{D}_i') \\ &= t^{m(n-i)}D(B, \mathscr{D}_i'). \end{aligned}$$

LEMMA 8. *If B contains no point of $\mathscr{D}_n \sim \mathscr{D}_i'$ then*

$$D(\tfrac{1}{2}*B, \mathscr{D}_n) = D(\tfrac{1}{2}*B, \mathscr{D}_{n-1}') = \ldots = D(\tfrac{1}{2}*B, \mathscr{D}_i').$$

Proof. If $x \in \mathscr{D}_n \sim \mathscr{D}_i'$ then

$$\big((\tfrac{1}{2}*B)+x)\big) \cap (\tfrac{1}{2}*B) = \emptyset,$$

or else $x \in B$ (using the convexity and symmetry of B). The result now follows from the definition of $D(\tfrac{1}{2}*B, \mathscr{D}_i')$.

Conclusion of the proof of theorem 5. From the definition of λ_{i+1} and the assumptions about the basis vectors $e_1, \ldots, e_n$ (i. e., that they satisfy lemma 6), no point of $\mathscr{D}_n \sim \mathscr{D}_i'$ lies in $\lambda_{i+1}*B$ for $0 \leqslant i \leqslant n-1$. Hence by lemma 8,

$$D(\tfrac{1}{2}\lambda_{i+1}*B, \mathscr{D}_{i+1}') = D(\tfrac{1}{2}\lambda_{i+1}*B, \mathscr{D}_i'),$$

and therefore

$$\begin{aligned} D(\tfrac{1}{2}*B, \mathscr{D}_n) &= D(\tfrac{1}{2}\lambda_n*B, \mathscr{D}_{n-1}') \\ &\geqslant \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^m D(\tfrac{1}{2}\lambda_{n-1}*B, \mathscr{D}_{n-2}') \quad \text{(using lemma 7)} \\ &\geqslant \left(\frac{\lambda_n}{\lambda_{n-1}}\left(\frac{\lambda_{n-1}}{\lambda_{n-2}}\right)^2 \cdots \left(\frac{\lambda_2}{\lambda_1}\right)^{n-1}\right)^m D(\tfrac{1}{2}\lambda_1*B, \mathscr{D}_0') \quad (\text{where } \mathscr{D}_0' = \{o\}) \\ &= (\lambda_1\lambda_2 \ldots \lambda_n)^m \mu(\tfrac{1}{2}\lambda_1*B)\lambda_1^{-mn} \quad \text{(using (18))} \\ &= (\lambda_1\lambda_2 \ldots \lambda_n)^m 2^{-mn}\mu(B). \end{aligned}$$

But, by (18),

$$D(\tfrac{1}{2}\lambda_n*B, \mathscr{D}_n) \leqslant \Delta(\mathscr{D}_n) = \Delta(\mathscr{D}).$$

Hence

$$\Delta(\mathscr{D}) \geqslant (\lambda_1\lambda_2 \ldots \lambda_n)^m 2^{-mn}\mu(B),$$

and theorem 5 is proved.

6.3. The product of successive minima; a lower bound. Our next theorem gives a lower bound for the product $\lambda_1 \ldots \lambda_n$ of successive minima, but the theorem only applies to convex bodies of a particular kind.

Suppose that v is a complex prime divisor of K and that B_v is a convex body at v. We call B_v *c-convex* if, when B_v is viewed in $\boldsymbol{C}^n$, we have

$$x \in B_v \Rightarrow ix \in B_v.$$

This condition means that if the real point $(t_1, 0, \ldots, t_n, 0)$ belongs to B_v (viewed in $\boldsymbol{R}^{2n}$) then the real point $(0, t_1, \ldots, 0, t_n)$ also belongs to B_v.

An example of a c-convex body is the set

$$\{z \in \boldsymbol{C}^n \mid |Az| \leqslant 1\},$$

where A is a non-singular (complex) linear transformation of $\boldsymbol{C}^n$.

THEOREM 6. *Let B be a bounded, open, symmetric convex body in $\mathscr{A}$ of the shape*

$$B = \prod_v B_v,$$

where B_v is a convex body at v for each v, and B_v is c-convex if v is complex. Assume also that $B_v = \mathfrak{O}_v^n$ for almost all $v \notin S_\infty$. Then the successive minima $\lambda_1, \ldots, \lambda_n$ of B with respect to the lattice Λ satisfy

$$(\lambda_1 \ldots \lambda_n)^m \mu(B) \geqslant 2^{n(r+s)} (n!)^{-m} |d_K|^{-n/2} \Delta(\Lambda). \tag{22}$$

Proof. The general shape of the body B we are considering and the inequality (22) are both invariant under any lattice transformation of $\mathscr{A}$. Hence we may reduce the proof to the case $\Lambda = \mathscr{D}$, and we may assume that the basis $\{e_1, \ldots, e_n\}$ for $\mathscr{D}$ satisfies the conditions of lemma 6.

From lemma 6 we know that the set of points $t_1 * e_1 + \ldots + t_n * e_n$ with $\sum_{j=1}^{n} \lambda_j |t_j| < 1$ is contained in B. Hence, if

$$E = \Big\{(t_1, \ldots, t_n) \in \boldsymbol{R}^n \Big| \sum_{j=1}^{n} \lambda_j |t_j| < 1\Big\},$$

then we have $E \subset B_v$ for each $v \in S_\infty$.

Consider a complex prime divisor v of K. Since B_v is c-convex it follows that $iE \subset B_v$, where we have viewed B_v in $\boldsymbol{C}^n$. Viewing B_v in $\boldsymbol{R}^{2n}$, we therefore have the points $(t_1, 0, \ldots, t_n, 0)$ and $(0, u_1, \ldots, 0, u_n)$ belonging to B_v whenever

$$\sum_{j=1}^{n} \lambda_j |t_j| < 1 \quad \text{and} \quad \sum_{j=1}^{n} \lambda_j |u_j| < 1. \tag{23}$$

But B_v is convex, so the point

$$\tfrac{1}{2}\{(t_1, 0, \ldots, t_n, 0) + (0, u_1, \ldots, 0, u_n)\} = \tfrac{1}{2}(t_1, u_1, \ldots, t_n, u_n)$$

is contained in B_v whenever (23) holds. In other words,

$$\psi(\tfrac{1}{2}E \times \tfrac{1}{2}E) \subset B_v,$$

where ψ is the permutation of variables

Thus we have

$$E \subset B_v \qquad (v \text{ real}),$$

$$\psi(\tfrac{1}{2}E \times \tfrac{1}{2}E) \subset B_v \qquad (v \text{ complex}).$$

Now the linear transformation ψ of $\boldsymbol{R}^{2n}$ has determinant one, and so by taking measures we obtain

$$\begin{aligned}\prod_{v\in S_\infty} \mu_v(B_v) &\geqslant (\tfrac{1}{2})^{2ns}(\mu_\infty(E))^m \\ &= 2^{-2ns}\left(\frac{2^n}{n!}\right)^m (\lambda_1 \ldots \lambda_n)^{-m} \\ &= 2^{nr}(n!)^{-m}(\lambda_1 \ldots \lambda_n)^{-m},\end{aligned}$$

where μ_∞ is Lebesgue measure on $\boldsymbol{R}^n$.

For each $v \notin S_\infty$, the local factor B_v of B is an $\mathcal{O}_v$-module and $e_1, \ldots, e_n \in B_v$, so that $\mathcal{O}_v^n \subset B_v$. Hence we have

$$\begin{aligned}\mu(B) &= \prod_v \mu_v(B_v) \\ &\geqslant 2^{nr}(n!)^{-m}(\lambda_1 \ldots \lambda_n)^{-m}\prod_{v\notin S_\infty}\mu_v(\mathcal{O}_v^n) \\ &= 2^{nr}(n!)^{-m}(\lambda_1 \ldots \lambda_n)^{-m} \\ &= 2^{n(r+s)}(n!)^{-m}|d_K|^{-n/2}(\lambda_1 \ldots \lambda_n)^{-m}\Delta(\mathcal{D}).\end{aligned}$$

This proves theorem 6.

6.4. Applications to algebraic number theory. We now apply our results on successive minima to some arithmetic questions in K. The first application concerns ideals of K. It is well-known that every ideal of K is a free $\boldsymbol{Z}$-module of rank m. By a *basis* for the ideal I we shall always mean a module basis for I over $\boldsymbol{Z}$.

The main result of this section is

THEOREM 7. *Let* $\boldsymbol{i}$ *be an idele of* K *with* $\|\boldsymbol{i}\| = 1$, *and let* I *be the corresponding ideal of* K:

$$I = \{a \in K \mid |a|_v \leqslant |i_v|_v \ (v \notin S_\infty)\}.$$

Then there exist $a_1, \ldots, a_m \in I$ *which are linearly independent over* $\boldsymbol{Q}$ *and satisfy*

$$|a_i|_v \leqslant \left(\frac{2}{\pi}\right)^s |d_K|^{1/2}|i_v|_v \qquad (v \in S_\infty)$$

for each $i = 1, 2, \ldots, m$.

The crux of the argument used to prove theorem 7 is the application of theorem 5 on successive minima so as to obtain points of the ideal I

which are linearly independent over Q. Since we are interested in linear independence over Q, and not over K, we need to consider adeles of Q rather than adeles of K. We link the two sorts of adeles as follows.

We are interested in two fields K and Q at once, so we index objects associated with the various adele spaces by a superscript indicating the field and a subscript indicating the dimension. Thus we write $\mathscr{A}_n^K$, $\mathscr{D}_n^K$, μ_n^K and $\mathscr{G}_n^K$ instead of $\mathscr{A}$, $\mathscr{D}$, μ and $\mathscr{G}$, and similarly for Q.

Let $\{\omega_1, \ldots, \omega_m\}$ be a field basis for K over Q, and let $\{\omega_1^*, \ldots, \omega_m^*\}$ denote the complementary basis. For each prime divisor p of Q, define K_p as the direct product

$$K_p = \prod_{v|p} K_v,$$

and define a mapping $\varphi_p\colon K_p \to Q_p^m$ as follows. For each $x = (x_v)_{v|p} \in K_p$ define

$$\varphi_p(x) = \Big(\sum_{v|p} S_{v|p}(x_v\omega_1^*), \ldots, \sum_{v|p} S_{v|p}(x_v\omega_m^*)\Big), \tag{24}$$

where $S_{v|p}$ denotes the local trace at v. We now use φ_p to define a mapping $\varphi\colon \mathscr{A}_1^K \to \mathscr{A}_m^Q$. For $\boldsymbol{x} = (x_v) \in \mathscr{A}_1^K$ we define $\varphi(\boldsymbol{x})$ to be the adele of $\mathscr{A}_m^Q$ whose p-component is given by the right-hand side of (24). Then (see, for example, 5—6—3 of Weiss [16]) φ is a topological Q-linear isomorphism of $\mathscr{A}_1^K$ onto $\mathscr{A}_m^Q$.

Each $a \in K$ is uniquely representable in the form

$$a = \xi_1\omega_1 + \ldots + \xi_m\omega_m,$$

where $\xi_1, \ldots, \xi_m \in Q$, and clearly

$$\varphi_p(a) = (\xi_1, \ldots, \xi_m) \in Q^m.$$

Viewing a in $\mathscr{D}_1^K$ we see at once that

$$\varphi(\mathscr{D}_1^K) = \mathscr{D}_m^Q. \tag{25}$$

Since φ is a homeomorphism the set function $\mu_1^K\varphi^{-1}$ is a non-trivial Haar measure on $\mathscr{A}_m^Q$, and hence there is some constant $\varrho > 0$ such that

$$\mu_1^K\big(\varphi^{-1}(T)\big) = \varrho\mu_m^Q(T)$$

for every measurable $T \subset \mathscr{A}_m^Q$. But if $\mathscr{F}$ is a fundamental domain for the lattice $\mathscr{D}_1^K$ in $\mathscr{A}_1^K$ then $\varphi(\mathscr{F})$ is a fundamental domain for $\varphi(\mathscr{D}_1^K) = \mathscr{D}_m^Q$ in $\mathscr{A}_m^Q$. Since

$$\mu_1^K(\mathscr{F}) = \Delta(\mathscr{D}_1^K) = 2^{-s}|d_K|^{1/2}$$

and $\mu_m^Q\big(\varphi(\mathscr{F})\big) = \Delta(\mathscr{D}_m^Q) = 1$, we have

$$\varrho = 2^{-s}|d_K|^{1/2}.$$

As a final observation about φ we note that

$$\varphi(t*\boldsymbol{x}) = t*\varphi(\boldsymbol{x}) \tag{26}$$

for every $t \in \boldsymbol{R}$, $x \in \mathscr{A}_1^K$. It follows that if B is convex in $\mathscr{A}_1^K$ then $\varphi(B)$ is convex in $\mathscr{A}_m^{\boldsymbol{Q}}$. If in addition B is bounded and open in $\mathscr{A}_1^K$, then $\varphi(B)$ is bounded and open in $\mathscr{A}_m^{\boldsymbol{Q}}$.

Remark. Before proving theorem 7 we make some observations about linear transformations and convex bodies in $\mathscr{A}_n^K$.

We suppose here that $\{\omega_1, \ldots, \omega_m\}$ is an integral basis for K over $\boldsymbol{Q}$. For each prime divisor p of $\boldsymbol{Q}$, we define

$$K_p^n = \prod_{v|p} K_v^n,$$

and define φ_{np}: $K_p^n \to \boldsymbol{Q}_p^{mn}$ as the n-dimensional analogue of φ_p, obtained by using φ_p n times. Since $\{\omega_1, \ldots, \omega_m\}$ is an integral basis it follows that, for $p \neq \infty$,

$$\varphi_{np}\Big(\prod_{v|p} \mathcal{O}_v^n\Big) = \mathcal{O}_p^{mn}. \tag{27}$$

Now let μ_v and μ_p be the local measures on K_v^n and $\boldsymbol{Q}_p^{mn}$, normalized as in 2.2, and consider the product measure $\prod_{v|p} \mu_v$ on K_p^n. Since φ_{np} is a topological isomorphism the set-function $\mu_p \varphi_{np}$ is a non-trivial Haar measure on K_p^n, and from (27) it follows that

$$\mu_p \varphi_{np} = \prod_{v|p} \mu_v \tag{28}$$

for all $p \neq \infty$. The maps φ_{np} lead at once to an n-dimensional analogue φ_n: $\mathscr{A}_n^K \to \mathscr{A}_{mn}^{\boldsymbol{Q}}$ of φ, which is a $\boldsymbol{Q}$-linear topological isomorphism of $\mathscr{A}_n^K$ onto $\mathscr{A}_{mn}^{\boldsymbol{Q}}$.

Now let $H = (H_v) \in \mathscr{G}_n^K$. Then we can write $H = (H_p)$, where H_p: $K_p^n \to K_p^n$ is defined by

$$H_p\colon\ (x_v)_{v|p} \to (H_v x_v)_{v|p}.$$

For each p, the mapping

$$\hat{H}_p = \varphi_{np} H_p \varphi_{np}^{-1}\colon\ \boldsymbol{Q}_p^{mn} \to \boldsymbol{Q}_p^{mn}$$

is $\boldsymbol{Q}_p$-linear and can therefore be represented by a non-singular $mn \times mn$ matrix (which we also denote by $\hat{H}_p$) over $\boldsymbol{Q}_p$. Thus the mapping $\hat{H}$: $\mathscr{A}_{mn}^{\boldsymbol{Q}} \to \mathscr{A}_{mn}^{\boldsymbol{Q}}$ defined by

$$\hat{H}(x) = \varphi_n H \varphi_n^{-1}(x)$$

is just the lattice transformation $(\hat{H}_p)$ of $\mathscr{A}_{mn}^{\boldsymbol{Q}}$. Thus each $H \in \mathscr{G}_n^K$ determines a corresponding $\hat{H} \in \mathscr{G}_{mn}^{\boldsymbol{Q}}$.

It is not true, however, that every $A \in \mathscr{G}_{mn}^{\boldsymbol{Q}}$ arises as an $\hat{H}$ for some $H \in \mathscr{G}_n^K$. For suppose that K is a Galois extension of $\boldsymbol{Q}$ which is not totally ramified at some prime divisor q of $\boldsymbol{Q}$, so that all the inertial degrees

$f(v|q)$ are equal to some $f>1$, and let A be the principal lattice transformation of $\mathscr{G}^{Q}_{mn}$ defined by

$$A = \operatorname{diag}(q, 1, \ldots, 1).$$

If we had $A = \hat{H}$ for some $H \in \mathscr{G}^{K}_{n}$, then in particular

$$\varphi_{nq}^{-1} A \varphi_{nq} = \bigoplus_{v|q} H_v$$

as linear transformations of K^n_q. But $\bigoplus_{v|q} H_v$ multiplies measure in K^n_q by

$$\prod_{v|q} |\det H_v|_v = (q^f)^a \quad (\text{some } a \in \mathbf{Z}),$$

whereas (28) shows that $\varphi_{nq}^{-1} A \varphi_{nq}$ multiplies measure in K^n_q by

$$|\det \dot{A}|_q = q^{-1}.$$

This proves our contention.

It is now easy to see that not every bounded, open, symmetric convex body in $\mathscr{A}^K_n$ is of the shape

$$B_\infty \times \prod_{v \notin S_\infty} A_v(\mathscr{O}^n_v) \tag{29}$$

with $A_v \in \mathscr{G}_v$ and $A_v \in U_v$ for almost all v. For example, let K, A, and q be as in the last paragraph and consider the convex body

$$B = A\left[C_\infty \times \prod_{p \neq \infty} \mathscr{O}^{mn}_p\right]$$

in A^{Q}_{mn}, where C_∞ is the cube $\max |x_i| < 1$ in $\mathbf{R}^{mn}$. Then $\varphi_n^{-1}(B)$ is bounded, open, symmetric and convex in $\mathscr{A}^K_n$, but is not of the shape (29). On the other hand, it is easily seen that every open symmetric convex body in $\mathscr{A}^K_n$ has the shape $\prod_p C_p$ for some sets $C_p \subset K^n_p$.

Proof of theorem 7. Let B be the open box in $\mathscr{A}^K_1$ determined by $\mathfrak{i}$ (that is, the interior of the closed box determined by $\mathfrak{i}$); then

$$\mu^K_1(B) = 2^n \pi^s.$$

From (25) and (26) it is clear that

$$\lambda_1(B, \mathscr{D}^K_1) = \lambda_1\big(\varphi(B), \mathscr{D}^Q_m\big),$$

where λ_1 denotes the first minimum. By the product formula $t*B$ contains no non-zero element of $\mathscr{D}^K_1$ if $0 < t \leqslant 1$. Hence

$$\lambda_1(B, \mathscr{D}^K_1) = \lambda_1\big(\varphi(B), \mathscr{D}^Q_m\big) \geqslant 1.$$

But $\varphi(B)$ is a bounded, open, symmetric convex body in $\mathscr{A}^Q_m$ and hence, writing $\lambda_i = \lambda_i\big(\varphi(B), \mathscr{D}^Q_m\big)$, we have, by theorem 5,

$$\mu^Q_m\big(\varphi(B)\big)(\lambda_1 \lambda_2 \ldots \lambda_m) \leqslant 2^m \Delta(\mathscr{D}^Q_m) = 2^m.$$

But

$$\mu_m^Q\big(\varphi(B)\big) = \varrho^{-1}\cdot\mu_1^K(B) = 2^s|d_K|^{-1/2}\cdot 2^r\pi^s,$$

so it follows that

$$\lambda_m \leqslant \left(\frac{2}{\pi}\right)^s |d_K|^{1/2}(\lambda_1\lambda_2\ldots\lambda_{m-1})^{-1}$$

$$\leqslant \left(\frac{2}{\pi}\right)^s |d_K|^{1/2} = t_0, \text{ say}.$$

This last inequality implies that for every $\varepsilon > 0$ there exist m points, say $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m$, of $\mathscr{D}_m^Q$ which are linearly independent over $\boldsymbol{Q}$ and lie in the body $(t_0+\varepsilon)*\varphi(B)$. But this means that there exist m points $\varphi^{-1}(\boldsymbol{d}_1), \ldots, \varphi^{-1}(\boldsymbol{d}_m)$ in $\varphi^{-1}(\mathscr{D}_m^Q) = \mathscr{D}_1^K$ which are linearly independent over $\boldsymbol{Q}$ and lie in the body

$$\varphi^{-1}\big((t_0+\varepsilon)*\varphi(B)\big) = (t_0+\varepsilon)*B.$$

Using the discreteness of $\mathscr{D}_1^K$ and the boundedness of B it is easily seen that there exist $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m \in \mathscr{D}_1^K$ which are linearly independent over $\boldsymbol{Q}$ and lie in the body $t_0*\bar{B}$. Viewing $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m$ in K gives the result.

COROLLARY. *Let* $\boldsymbol{i}$, *I be as in theorem* 7, *and denote by M the field constant*

$$M = \frac{m}{2}\left(\frac{2}{\pi}\right)^s |d_K|^{1/2}. \tag{30}$$

Then there exists a basis $\{\omega_1, \ldots, \omega_m\}$ *for I such that*

$$|\omega_i|_v \leqslant M|i_v|_v,$$

$$|\omega_i|_v \leqslant \frac{m}{2} M|i_v|_v,$$

for each $i = 1, 2, \ldots, m$.

Proof. We view I as a lattice in $\boldsymbol{R}^m$ (see 2.2). For $a \in K$ define

$$F(a) = \max_{v\in S_\infty}\left(\frac{|a|_v}{M}\right)^{1/m_v},$$

where $m_v = 1$ or 2 according as v is real or complex. Since K is dense in $\boldsymbol{R}^m$, F can be extended to convex distance function F_1 on $\boldsymbol{R}^m$. Since $F_1(a_i) = F(a_i) \leqslant 1$ for each of the a_i given by the theorem, the lemma follows at once from lemma 8, chapter V of Cassels [3].

In the first theorem of [11], Mahler proved a result of the same form as the corollary to theorem 7, but with our constant M replaced by C for real prime divisors, and our constant $\frac{m}{2}M$ replaced by C^2 for complex prime divisors (remembering that Mahler does not use normalized valuations), where C is the field constant defined by (26) in [11]. For $m = 2, 3, 4$, Mahler's result is better than ours. For large m, Luthar [6] has shown that Mahler's C can be replaced by

$$C' = m!\gamma_m m^{1-m}\pi^{-s}2^{-(r+s/m)}|d_K|^{1/2},$$

where γ_m is defined in section 1 of [11]. Since $\gamma_m \leqslant 2m!$ (see (2) of [11]). C' is at worst

$$C_1 = 2(m!)^2 m^{1-m}\pi^{-s}2^{-(r+s/m)}|d_K|^{1/2},$$

and $\dfrac{M}{C_1}, \dfrac{\frac{1}{2}mM}{C_1^2}$ are certainly much less than 1 for large m. However, this comparison is unsatisfactory because of the probable weakness of our estimate of γ_m.

Corresponding to theorem 3 of Mahler [11] we have

THEOREM 8. *Let* $\mathfrak{a}$ *be any one-dimensional adele of* K *and* $\mathfrak{i}$ *an idele of volume one. Then there exists* $\gamma \in K$ *such that*

$$\begin{aligned} |\gamma - a_v|_v &\leqslant M|i_v|_v && (v \textit{ real}),\\ |\gamma - a_v|_v &\leqslant \frac{m}{2} M|i_v|_v && (v \textit{ complex}),\\ |\gamma - a_v|_v &\leqslant |i_v|_v && (v \notin S_\infty), \end{aligned}$$

where M *is given by* (30).

Proof. Using theorem 2 it is easily shown that there exists $\beta \in K$ such that

$$|\beta - a_v|_v \leqslant |i_v|_v \quad (v \in S_\infty).$$

By theorem 7 there exist $\gamma_1, \ldots, \gamma_m \in K$ which are linearly independent over $\mathbf{Z}$ and satisfy

$$\begin{aligned} |\gamma_i|_v &\leqslant \left(\frac{2}{\pi}\right)^s |d_K|^{1/2}|i_v|_v && (v \in S_\infty),\\ |\gamma_i|_v &\leqslant |i_v|_v && (v \notin S_\infty), \end{aligned}$$

for $1 \leqslant i \leqslant n$. The inequalities

$$|\xi - a_v|_v \leqslant |i_v|_v \quad (v \notin S_\infty)$$

are satisfied by all elements

$$\xi = \beta + x_1\gamma_1 + \ldots + x_m\gamma_m$$

of K with $x_1, \ldots, x_m \in \mathbf{Z}$.

Let upper indices denote conjugates over $\mathbf{Q}$. Since $\gamma_1, \ldots, \gamma_m$ are linearly independent over $\mathbf{Q}$, the discriminant

$$\Delta(\gamma_1, \ldots, \gamma_m) = (\det[\gamma_i^j])^2$$

is real and non-zero. Now use exactly the same argument as in section 11 of [11], but using $\gamma_1, \dots, \gamma_m$ instead of Mahler's $a_1, \dots, a_n$.

Theorem 8 can plainly be proved for n-dimensional adeles of K; this is also true of the corollaries which follow, though we only prove them for the one-dimensional case.

Corollary 1. *Let M be the constant given by* (30) *and $\mathbf{i}$ an idele of volume one. The closed box B^* with sides of length*

$$\begin{array}{rl} M\,|i_v|_v & (v \text{ real}), \\ \tfrac{1}{2} m M\,|i_v|_v & (v \text{ complex}), \\ |i_v|_v & (v \notin S_\infty), \end{array}$$

is a $\mathscr{D}$-covering of $\mathscr{A}$. B^* has measure

$$2^r \pi^s \left(\frac{m}{2}\right)^s M^{r+s}.$$

Corollary 2. *Put*

$$W = \left(\frac{m}{\pi}\right)^s M^{r+s} |d_K|^{1/2} \tag{31}$$

where M is the constant given by (30). *Then any closed box determined by an idele $\mathbf{j}$ of volume not less than W is a $\mathscr{D}$-covering.*

Proof. For any idele $\mathbf{i}$ of K, we denote by $B(\mathbf{i})$ the closed box determined by $\mathbf{i}$.

Let $\mathbf{l}$ be an idele which determines the box B^* of corollary 1. Then the idele $\mathbf{j}\mathbf{l}^{-1}$ has volume

$$\|\mathbf{j}\mathbf{l}^{-1}\| \geqslant \left(\frac{2}{\pi}\right)^s |d_K|^{1/2},$$

and hence $B(\mathbf{j}\mathbf{l}^{-1})$ has measure not less than

$$2^r \pi^s \left(\frac{2}{\pi}\right)^s |d_K|^{1/2} = 2^m \Delta(\mathscr{D}).$$

Hence there exists $\boldsymbol{\alpha} \in \mathscr{D}$, $\boldsymbol{\alpha} \neq \mathbf{O}$ with $\boldsymbol{\alpha} \in B(\mathbf{j}\mathbf{l}^{-1})$.

Now let $\mathscr{F}$ be a fundamental domain for $\mathscr{D}$ with $\mathscr{F} \subset B^*$. Then

$$a\mathscr{F} \subset B(a\mathbf{l}) \subset B(\mathbf{j}).$$

But $a\mathscr{F}$ is a fundamental domain for $a\mathscr{D} = \mathscr{D}$.

Our next corollary is a quantitative form of a well-known density theorem in algebraic number theory.

Corollary 3. *Let* $\boldsymbol{i}$ *be an idele of* K, *and* W *the constant given by* (31). *Then the number* $M(\boldsymbol{i})$ *of field elements bounded by* $\boldsymbol{i}$ *satisfies*

$$M(\boldsymbol{i}) \leqslant 2^{r+s}\pi^s|d_K|^{-1/2}\left(1+\left(\frac{W}{\|\boldsymbol{i}\|}\right)^{1/m}\right)^m\|\boldsymbol{i}\|.$$

Proof. Let $\lambda = \left(\frac{W}{\|\boldsymbol{i}\|}\right)^{1/m}$. Then $\|\lambda*\boldsymbol{i}\| \geqslant W$, and hence by corollary 2, the closed box $B(\lambda*\boldsymbol{i})$ determined by $\lambda*\boldsymbol{i}$ contains some fundamental domain $\mathscr{F}_0$ for $\mathscr{D}_1 = K$. Then

$$\begin{aligned}\bigcup_{a\in K\cap B(\boldsymbol{i})}(a+\mathscr{F}_0) &\subset B(\boldsymbol{i})+\mathscr{F}_0\\ &\subset B(\boldsymbol{i})+\lambda*B(\boldsymbol{i})\\ &\subset (1+\lambda)*B(\boldsymbol{i}).\end{aligned}$$

Hence, taking measures, we have

$$M(\boldsymbol{i})\mu(\mathscr{F}_0) \leqslant (1+\lambda)^m\|\boldsymbol{i}\|2^r\pi^s.$$

Substituting $\mu(\mathscr{F}_0) = 2^{-s}|d_K|^{1/2}$, $\lambda = (W/\|\boldsymbol{i}\|)^m$, we obtain the result.

From the inequality of corollary 3 it is easily deduced that there is a field constant δ_K (in fact,

$$\delta_K = 2^{r+s}\pi^s|d_K|^{-1/2}W^{1/m}[(1+W^{1/m})^m-1]$$

will do) such that

$$M(\boldsymbol{i}) \leqslant 2^{r+s}\pi^s|d_K|^{-1/2}\|\boldsymbol{i}\|+\delta_K\|\boldsymbol{i}\|^{1-1/m}$$

for every idele $\boldsymbol{i}$ of K; the second term on the right-hand side here is of the same order when $\|\boldsymbol{i}\| \to \infty$ as the error term in the asymptotitc result

$$\left|M(\boldsymbol{i})-2^{r+s}\pi^s|d_K|^{-1/2}\|\boldsymbol{i}\|\right| = O(\|\boldsymbol{i}\|^{1-1/m}) \quad (\|\boldsymbol{i}\| \to \infty)$$

proved in chapter V, section 2 of Lang [5].

Our last corollary is a quantitative version of the very strong approximation theorem.

Corollary 4. *Let* τ *be any prime divisor of* K, $\boldsymbol{a}$ *any adele and* $\boldsymbol{i}$ *any idele. Then there exists* $\gamma\in K$ *such that*

$$|\gamma-a_v|_v \leqslant |i_v|_v \quad (v \neq \tau).$$

Furthermore, γ *can be chosen so that*

$$|\gamma-a_\tau|_\tau \leqslant W_0,$$

where W_0 *is the least element of* $|K_\tau|_\tau$ *such that*

$$W_0 \geqslant \frac{W}{\prod_{v\neq\tau}|i_v|_v},$$

where W *is given by* (31).

Proof. Let the idele $\boldsymbol{j}$ be defined by

$$j_v = i_v \quad (v \neq \tau),$$
$$|j_\tau|_\tau = W_0.$$

Then $\|\boldsymbol{j}\| \geqslant W$ and so by corollary 2 the box $\boldsymbol{\alpha}+B(\boldsymbol{j})$ centered about $\boldsymbol{\alpha}$ is a $\mathscr{D}$-covering and therefore contains a field element.

7. T-adeles

We now discuss briefly now our analogue of the geometry of numbers can adapted to the T-adeles considered by Cantor [2]. As Cantor points out, by taking K as Q and T consisting of ∞ and a single p-adic prime divisor of Q we arrive at the set-up of Lutz [7] as a special case. To conclude our discussion we derive some classical results, and connect our point of view with that of Rogers and Swinnerton-Dyer [13].

7.1. The general theory for T-adeles. Let T be any subset of Ω. If $\boldsymbol{x}\in\mathscr{A}$, *the T-part* $(\boldsymbol{x})_T$ of $\boldsymbol{x}$ is defined by

$$(\boldsymbol{x})_T = (x_v)_{v\in T}.$$

If $G \subset \mathscr{A}$, the T-part is defined similarly:

$$(G)_T = \{(\boldsymbol{x})_T |\ \boldsymbol{x}\in G\}. \tag{32}$$

The *space $\mathscr{A}^T$ of (n-dimensional) T-adeles* is just the T-part of $\mathscr{A}$: $\mathscr{A}^T = (\mathscr{A})_T$. Similarly we define $\mathscr{L}^T$ and $\mathscr{G}^T$ as the T-parts of $\mathscr{L}$ and $\mathscr{G}$ respectively.

The definitions just stated are valid for any $T \subset \Omega$, but from now on we assume that $T \supset S_\infty$ and $T \neq \Omega$.

Clearly $\mathscr{A}^T$ is a locally compact additive group, since it is just the restricted direct product of the groups K_v^n $(v\in T)$ with respect to the subgroups $\mathcal{O}_v^n (v\in T, v\notin S_\infty)$. A metric for $\mathscr{A}^T$ is defined by analogy with (7) in the obvious way. Boxes in $\mathscr{A}^T$ are also defined in the obvious way.

The vector space K^n is naturally embedded along the diagonal of $\mathscr{A}^T$ as a subgroup $\mathscr{D}$ of $\mathscr{A}^T$, but since $T \neq \Omega$, $\mathscr{D}$ is dense in $\mathscr{A}^T$. Let K^T denote the ring of T-integers of K:

$$K^T = \{a\in K|\ |a|_v \leqslant 1\ (v\notin T)\}.$$

Then $(K^T)^n$, viewed as a subgroup $\mathscr{D}^T$ of $\mathscr{D}$, is discrete in $\mathscr{A}^T$. If $\mathscr{F}_0$ is the fundamental domain for $\mathscr{D}$ in $\mathscr{A}$ given by (9), then $(\mathscr{F}_0)_T$ is a fundamental domain for $\mathscr{D}^T$ in $\mathscr{A}^T$. It follows that $\mathscr{A}^T/\mathscr{D}^T$ is compact.

The Haar measure μ^T on $\mathscr{A}^T$ is normalized so that

$$\mu^T\{\boldsymbol{x}\in\mathscr{A}^T|\ |x_v|_v \leqslant 1\ \ (v\in T)\} = (2^r\pi^s)^n.$$

Then if $A \in \mathscr{G}^T$ and $X \subset \mathscr{A}^T$ is measurable, we have

$$\mu^T(A(X)) = \prod_{v \in T} |\det A_v|_v \mu^T(X).$$

From the remarks above about fundamental domains we have

$$\Delta(\mathscr{D}^T) = 2^{-ns}|d_K|^{n/2}.$$

A *convex body* in $\mathscr{A}^T$ is defined as the T-part of a convex body in $\mathscr{A}$.

The obvious definition for a lattice in $\mathscr{A}^T$ is a set of the shape $A(\mathscr{D}^T)$ with $A \in \mathscr{G}^T$, but this definition turns out to be unnecessarily restrictive. A more satisfactory definition, which includes the T-ideals of K (for $T \in \Sigma$) as special cases, is given below. We note that the meaning of $\mathscr{A}^{\sim T}$ ($\sim$ means set-theoretic complement) is clear from the general definition (32). *Boxes* in $\mathscr{A}^{\sim T}$ are defined in the obvious way; the box

$$\prod_{v \notin T} \mathscr{O}_v^n$$

in $\mathscr{A}^{\sim T}$ is denoted by $\mathscr{O}^{\sim T}$.

DEFINITION 6. A *lattice in* $\mathscr{A}^T$ (a *T-lattice*) is a subset of $\mathscr{A}^T$ of the shape

$$((\mathscr{A}^T \times B^{\sim T}) \cap \Lambda)_T,$$

where $B^{\sim T}$ is a box in $\mathscr{A}^{\sim T}$ and Λ is a lattice in $\mathscr{A}$.

It is easily shown that every T-lattice Λ^T can be written in the form

$$\Lambda^T = (\mathscr{A}_T \cap \Lambda)_T, \tag{33}$$

for some lattice Λ in $\mathscr{A}$, where

$$\mathscr{A}_T = \mathscr{A}^T \times \mathscr{O}^{\sim T}.$$

It then follows easily that every T-lattice Λ^T is a discrete K^T-module which contains n points which are linearly independent over K^T and has compact quotient $\mathscr{A}^T/\Lambda^T$. Unfortunately, T-lattices need not be free over K^T, that is, of the shape $A(\mathscr{D}^T)$ with $A \in \mathscr{G}^T$. But if K^T is a principal ideal domain then every T-lattice is free.

If $\mathscr{F}^T$ is a fundamental domain for the T-lattice Λ^T in $\mathscr{A}^T$, and we write Λ^T in the shape (33) for some lattice Λ in $\mathscr{A}$, then $\mathscr{F}^T \times \mathscr{O}^{\sim T}$ is a fundamental domain for Λ in $\mathscr{A}$; in particular,

$$\Delta(\Lambda^T) = \Delta(\Lambda).$$

We now have the following version of Minkowski's convex body theorem for $\mathscr{A}^T$.

THEOREM 9. *Let Λ^T be a T-lattice and B^T a convex body in $\mathscr{A}^T$. If*

$$\mu^T(B^T) > k \cdot 2^{mn} \Delta(\Lambda^T)$$

for some $k \in \mathbf{Z}^+$ *then there exist at least* k *distinct pairs* $\pm \boldsymbol{x}_i$ $(1 \leqslant i \leqslant k)$ *of non-zero points of* Λ^T *which lie in* B^T.

Proof. Write

$$\Lambda^T = (\mathscr{A}_T \cap \Lambda)_T$$

for some lattice Λ in $\mathscr{A}$. Then $\Delta(\Lambda) = \Delta(\Lambda^T)$. Now put

$$B = B^T \times \mathfrak{O}^{\sim T}.$$

Then B is a convex body in $\mathscr{A}$ and $\mu(B) = \mu^T(B^T)$. The result now follows from the convex body theorem in $\mathscr{A}$ (theorem 2).

Theorem 9 is easily extended to the case when B is compact.

We now give a sketch of the theory of successive minima in $\mathscr{A}^T$.

Definition 7. Let B^T be an open subset of $\mathscr{A}^T$ and let Λ^T be a lattice in $\mathscr{A}^T$. For $i = 1, 2, \ldots, n$, *the i-th T-minimum* $\lambda_i^T = \lambda_i^T(B^T, \Lambda^T)$ *of* B^T *with respect to* Λ^T is defined to be

$$\lambda_i^T = \inf\{t > 0 \mid (t * B^T) \cap \Lambda^T \text{ contains } i \text{ points which are linearly independent over } K^T\}.$$

Consider now a bounded, open, symmetric convex body B^T in $\mathscr{A}^T$, and a lattice Λ^T in $\mathscr{A}^T$. Then

$$\Lambda^T = (\mathscr{A}_T \cap \Lambda)_T$$

for some lattice Λ in $\mathscr{A}$, and

$$\Delta(\Lambda) = \Delta(\Lambda^T).$$

Also $B = B^T \times \mathfrak{O}^{\sim T}$ is a convex body in $\mathscr{A}$ and

$$\mu(B) = \mu^T(B^T).$$

Suppose that $\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_i^T \in \Lambda^T \cap (t * B^T)$ are linearly independent over K^T. Since $\Lambda^T = (\mathscr{A}_T \cap \Lambda)_T$, there exist $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i \in \mathscr{A}_T \cap \Lambda$ such that

$$\boldsymbol{x}_j^T = (\boldsymbol{x}_j)_T \quad (1 \leqslant j \leqslant i);$$

certainly $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i \in \Lambda \cap (t * B)$. Suppose that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i$ are linearly dependent over K. Then

$$a_1 \boldsymbol{x}_1 + \ldots + a_i \boldsymbol{x}_i = \boldsymbol{o}$$

for some $a_1, \ldots, a_i \in K$. Using the strong approximation theorem we can find a non-zero $a \in K$ such that all of $aa_1, \ldots, aa_i$ belong to K^T. But then it follows from that

$$aa_1 \boldsymbol{x}_1^T + \ldots + aa_i \boldsymbol{x}_i^T = \boldsymbol{o},$$

which contradicts our assumption about $\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_i^T$. Hence we must have $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i$ linearly independent over K, and so

$$\lambda_i^T(B^T, \Lambda^T) \geqslant \lambda_i(B, \Lambda).$$

Conversely, suppose that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i \in \Lambda \cap (t*B)$ are linearly independent over K. Then it is easily seen that $(\boldsymbol{x}_1)_T, \ldots, (\boldsymbol{x}_i)_T \in \Lambda^T \cap (t*B^T)$. Now let $\Lambda = A\mathscr{D}$, where $A \in \mathscr{G}$. Then

$$\boldsymbol{x}_j = A\boldsymbol{d}_j \quad (1 \leqslant j \leqslant i)$$

for some $\boldsymbol{d}_j \in \mathscr{D}$, and hence

$$(\boldsymbol{x}_j)_T = (A)_T \boldsymbol{d}_j \quad (1 \leqslant j \leqslant i).$$

Using this, together with the fact that A is a lattice transformation, it is easily shown that $(\boldsymbol{x}_1)_T, \ldots, (\boldsymbol{x}_i)_T$ are linearly independent over K, and hence over K^T. Thus we must have $\lambda_i^T(B^T, \Lambda^T) \leqslant \lambda_i(B, \Lambda)$. It is now clear that

$$\lambda_i^T(B^T, \Lambda^T) = \lambda_i(B, \Lambda).$$

Using theorem 5 we obtain immediately

THEOREM 10. *Let B^T be a bounded, open, symmetric convex body in $\mathscr{A}^T$. Then the successive T-minima $\lambda_1^T, \ldots, \lambda_n^T$ of B^T with respect to the T-lattice Λ^T satisfy*

$$(\lambda_1^T \ldots \lambda_n^T)^m \mu^T(B^T) \leqslant 2^{mn} \Delta(\Lambda^T).$$

Similarly, theorem 6 leads to

THEOREM 11. *Let B^T be a bounded, open, symmetric convex body in $\mathscr{A}^T$ of the shape*

$$B^T = \prod_{v \in T} B_v,$$

where B_v is a convex body at v for each $v \in T$, and B_v is c-convex if v is complex. Assume also that $B_v = \mathcal{O}_v^n$ for almost all $v \notin S_\infty$. Then the successive T-minima $\lambda_1^T, \ldots, \lambda_n^T$ of B^T with respect to the T-lattice Λ^T satisfy

$$(\lambda_1^T \ldots \lambda_n^T)^m \mu^T(B^T) \geqslant 2^{n(r+s)}(n!)^{-m}|d_K|^{-n/2} \Delta(\Lambda^T).$$

Remarks. Theorems 10 and 11 can be applied to arithmetic questions in $\mathscr{A}^T$ exactly as was done with theorems 5 and 6 in $\mathscr{A}$. In particular, from theorem 10 we obtain quantitative versions of lemmas 2.6, 2.7 and 2.8 of Cantor [2].

We note also that our theorems on successive minima (both in $\mathscr{A}$ and $\mathscr{A}^T$) can be applied to convex bodies determined by local distance functions (c.f. theorem 4), and lead to results about the number of linearly independent solutions of certain inequalities (see chapter 7 of [12] for the details). These results contain as special cases (when $K = \boldsymbol{Q}$, $T = \{\infty, p\}$) many of the theorems in chapter 2 of Lutz [7].

7.2. Two special cases. We first consider the case $K = \boldsymbol{Q}$, $T = \{\infty\}$. Thus $\mathscr{A}^T \cong \boldsymbol{R}^n$ algebraically and topologically, so we are in the situation of the classical geometry of numbers. Since $K^T = \boldsymbol{Z}$ is a principal ideal domain it follows from the remarks about T-lattices in 7.1 that a subset L of $\boldsymbol{R}^n$ is a T-lattice if and only if $L = A(\boldsymbol{Z}^n)$ for some $A \in \mathscr{G}^T$, that is, if and only if L is a lattice in the classical sense. Clearly the notions of convex bodies in $\mathscr{A}^T$ and $\boldsymbol{R}^n$ are the same. Hence theorem 9 yields the classical version of Minkowski's convex body theorem, and theorems 10 and 11 specialize to $\boldsymbol{R}^n$ giving the classical estimates for the product of successive minima (with the obvious notation):

$$\frac{2^n}{n!}\Delta(\Lambda) \leqslant \lambda_1\lambda_2 \ldots \lambda_n \mu(B) \leqslant 2^n \Delta(\Lambda).$$

Finally, we look at Rogers and Swinnerton-Dyer [14]. Let $\{\omega_1, \ldots, \omega_m\}$ be an integral basis for K over $\boldsymbol{Q}$. Then the algebra

$$K^* = \boldsymbol{R}\omega_1 + \ldots + \boldsymbol{R}\omega_m$$

of [13] is clearly isomorphic to K_∞, and so $(K^*)^n$ is isomorphic to $K_\infty^n = \mathscr{A}^{S_\infty}$.

Let $\mathcal{O} = K^{S_\infty}$. A subset Λ of $(K^*)^n$ is a lattice in the sense of [13] if and only if Λ is discrete in $(K^*)^n$, $(K^*)^n/\Lambda$ is compact, and

$$\Lambda = \mathcal{O}b_1 + \ldots + \mathcal{O}b_n$$

for some $b_1, \ldots, b_n \in \Lambda$ which are linearly independent over K^* (and so certainly over $\mathcal{O}$). It now follows that if $\Gamma \subset (K^*)^n$ is a lattice in the sense of [13], then Γ, when viewed in $\mathscr{A}^{S_\infty}$, is of the form $A(\mathcal{O}^n)$ for some $A \in \mathscr{G}^{S_\infty}$, and so certainly Γ is an S_∞-lattice.

Hence the situation in [13] is a special case of our T-adele theory when $T = S_\infty$. Theorems 2 and 3 (applied to convex bodies) of [13] can now be derived from the corresponding results in 7.1 for S_∞-adeles.

Part II

1. Introduction

In part I (henceforth GNI) I set up the basic tools for an analogue of the geometry of numbers for adele spaces, and carried the analogue as far as a discussion of successive minima. This paper carries on from GNI, and deals with compactness for lattices and related topological questions. The notation of GNI is used freely here.

Recall that a lattice in $\mathscr{A}$ is a subset of the shape $A\mathscr{D}$, where A is a lattice transformation (i. e., $A \in \mathscr{G}$) and $\mathscr{D}$ is the image of K^n along the diagonal of $\mathscr{A}$. We shall see that the lattice $A\mathscr{D}$ can be identified with the coset $A\mathscr{P}$ of $\mathscr{P}$ in $\mathscr{G}$, where $\mathscr{P}$ is the group of principal lattice transformations. Hence we call the left coset space $\mathscr{G}/\mathscr{P}$ the lattice space of $\mathscr{A}$. My main object in this paper is to study the topological structure of $\mathscr{G}/\mathscr{P}$ and thereby obtain analogues of Mahler's compactness theorem for lattices in $\mathscr{A}$.

In section 2 I look at two naturally arising topologies on $\mathscr{G}$ and the relationship between them. In section 3 I take quotient topologies on $\mathscr{G}/\mathscr{P}$ and determine conditions under which a subset of $\mathscr{G}/\mathscr{P}$ is compact; these results are essentially analogues of Mahler's compactness theorem. Section 4 deals briefly with the connection between the topologies already introduced on $\mathscr{G}/\mathscr{P}$ and the topology used in Chabauty [4]. Finally, in section 5, I consider the T-adele case and the connection with the classical version of Mahler's theorem. The proofs of the main results in section 3 are independent of any earlier analogues of the geometry of numbers, but the proof of theorem 3 is based directly on section 16 of Cassels [17].

In sections 4 and 5 several details have been omitted, but can be found in chapters 8 and 9 of my thesis [12].

2. Topology in $\mathscr{G}$

2.1. Two topologies on $\mathscr{G}$. We recall from 2.2 of GNI that for each prime divisor v of K, $\mathscr{L}_v$ is the ring of all $n \times n$ matrices over the completion K_v, with the topology determined by the metric D_v' given by (3) of GNI. For each finite v, $\mathscr{I}_v$ denotes the subring of $\mathscr{L}_v$ containing those matrices with (v-adic) integer components. Te restricted direct product of the $\mathscr{L}_v$ with respect to the $\mathscr{I}_v$ (as topological rings) is denoted by $\mathscr{L}$; $\mathscr{L}$ is a locally compact topological ring with metric D' (see (8) of GNI).

The group of units of $\mathscr{L}_v$ is denoted by $\mathscr{G}_v$, and, for v finite, the subgroup U_v of $\mathscr{G}_v$ is defined by

$$U_v = \{A \in \mathscr{I}_v | \ |\det A|_v = 1\}.$$

The restricted direct product of the $\mathscr{G}_v$ with respect to the U_v is denoted by $\mathscr{G}$. There are two obvious ways of topologizing $\mathscr{G}$: one by regarding it as a subset of $\mathscr{L}$, the other by taking a restricted direct product topology.

The additive topology. We call the relative topology induced on $\mathscr{G}$ from $\mathscr{L}$ the *additive topology*. Clearly $\mathscr{L}$ can be identified (additively) with $\mathscr{A}^n$, and the topology on $\mathscr{L}$ agrees with the product topology on $\mathscr{A}^n$ Under this identification the group $\mathscr{P}$ of principal lattice transformations (see 3.2 of GNI is identified with a subset of $\mathscr{D}^n$. It follows that $\mathscr{P}$ is discrete in $\mathscr{G}$ under the additive topology.

The multiplicative topology. For each v, $\mathscr{G}_v$ is a multiplicative topological group under the relative topology induced from $\mathscr{L}_v$, and (for v finite) U_v is compact and open in $\mathscr{G}_v$. Hence under the corresponding restricted direct product topology $\mathscr{G}$ is a locally compact multiplicative topological group. We call this restricted direct product topology on $\mathscr{G}$ the *multiplicative topology*.

For each v we have a metric D_v for the group $\mathscr{G}_v$ which is invariant under right multiplicative translations (see lemma 1 of GNI). For $A, B \in \mathscr{G}$, define

$$D(A, B) = \sup_v \frac{D_v(A_v, B_v)}{\log\log Nv^*},$$

where Nv^* is defined by

$$\begin{aligned} \log\log Nv^* &= 1 \qquad (v \in S_\infty), \\ Nv^* &= Nv \quad (v \notin S_\infty). \end{aligned}$$

LEMMA 1. *D is a metric on $\mathscr{G}$, and is invariant under right multiplicative translations:*

$$D(AM, BM) = D(A, B)$$

for all A, B, $M \in \mathscr{G}$. The D-topology on $\mathscr{G}$ coincides with the multiplicative topology.

Proof. The pattern of the proof is similar to that of the corresponding proofs for d and D' (c. f. lemma 2 of GNI). The details are given in theorem 5, chapter 3 of [12].

2.2. Comparison of the two topologies. The additive topology on $\mathscr{G}$ is weaker than the multiplicative topology since, roughly speaking, $A \in \mathscr{G}$ is close to $B \in \mathscr{G}$ under the multiplicative topology if and only if A is close to B and A^{-1} is close to B^{-1} under the additive topology. Moreover, the two topologies are distinct; this is well-known for $n = 1$, in which case $\mathscr{G}$ is the idele group of K, the multiplicative topology is the standard idele topology, and the additive topology is the topology induced from the adele ring $\mathscr{A}_1$ (see, for example, section 16 of Cassels [17]). Note that since $\mathscr{P}$ is discrete in $\mathscr{G}$ under the additive topology it must be discrete in $\mathscr{G}$ under the multiplicative topology.

Recall that for each $A \in \mathscr{G}$ we define

$$\|A\| = \prod_v |\det A_v|_v .$$

The following theorem gives us more precise information about the relationship between our two topologies. The proof of this theorem is based on section 16 of Cassels [17].

THEOREM 1. *Let* $\Delta \geqslant \delta > 0$, *and define*

$$\mathscr{C}(\delta) = \{A \in G \mid \|A\| \geqslant \delta\},$$
$$\mathscr{C}(\delta, \Delta) = \{A \in G \mid \delta \leqslant \|A\| \leqslant \Delta\}.$$

Then

(i) *each of* $\mathscr{C}(\delta)$ *and* $\mathscr{C}(\delta, \Delta)$ *is closed in* $\mathscr{L}$;

(ii) *the additive and multiplicative topologies coincide on both* $\mathscr{C}(\delta)$ *and* $\mathscr{C}(\delta, \Delta)$.

Proof. Let $A \in \mathscr{L}$ and suppose that $\|A\| < \delta$. Then we can choose $S \in \Sigma$ such that S contains all finite v for which A_v is not integral, and, furthermore, such that

$$\prod_{v \in S} |\det A_v|_v < \delta .$$

Put

$$W = \{B \in \mathscr{L} \mid \|A_v - B_v\|_v < \varepsilon (v \in S),\ B_v \in \mathscr{J}_v\ (v \notin S)\}.$$

Then W is a neighbourhood of A in $\mathscr{L}$, and if ε is small enough we have

$$\prod_{v \in S} |\det B_v|_v < \delta$$

for all $B \in W$. Thus $W \cap \mathscr{C}(\delta) = \emptyset$, so $\mathscr{C}(\delta)$ is closed in $\mathscr{L}$.

Now suppose that $\|A\| = \Delta_0 > \Delta$. Then we can choose $S \in \Sigma$ such that S contains all finite v for which A_v is not integral, and, furthermore, such that if $v \notin S$ then

$$|x|_v < 1 \Rightarrow |x|_v < \tfrac{1}{2} \Delta_0^{-1} \delta \qquad (x \in K_v^n);$$

this is possible as the finite v are discrete. Now choose $\varepsilon > 0$ so that if

$$\|A_v - B_v\|_v < \varepsilon \qquad (v \in S)$$

then

$$\Delta < \prod_{v \in S} |\det B_v|_v < 2\,\Delta_0.$$

Then

$$W = \{B \in \mathscr{L} \mid \|A_v - B_v\|_v < \varepsilon \ (v \in S),\ B_v \in \mathscr{J}_v \ (v \notin S)\}$$

is a neighbourhood of A in $\mathscr{L}$ satisfying

$$W \cap \mathscr{C}(\delta, \Delta) = \emptyset.$$

It now follows that $\mathscr{C}(\delta, \Delta)$ is closed in $\mathscr{L}$, and (i) is proved.

Certainly the additive topology is weaker than the multiplicative topology on $\mathscr{C}(\delta)$. To prove (ii) for $\mathscr{C}(\delta)$ it is therefore sufficient to show that if $A \in \mathscr{C}(\delta)$ and we are given a neighbourhood $\mathscr{N}$ of A of the multiplicative topology then there is a neighbourhood $\mathscr{N}'$ of the additive topology with

$$N' \cap \mathscr{C}(\delta) \subset \mathscr{N} \cap \mathscr{C}(\delta).$$

This we now do.

Take $A \in \mathscr{C}(\delta)$ and a multiplicative neighbourhood $\mathscr{N}$ of A. Then $\mathscr{N}$ contains a multiplicative neighbourhood of the shape

$$\mathscr{N}_1 : \begin{array}{ll} \|X_v - A_v\|_v < \varepsilon & (v \in S), \\ X_v \in U_v & (v \notin S), \end{array}$$

for some $\varepsilon > 0$ and some $S \in \Sigma$ such that

$$v \notin S \Rightarrow A_v \in U_v.$$

Now choose $S_1 \supset S$ such that

$$(1) \qquad v \notin S_1,\ a \in \mathcal{O}_v,\ |a|_v \neq 1 \Rightarrow |a|_v(\|A\|+1) < \delta;$$

this is possible by the properties of the discrete prime divisors. Now S_1 is finite. Using this, the continuity of $|\det\ |_v$ for each v, and the discreteness of the $v \notin S_\infty$, we can choose $\varepsilon_1 > 0$ such that

(i) $\varepsilon_1 < \varepsilon$;

(ii) $\|X_v - A_v\|_v < \varepsilon_1 \quad (v \in S_1 \sim S)$
$\Rightarrow X_v \in U_v$;

(iii) $\|X_v - A_v\|_v < \varepsilon_1 \quad (v \in S_1)$
$\Rightarrow \Big| \prod_{v \in S_1} |\det X_v|_v - \prod_{v \in S_1} |\det A_v|_v \Big| \leqslant 1.$

Now consider the additive neighbourhood

$$\mathscr{N}_2' : \begin{array}{ll} \|X_v - A_v\|_v < \varepsilon_1 & (v \in S_1), \\ X_v \in \mathscr{J}_v & (v \notin S_1), \end{array}$$

and the corresponding multiplicative neighbourhood

$$\mathscr{N}_2 : \begin{array}{ll} \|X_v - A_v\|_v < \varepsilon_1 & (v \in S_1), \\ X_v \in U_v & (v \notin S_1), \end{array}$$

of A. Using (i) and (ii) above it follows that $\mathscr{N}_2 \subset N_1$. We complete the proof by showing that

$$\mathscr{N}_2' \cap \mathscr{C}(\delta) = \mathscr{N}_2 \cap \mathscr{C}(\delta).$$

Clearly

$$\mathscr{N}_2 \cap \mathscr{C}(\delta) \subset \mathscr{N}_2' \cap \mathscr{C}(\delta).$$

Now suppose that $X \in \mathscr{N}_2' \cap \mathscr{C}(\delta)$ but $X \notin \mathscr{N}_2 \cap \mathscr{C}(\delta)$. Then there exists $\tau \notin S_1$ such that $|\det X_\tau|_\tau < 1$. Then

$$\begin{aligned} \|X\| &\leqslant \prod_{v \in S_1} |\det X_v|_v \cdot |\det X_\tau|_\tau \\ &\leqslant (\|A\|+1) \cdot |\det X_\tau|_\tau \quad \text{(using (iii) above)} \\ &< \delta \quad \text{(using (1))}. \end{aligned}$$

This contradicts our assumption that $X \in \mathscr{C}(\delta)$. Hence we must have

$$\mathscr{N}_2' \cap \mathscr{C}(\delta) \subset \mathscr{N}_2 \cap \mathscr{C}(\delta).$$

We have thus shown that the two topologies coincide on the set $\mathscr{C}(\delta)$. But $\mathscr{C}(\delta, \Delta)$ is a subset of $\mathscr{C}(\delta)$, so the two topologies must coincide on $\mathscr{C}(\delta, \Delta)$. This proves (ii).

3. Compactness for lattices

3.1. Two topologies on the lattice space. Suppose that $A \in \mathscr{G}$. Then clearly $A\mathscr{D} = \mathscr{D}$ if and only if $A \in \mathscr{P}$. It follows that the lattices $A\mathscr{D}$, $B\mathscr{D}$ are equal if and only if the cosets $A\mathscr{P}$, $B\mathscr{P}$ coincide. Thus we can identify the lattice $A\mathscr{D}$ with the left coset $A\mathscr{P}$ of $\mathscr{P}$ in $\mathscr{G}$. We shall use the symbols $A\mathscr{D}$ and $A\mathscr{P}$ interchangeably; it will always be clear from the context whether we are thinking of a lattice as a point set in $\mathscr{A}$ or a coset of $\mathscr{P}$ in $\mathscr{G}$. We call the left coset space $\mathscr{G}/\mathscr{P}$ the *lattice space of* $\mathscr{A}$.

The way to topologize the lattice space of $\mathscr{A}$ is now clear: we take quotient topologies on $\mathscr{G}/\mathscr{P}$ corresponding to the additive and multiplicative topologies on $\mathscr{G}$.

The additive quotient topology. Consider $\mathscr{G}$ with the additive topology. We call the corresponding quotient topology on $\mathscr{G}/\mathscr{P}$ the *additive quotient topology*. A fundamental system of neighbourhoods of the lattice $\Lambda = A\mathscr{P}$ for this topology is given by $\{I(\Lambda; \varepsilon) \mid \varepsilon > 0\}$, where

$$I(\Lambda; \varepsilon) = \{M\mathscr{P} \in \mathscr{G}/\mathscr{P} \mid D'(M, A) < \varepsilon\}.$$

Intuitively, the lattice Λ_1 is close to the lattice

$$\Lambda = K\boldsymbol{a}_1 \oplus \ldots \oplus K\boldsymbol{a}_n$$

under the additive quotient topology if a basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ for Λ_1 can be chosen so that $\boldsymbol{b}_i$ is close to $\boldsymbol{a}_i$ in $\mathscr{A}$ for $1 \leqslant i \leqslant n$.

The multiplicative quotient topology. Suppose now that $\mathscr{G}$ has the multiplicative topology. We call the resulting quotient topology on $\mathscr{G}/\mathscr{P}$ the *multiplicative quotient topology*; this topology is stronger than the additive quotient topology. We now use the translation invariance of the metric D of lemma 1 to define a metric for the multiplicative quotient topology.

Let $\Lambda_1 = A_1\mathscr{P}, \Lambda_2 = A_2\mathscr{P}$ be lattices, and define

$$\begin{aligned} D^*(\Lambda_1, \Lambda_2) &= \inf\{D(L_1, L_2) \mid L_1 \in \Lambda_1, L_2 \in \Lambda_2\} \\ &= \inf\{D(A_1P_1, A_2P_2) \mid P_1, P_2 \in \mathscr{P}\}. \end{aligned}$$

THEOREM 2. *D^* is a metric on $\mathscr{G}/\mathscr{P}$ and the D^*-topology coincides with the multiplicative quotient topology.*

Proof. Since $\mathscr{P}$ is a group and D is right invariant we have

$$\begin{aligned} D^*(A\mathscr{P}, B\mathscr{P}) &= \inf\{D(A, B\mathscr{P}) \mid P \in \mathscr{P}\} \\ &= \inf\{D(A\mathscr{P}, B) \mid P \in \mathscr{P}\} \end{aligned}$$

for all $A, B \in \mathscr{G}$. Hence if $A\mathscr{P}, B\mathscr{P} \in \mathscr{G}/\mathscr{P}$ and $D^*(A\mathscr{P}, B\mathscr{P}) = 0$ then for each $n \in \boldsymbol{Z}^+$ there exists $P_n \in \mathscr{P}$ such that

$$D(A, BP_n) < \frac{1}{n}.$$

Hence, under the D-topology on $\mathscr{G}$, $\lim BP_n$ exists and is equal to A, and so $\lim P_n = B^{-1}A$. But $P_n \in \mathscr{P}$ for all n and $\mathscr{P}$ is discrete, so $\lim P_n = B^{-1}A \in \mathscr{P}$. Thus

$$D^*(A\mathscr{P}, B\mathscr{P}) = 0 \Rightarrow A\mathscr{P} = B\mathscr{P}.$$

Now let $A, B, C \in \mathscr{G}$. Then

$$\begin{aligned} D^*(A\mathscr{P}, B\mathscr{P}) &= \inf_{P_1, P_2 \in \mathscr{P}} D(AP_1, BP_2) \\ &\leqslant \inf_{P_1, P_2 \in \mathscr{P}} \big(D(AP_1, C) + D(C, BP_2)\big) \\ &= \inf_{P \in \mathscr{P}} D(AP, C) + \inf_{P \in \mathscr{P}} D(C, BP) \\ &= D^*(A\mathscr{P}, C\mathscr{P}) + D^*(C\mathscr{P}, B\mathscr{P}). \end{aligned}$$

Hence the triangle law holds for D^*. The other properties needed for D^* to be a metric are obvious.

Let $\psi: \mathscr{G} \to \mathscr{G}/\mathscr{P}$ be the natural mapping and denote by $\mathscr{S}(\Lambda; \varrho)$ the open D-sphere of radius $\varrho > 0$ about $\Lambda \in \mathscr{G}$, and by $\mathscr{S}^*(\Lambda\mathscr{P}; \varrho)$ the open D^*-sphere of radius ϱ about $\Lambda\mathscr{P} \in \mathscr{G}/\mathscr{P}$. Then it follows easily from the definition of D^* that

$$\mathscr{S}^*(\Lambda\mathscr{P}; \varrho) = \psi\big(\mathscr{S}(\Lambda; \varrho)\big).$$

This establishes the equality of topologies.

3.2. An important lemma. Our general aim is to give simple conditions on a subset of $\mathscr{G}/\mathscr{P}$ which ensure that it (or its closure) is compact, or, more specifically, to determine conditions under which a sequence of lattices has a convergent subsequence (under either topology). Our treatment depends on the theory of successsive minima developed in section 6 of GNI in much the same way as Mahler's original compactness theorem depends on the classical theory of successive minima. The results we need from GNI are contained in

LEMMA 2. *Let B be an open box in $\mathscr{A}$ and Λ a lattice. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the successive minima of B with respect to Λ. Then there exist contants θ_1, θ_2 (depending only on K and n) such that*

$$\theta_2 \Delta(\Lambda) \leqslant (\lambda_1 \lambda_2 \ldots \lambda_n)^m \mu(B) \leqslant \theta_2 \Delta(\Lambda).$$

Proof. See theorems 5 and 6 of GNI.

The vital lemma for our purposes is

LEMMA 3. *Let B be an open box and L a collection of lattices. The following two statements about L are equivalent.*

(i) *There exist $c < \infty$ and $\gamma > 0$ such that $\Delta(\Lambda) \leqslant c$ and*

$$(\gamma * B) \cap \Lambda = \{o\}$$

for all $\Lambda \in L$.

(ii) *There exist $\Delta_0 > 0$ and $\beta < \infty$ such that $\Delta(\Lambda) \geqslant \Delta_0$ and the box $\beta * B$ contains a basis for Λ, for all $\Lambda \in L$.*

Proof. Let $\Lambda \in L$ and let $\lambda_1, \ldots, \lambda_n$ be the successive minima of B with respect to Λ. Clearly (i) and (ii) are equivalent to

(i′) $\Delta(\Lambda) \leqslant c$, $\lambda_1 \geqslant \gamma > 0$ $(\Lambda \in L)$,

and

(ii′) $\Delta(\Lambda) \geqslant \Delta_0 > 0$, $\lambda_n \leqslant \beta$ $(\Lambda \in L)$,

respectively.

Suppose that (i′) holds. Then by lemma 2,

$$\begin{aligned} \Delta(\Lambda) &\geqslant \mu(B)(\lambda_1 \lambda_2 \ldots \lambda_n)^m \theta_2^{-1} \\ &\geqslant \mu(B) \gamma^{mn} \theta_2^{-1} = \Delta_0, \text{ say}. \end{aligned}$$

Again by lemma 2 we have

$$\lambda_n \leqslant (\lambda_1\lambda_2 \ldots \lambda_{n-1})^{-1}\left(\frac{\Delta(\Lambda)}{\theta_2\mu(B)}\right)^{1/m}$$
$$\leqslant \gamma^{-n+1}\left(\frac{c}{\theta_2\mu(B)}\right)^{1/m} = \beta, \text{ say}.$$

Thus (i′) implies (ii′).

Suppose now that (ii′) holds. By lemma 2,

$$\lambda_1 \geqslant (\lambda_2\lambda_3 \ldots \lambda_n)^{-1}\left(\frac{\theta_1\Delta(\Lambda)}{\mu(B)}\right)^{1/m}$$
$$\geqslant \beta^{-n+1}\left(\frac{\theta_1\Delta_0}{\mu(B)}\right)^{1/m} = \gamma.$$

Also

$$\Delta(\Lambda) \leqslant \theta_1^{-1}(\lambda_1\lambda_2 \ldots \lambda_n)^m\mu(B)$$
$$\leqslant \theta_1^{-1}\beta^{mn}\mu(B) = c.$$

Thus (ii′) implies (i′).

3.3. An analogue of Mahler's compactness theorem. Our main result is

THEOREM 3. *Let B be an open box, Y and γ positive real numbers, and suppose that the subset $\mathscr{M}$ of $\mathscr{G}$ satisfies*

(i) $\|M\| \leqslant Y$ *for all* $M \in \mathscr{M}$;

(ii) $(\gamma * B) \cap M\mathscr{D} = \{o\}$ *for all* $M \in \mathscr{M}$.

Let $\psi\colon \mathscr{G} \to \mathscr{G}/\mathscr{P}$ be the natural mapping. Then $\mathscr{M}/\mathscr{P} = \psi(\mathscr{M})$ is bounded in $\mathscr{G}/\mathscr{P}$ in terms of D^*. Equivalently, $\overline{\mathscr{M}}/\mathscr{P} = \psi(\overline{\mathscr{M}})$ is compact under the multiplicative quotient topology, where $\overline{\mathscr{M}}$ denotes the closure of $\mathscr{M}$ in $\mathscr{G}$ under the multiplicative topology.

Proof. The equivalence of the theorem is immediate from the definition of the quotient topology, so it is enough to prove that $\overline{\mathscr{M}}/\mathscr{P}$ is compact.

We note first that $\overline{\mathscr{M}}$ satisfies (i) and (ii); this follows easily from the continuity of $\|\ \|$ under the multiplicative topology. From lemma 3 we have: there exist $\Delta_0 > 0$ and $\beta < \infty$ such that

(iii) $\|M\| \geqslant \Delta_0$ for every $M \in \overline{\mathscr{M}}$;

(iv) the box $\beta * B$ contains a basis for $M\mathscr{D}$, for every $M \in \overline{\mathscr{M}}$.

Hence, by (i) and (iii), $\overline{\mathscr{M}} \subset \mathscr{C}(\delta, \Delta)$ for some δ, Δ with $\Delta \geqslant \delta > 0$, and so by theorem 1 (ii) the additive and multiplicative topologies coincide on $\overline{\mathscr{M}}$.

The mapping ψ is continuous and so the result will be proved if we can find an additively compact subset $\mathscr{W}$ of $\mathscr{L}$ such that $\psi(\overline{\mathscr{M}} \cap \mathscr{W})$

$= \bar{\mathcal{M}}/\mathcal{P}$, that is, we must find an additively compact subset $\mathcal{W}$ of $\mathcal{L}$ such that if $M \in \bar{\mathcal{M}}$ then $MP \in \mathcal{W}$ for some $P \in \mathcal{P}$. But this is equivalent to finding a compact subset $\mathcal{W}'$ of $\mathcal{A}$ such that any lattice $M\mathcal{D}$ with $M \in \bar{\mathcal{M}}$ has a basis in $\mathcal{W}'$. The existence of such a set $\mathcal{W}'$ follows from (iv) above.

COROLLARY (analogue of Mahler's compactness theorem). *Let* $\{\Lambda_i\}$ *be a sequence of lattices such that*

(i) $\Delta(\Lambda_i) \leqslant Y$ *for some constant* $Y < \infty$ *and all* i;

(ii) $\Lambda_i \cap \mathcal{N} = \{\boldsymbol{o}\}$ *for some open neighbourhood* $\mathcal{N}$ *of* $\boldsymbol{o}$ *in* $\mathcal{A}$ *and all* i.

Then there exists a subsequence $\{\Lambda_{i_r}\}$ *of* $\{\Lambda_i\}$ *and a lattice* Λ *such that* $\Lambda_{i_r} \to \Lambda$ *as* $r \to \infty$ *under either topology on* $\mathcal{G}/\mathcal{P}$. *Also* $\Delta(\Lambda) \leqslant Y$ *and* $\Lambda \cap \mathcal{N} = \{\boldsymbol{o}\}$.

Proof. Theorem 3 gives the result at once for the multiplicative quotient topology, and this in turn implies the result for the weaker additive quotient topology.

We remark that if the hypotheses of the above corollary are replaced by the equivalent condition (ii) of lemma 3 then the result of the corollary can be established for the additive quotient topology without mentioning successive minima — the proof uses only properties of compact subsets of metric spaces and part (i) of theorem 1. But in order to establish the conclusion of the corollary either with the hypotheses of theorem 3 or for the multiplicative quotient topology it seems necessary to use successive minima.

Our analogue of Mahler's compactness theorem can be used to furnish results about the existence and properties of critical lattices and also leads to a method for the effective determination of the lattice constant of a convex body. The ideas involved are exactly similar to those in the classical case (see V. 5, V. 6 of Cassels [3]).

4. The Chabauty topology

In [4] Chabauty defines a topology on the space of all discrete subgroups of a locally compact, σ-compact, unimodular topological group. In the present context we can describe this topology as follows.

Suppose that Λ is a lattice in $\mathcal{A}$. Let C be a compact subset of $\mathcal{A}$ and let $\mathcal{N}$ be a neighbourhood of $\boldsymbol{o}$ in $\mathcal{A}$. The corresponding neighbourhood $\mathcal{H}(\Lambda; C, \mathcal{N})$ of Λ is the set of all lattices Λ' such that

$$\Lambda' \cap C \subset \Lambda + \mathcal{N}, \qquad \Lambda \cap C \subset \Lambda' + \mathcal{N}.$$

The collection

$$\{\mathcal{H}(\Lambda; C, \mathcal{N}) \mid C \text{ compact, } \mathcal{N} \text{ a neighbourhood of } \boldsymbol{o}\}$$

is a fundamental system of neighbourhoods of Λ for some topology on $\mathscr{G}/\mathscr{P}$; we call this topology the *Chabauty topology.*

The Chabauty topology is weaker than the multiplicative quotient topology and stronger than the additive quotient topology (the details are given in theorem 3, chapter 8 of [12]). Also it can be shown that the Chabauty topology and the additive quotient topology are distinct by considering the case $n = 1$, $K = \boldsymbol{Q}$, when lattices are of the form $\boldsymbol{i}\boldsymbol{Q}$ with $\boldsymbol{i}$ an idele of $\boldsymbol{Q}$. For suppose that the rational primes are arranged in ascending order: $p_1, p_2, \ldots,$ and define a sequence $(\boldsymbol{i}_j)$ of ideles of $\boldsymbol{Q}$ by

$$\begin{aligned} i_{jp} &= p_j \quad \text{if } p = p_j, \\ i_{jp} &= 1 \quad \text{otherwise}. \end{aligned}$$

Then $\boldsymbol{i}_j\boldsymbol{Q} \to \boldsymbol{Q}$ as $j \to \infty$ under the additive quotient topology but not under the Chabauty topology (the details are given in theorem 4, chapter 8 of [12]). However, I have not been able to decide whether the Chabauty and multiplicative quotient topologies are distinct or not.

Since the Chabauty topology is weaker than the multiplicative quotient topology, our analogue of Mahler's compactness theorem (corollary to theorem 3) implies the corresponding result for the Chabauty topology. Alternatively, we have the following proof which uses neither the additive quotient topology nor the multiplicative quotient topology.

PROOF OF COROLLARY TO THEOREM 3 FOR THE CHABAUTY TOPOLOGY. Suppose that $\{\Lambda_i\}$ is a sequence of lattices satisfying the conditions of the corollary to theorem 3. Then theorem 1 of Chabauty [4] guarantees a subsequence of $\{\Lambda_i\}$ which converges under the Chabauty topology to a discrete subgroup Λ of $\mathscr{A}$ with $\Delta(\Lambda) < \infty$. For notational convenience, we suppose that $\Lambda_i \to \Lambda$. It is easily checked, using the definition of Chabauty topology convergence, that each $\boldsymbol{x} \in \Lambda$ is the limit of a sequence $\{\boldsymbol{x}_i\}$ with $\boldsymbol{x}_i \in \Lambda_i$, and, conversely, that every convergent sequence $\{\boldsymbol{x}_i\}$ with $\boldsymbol{x}_i \in \Lambda_i$ converges to a point of Λ.

We must show that Λ is a lattice in $\mathscr{A}$. Suppose that $\boldsymbol{x} \in \Lambda$ and $a \in K$. Then $\boldsymbol{x} = \lim \boldsymbol{x}_i$ for some $\boldsymbol{x}_i \in \Lambda_i$, and hence $a\boldsymbol{x} = \lim a\boldsymbol{x}_i$. Since $a\boldsymbol{x}_i \in \Lambda_i$ for each i, it follows that $a\boldsymbol{x} \in \Lambda$ and so Λ is a vector space over K. Also, if $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_k \in \Lambda$ are linearly independent over K, then we can find a neighbourhood $\mathscr{N}$ of $\boldsymbol{o}$ such that if $\boldsymbol{p}_i \in \boldsymbol{m}_i + \mathscr{N}$ $(1 \leqslant i \leqslant k)$ then $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k$ are also linearly independent over K. But there exist $t \in \boldsymbol{Z}^+$ and $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k \in \Lambda_t$ such that

$$\boldsymbol{p}_i \in \boldsymbol{m}_i + \mathscr{N} \quad (1 \leqslant i \leqslant k).$$

Since Λ_t is of dimension n over K, it follows that Λ has dimension at most n over K. It is easily seen that if Λ has dimension less than n over

K then any fundamental domain for Λ has infinite measure; consequently Λ has dimension n over K, since $\Delta(\Lambda) < \infty$. Since Λ is discrete in $\mathscr{A}$, it now follows from remark 2 following the corollary to theorem 4 of GNI that Λ is a lattice in $\mathscr{A}$. This completes the proof.

5. T-adeles

We now look at our theory of compactness for lattices in the T-adele space $\mathscr{A}^T$ (see 7.1 of GNI for the definition), restricting our attention to the case when $T \neq \Omega$ and $T \supset S_\infty$. We recall from (33) of GNI that a *T-lattice* is a subset Λ^T of $\mathscr{A}^T$ of the shape

$$\Lambda^T = (\mathscr{A}_T \cap \Lambda)_T,$$

where Λ is a lattice in $\mathscr{A}$. A *free* T-lattice is of the shape $A(\mathscr{D}^T)$ with $A \in \mathscr{G}^T$.

Additive quotient and multiplicative quotient topologies for T-lattices are only meaningful for free T-lattices. In general we take as a topology on T-lattices one corresponding to the Chabauty topology on $\mathscr{G}/\mathscr{P}$. Thus we take as a fundamental system of neighbourhoods of the T-lattice Λ^T the collection

$$\{\mathscr{H}(\Lambda^T; C^T, \mathscr{N}^T) \mid C^T \text{ compact in } \mathscr{A}^T, \mathscr{N}^T \text{ a neighbourhood of } \boldsymbol{o} \text{ in } \mathscr{A}^T\},$$

where $\mathscr{H}(\Lambda^T; C^T, \mathscr{N}^T)$ is the set of all T-lattices M^T such that

$$\Lambda^T \cap C^T \subset M^T + \mathscr{N}^T, \qquad M^T \cap C^T \subset \Lambda^T + \mathscr{N}^T.$$

We call the corresponding topology on T-lattices the *Chabauty topology*.

LEMMA 4. *Let Λ_r $(r = 1, 2, \ldots)$ and Λ be lattices in $\mathscr{A}$, and suppose that $\Lambda_r \to \Lambda$ as $r \to \infty$ under the Chabauty topology on lattices in $\mathscr{A}$. Define T-lattices Λ_r^T $(r = 1, 2, \ldots)$, Λ^T by*

$$\Lambda_r^T = (\mathscr{A}_T \cap \Lambda_r)_T, \qquad \Lambda^T = (\mathscr{A}_T \cap \Lambda)_T.$$

Then $\Lambda_r^T \to \Lambda^T$ as $r \to \infty$ under the Chabauty topology on T-lattices.

Proof. Let C^T be compact in $\mathscr{A}^T$ and $\mathscr{N}^T$ a neighbourhood of $\boldsymbol{o}$ in $\mathscr{A}^T$. Then $C = C^T \times \mathfrak{O}^{\sim T}$ is compact in $\mathscr{A}$, and $\mathscr{N} = \mathscr{N}^T \times \mathfrak{O}^{\sim T}$ is a neighbourhood of $\boldsymbol{o}$ in $\mathscr{A}$. Hence there exists $r_0 \in \boldsymbol{Z}^+$ such that $r \geqslant r_0$ implies

$$\Lambda_r \cap C \subset \Lambda + \mathscr{N} \quad \text{and} \quad \Lambda \cap C \subset \Lambda_r + \mathscr{N}.$$

By intersecting with $\mathscr{A}_T$ and taking T-parts, we obtain

$$\Lambda_r^T \cap C^T \subset \Lambda^T + \mathscr{N}^T, \qquad \Lambda^T \cap C^T \subset \Lambda_r^T + \mathscr{N}^T$$

for all $r \geqslant r_0$. This establishes the lemma.

We can now prove the following result about compactness for T-lattices.

THEOREM 4. *Let $\{\Lambda_r^T\}$ be a sequence of T-lattices. Suppose that $c > 0$ is a constant and $\mathscr{N}^T$ is a neighbourhood of $\mathbf{o}$ in $\mathscr{A}^T$ such that*

(i) $\Delta(\Lambda_r^T) \leqslant c$ $(r = 1, 2, \ldots)$;

(ii) $\Lambda_r^T \cap \mathscr{N}^T = \{\mathbf{o}\}$ $(r = 1, 2, \ldots)$.

Then there exist a T-lattice Λ^T and a subsequence $\{\Lambda_{r_s}^T\}$ of $\{\Lambda_r^T\}$ such that $\Lambda_{r_s}^T \to \Lambda^T$ as $s \to \infty$ under the Chabauty topology on T-lattices.

Proof. Let Λ_r $(r = 1, 2, \ldots)$ be lattices in $\mathscr{A}$ such that

$$\Lambda_r^T = (\mathscr{A}_T \cap \Lambda_r)_T$$

for all r, and write $\mathscr{N} = \mathscr{N}^T \times \mathscr{O}^{\sim T}$, so that $\mathscr{N}$ is a neighbourhood of $\mathbf{o}$ in $\mathscr{A}$. Then $\Delta(\Lambda_r) = \Delta(\Lambda_r^T) \leqslant c$ and $\Lambda_r \cap \mathscr{N} = \{\mathbf{o}\}$ for all r. The theorem now follows easily from theorem 3 and lemma 4.

Let us now restrict our attention to free T-lattices. We identify the space of free T-lattices with the left coset space $\mathscr{G}^T/\mathscr{P}^T$, where $\mathscr{P}^T$ is the image in $\mathscr{G}^T$ of the group of $n \times n$ matrices A over K with $|\det A|_v = 1$ for $v \notin T$. The additive and multiplicative topologies on $\mathscr{G}^T$ are defined by analogy with the corresponding topologies on $\mathscr{G}$. We then take quotient topologies on $\mathscr{G}^T/\mathscr{P}^T$ to obtain the additive quotient topology $\mathscr{T}^+$ and the multiplicative quotient topology $\mathscr{T}^\times$. It can be shown that

$$\mathscr{T}^+ \subset \mathscr{T}_c \subset \mathscr{T}^\times,$$

where $\mathscr{T}_c$ is the Chabauty topology restricted to free T-lattices. If T is finite then the additive and multiplicative topologies on $\mathscr{G}^T$ are both a finite product of the same local topologies, so it follows that $\mathscr{T}^+ = \mathscr{T}_c = \mathscr{T}^\times$. But if T is infinite then $\mathscr{T}^+ \neq \mathscr{T}_c$ (the details are given in theorem 8, chapter 9 of [12]).

Theorem 4 gives at once an analogue of Mahler's theorem for free T-lattices; it is easily seen that in this case the limit T-lattice is actually free.

Finally, we look at a couple of special cases. We saw in 7.2 of GNI that the classical geometry of numbers is simply the T-adele theory in the case $K = \mathbf{Q}$, $T = \{\infty\}$, and that a T-lattice is precisely a lattice in the classical sense. Moreover, convergence of a sequence of T-lattices under any one of our three topologies is equivalent to the usual definition of convergence of a sequence of lattices in $\mathbf{R}^n$. Thus in this case theorem 4 is just the classical version of Mahler's theorem. We also noted in 7.2 of GNI that the situation of Rogers and Swinnerton-Dyer [4] is a special case of the T-adele theory for $T = S_\infty$. Thus theorem 5 of [4] can be derived from theorem 4 in the case $T = S_\infty$.

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