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ON A THEOREM OF KŁOSOWSKA ABOUT GENERALISED CONVOLUTIONS

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Generalised convolutions, with associated concepts of infinite divisibility, stability, and domains of attraction, were introduced and studied by Urbanik [14]. Domains of attraction were characterised in terms of characteristic functions by the author [2]; a direct characterisation in terms of the measures themselves was subsequently given by Kłosowska [10]. The object of this note is to show that Kłosowska's result may be proved more simply (and somewhat extended) by using Tauberian theory.

1. Results. Let $\mathscr{P}$ be the class of probability laws on $[0, \infty)$, and $\circ$ a generalised convolution in Urbanik's sense. For $P \in \mathscr{P}$, define P^n as the n-th power of P under $\circ$, δ_x the unit mass at x, and T_t $(t > 0)$ the map with

$$\int_0^\infty f(x)(T_t P)(dx) = \int_0^\infty f(tx)\,P(dx) \qquad (P \in \mathscr{P})$$

for bounded continuous f. Urbanik's key axiom postulates the existence of constants c_n and a measure $M \in \mathscr{P}$ other than δ_0 (the *characteristic measure* of the algebra $(\mathscr{P}, \circ)$) with

$$\text{(U)} \qquad T_{1/c_n}\delta_1^n \Rightarrow M \qquad (n \to \infty)$$

(weak convergence).

As in [10], we confine our attention to *regular* algebras, i.e. those for which there exists a non-constant characteristic function $\Phi(P, t)$ with

$$\Phi(P \circ Q, t) = \Phi(P, t)\Phi(Q, t)$$

and weak convergence of measures $P_n \in \mathscr{P}$ equivalent to uniform convergence of $\Phi(P_n, t)$ on compact t-sets. Then Φ is an integral transform of Mellin-convolution type:

$$\Phi(P, t) = \int_0^\infty \Omega(xt)\,P(dx) \qquad (P \in \mathscr{P},\ t \geqslant 0).$$

Here the *kernel* Ω is continuous on $[0, \infty)$, $\Omega(0) = 1$, $|\Omega(\cdot)| \leqslant 1$, and $1 - \Omega$

varies regularly at zero (with index χ, say); χ is called the *characteristic exponent* of the algebra. Then $\Phi(M, t) = \exp(-t^\chi/v^\chi)$ for some scale-constant $v > 0$. Taking characteristic functions of (U), we get

$$[\Omega(t/c_n)]^n \to \exp(-t^\chi/v^\chi) \quad (n \to \infty).$$

Taking logarithms and writing $c(t)$ for $c_{[t]}$, we obtain

$$1/[1-\Omega(v/c(t))] \sim t \quad (t \to \infty).$$

As $1/[1-\Omega(1/t)]$ varies regularly at infinity with index χ, this shows [6] that $c(t)$ varies regularly with index $1/\chi$, and

$$c(1/[1-\Omega(v/t)]) \sim t \quad (t \to \infty).$$

A measure $P \neq \delta_0$ is said to be *stable* [14] if

$$T_{1/a_n} Q^n \Rightarrow P$$

for some measure $Q \in \mathscr{P}$ and sequence $\{a_n\}$; the Q which can arise belong to the *domain of attraction* of P. The stable measures P are those ([14], [2]) with characteristic functions

$$\Phi(P, t) = \exp(-ct^\lambda) \quad (c > 0,\ 0 < \lambda \leqslant \chi).$$

We may restrict attention to $c = 1$ (which amounts to a scale-change); $\lambda \in (0, \chi]$ is the *index* of P. Then

$$[\Phi(Q, t/a_n)]^n \to \Phi(P, t) = \exp(-t^\lambda) \quad (n \to \infty)$$

or, writing $a(t)$ for $a_{[t]}$,

$$1/[1-\Phi(Q, 1/a(t))] \sim t \quad (t \to \infty).$$

Here ([2], Lemma 7) $a(t)$ varies regularly with index $1/\lambda$, whence ([2], Proposition 1, or [6]) $1-\Phi(Q, 1/t)$ varies regularly with index λ, and

$$a(1/[1-\Phi(Q, 1/t)]) \sim t \quad (t \to \infty).$$

So Q belongs to a domain of attraction (of a stable law with index $\lambda \in (0, \chi]$) if and only if $1-\Phi(Q, t)$ varies regularly with index λ (see [2]). This may be translated into a direct condition on the measure Q. As in [10], we shall assume that

$$(1) \qquad \int_0^\infty x^\chi M(dx) < \infty,$$

which we shall need for the case $\lambda = \chi$; this holds for all known examples of generalised convolution algebras.

THEOREM. *Let L vary slowly: write*

$$(2) \qquad c = \lambda \int_0^\infty [1-\Omega(t)]\, dt/t^{1+\lambda} = \Gamma(1-\lambda/\chi)\left[v^\lambda \int_0^\infty x^\lambda M(dx)\right]^{-1}.$$

Then

$$(3)\qquad 1-\Phi(P,t)\sim ct^{\lambda}L(1/t)\quad (t\to 0)$$

is equivalent to

$$(4)\qquad 1-P(x)\sim L(x)/x^{\lambda}\quad (x\to\infty)$$

if $0<\lambda<\chi$, *and to*

$$(5)\qquad \int_0^x u^{\chi}P(du)\sim L(x)\quad (x\to\infty)$$

if $\lambda=\chi$.

COROLLARY (Kłosowska [10]). *The measure* P *lies in the domain of attraction of a stable law with index* $\lambda\in(0,\chi)$ *if and only if*

$$(6)\qquad [1-P(xu)]/[1-P(x)]\sim u^{-\lambda}\quad (x\to\infty)\ \textit{for all}\ u>0,$$

and of a stable law with index χ *if and only if*

$$(7)\qquad x^{\chi}[1-P(x)]/\int_0^x u^{\chi}P(du)\to 0\quad (x\to\infty).$$

2. Proofs.

LEMMA. *The Mellin transform of the characteristic measure is regular at least for* $0<\operatorname{Re}s<\chi$ *and is given by*

$$(8)\qquad \int_0^{\infty} x^{s}M(dx)=\Gamma(1-s/\chi)\Big[v^{s}s\int_0^{\infty}[1-\Omega(t)]\,dt/t^{1+s}\Big]^{-1}.$$

If (1) *holds, then* $\int_0^{\infty} x^{s}M(dx)$ *is regular for* $s=\chi$.

Proof. We have

$$\int_0^{\infty}[1-\Phi(M,t)]\,dt/t^{1+\lambda}=\int_0^{\infty}[1-\exp(-t^{\chi}/v^{\chi})]\,dt/t^{1+\lambda}$$
$$=\Gamma(1-\lambda/\chi)[\lambda v^{\lambda}]^{-1}.$$

By Fubini's theorem, the left-hand side is

$$\int_0^{\infty}t^{-1-\lambda}dt\int_0^{\infty}[1-\Omega(xt)]\,M(dx)=\int_0^{\infty}x^{\lambda}M(dx)\int_0^{\infty}[1-\Omega(u)]\,du/u^{1+\lambda}$$

for $0<\lambda<\chi$; the result for $0<\operatorname{Re}s<\chi$ follows by analytic continuation. When (1) holds, Kłosowska ([10], Lemma 1) shows that

$$[1-\Omega(x)]/x^{\chi}\to A>0\quad (x\to 0),$$
$$1/A=\int_0^{\infty}x^{\chi}M(dx).$$

Then $\Gamma(1-s/\chi)$ has a simple pole of residue 1 at $s=\chi$, and the integral

$$\int_0^\infty [1-\Omega(t)]\,dt/t^{1+s}$$

has a simple pole at $s=\chi$ of residue $-A$. Thus the right-hand side of (8) is regular at $s=\chi$ (with an appropriate choice of branch of v^s, unless we fix $v = 1$ by choice of scale). Then $\int_0^\infty x^s M(dx)$ is regular at $s=\chi$ (and, in particular, is defined in some open strip containing $s=\chi$).

Proof of the Theorem. Define $P^* \in \mathscr{P}$ by

$$P^*([a, b]) = P([a^{1/\chi}, b^{1/\chi}]).$$

By Fubini's theorem, we have

$$\begin{aligned}\int_0^\infty \Phi(P, vts^{1/\chi})\,M(dt) &= \int_0^\infty M(dt) \int_0^\infty \Omega(vts^{1/\chi}x)\,P(dx)\\ &= \int_0^\infty P(dx) \int_0^\infty \Omega(vts^{1/\chi}x)\,M(dx)\\ &= \int_0^\infty \exp(-sx^\chi)\,P(dx) = \int_0^\infty \exp(-sx)\,P^*(dx),\end{aligned}$$

the Laplace-Stieltjes transform $p^*(s)$ of P^*. So

$$\int_0^\infty [1-\Phi(P, vts^{1/\chi})]\,M(dt) = 1-p^*(s) \tag{9}$$

(cf. [14], proof of Theorem 8). That $P((x, \infty)) \sim L(x)/x^\lambda$ and

$$1-\Phi(P, t) = \int_0^\infty [1-\Omega(xt)]\,P(dx)$$

imply

$$1-\Phi(P, t) \sim \Big\{\lambda \int_0^\infty [1-\Omega(u)]\,du/u^{1+\lambda}\Big\}\, t^\lambda L(1/t) \qquad (t \to 0),$$

i.e. that (4) implies (3) in the case $0<\lambda<\chi$, follows by Abelian results for Mellin-Stieltjes convolutions as in [5], Section 2. In the reverse direction, Abelian arguments as in Section 1 of [5] and [1] show that (3) for $0<\lambda<\chi$ implies

$$\begin{aligned}\int_0^\infty [1-\Phi(P, vxt^{1/\chi})]\,M(dx) &\sim [1-\Phi(P, t^{1/\chi})]\,v^\lambda \int_0^\infty x^\lambda M(dx)\\ &\sim ct^{\lambda/\chi} L(1/t^{1/\chi})\,v^\lambda \int_0^\infty x^\lambda M(dx)\end{aligned}$$

as $t \to 0$, and similarly for $\lambda = \chi$ when (1) holds. Then, by (9),

$$(10) \qquad 1-p^*(t) \sim ct^{\lambda/\chi} L(1/t^{1/\chi}) v^\lambda \int_0^\infty x^\lambda M(dx) \quad (t \to 0).$$

If $0 < \lambda < \chi$, then Karamata's Tauberian theorem for Laplace-Stieltjes transforms ([17], XIII, Sections 5 and 6) shows that (10) is equivalent to

$$(11) \qquad 1-P^*(x) \sim cx^{-\lambda/\chi} L(x^{1/\chi}) v^\lambda \int_0^\infty x^\lambda M(dx)/\Gamma(1-\lambda/\chi),$$

which is (4) by the Lemma and the definition of P^*. If $\lambda = \chi$, then (10) is equivalent to

$$\int_0^x uP^*(du) \sim L(x) \quad (x \to \infty)$$

([4], Theorem A; $n = 0$, $\alpha = \beta = 1$), i.e. to (5) by the definition of P^*.

In the case $0 < \lambda < \chi$ an alternative procedure is available using the Wiener Tauberian theory. The lemma shows that $\int_0^\chi x^s M(dx)$ converges for $0 < \mathrm{Re}\, s < \chi$; this and the non-vanishing of $\Gamma(1-s/\chi)$ prove that the Wiener condition

$$\int_0^\chi [1-\Omega(t)]\, dt/t^{1+s} \neq 0 \quad (0 < \mathrm{Re}\, s < \chi)$$

holds. If (4) holds (we may take $1-P(x) = L(x)/x^\chi$), then

$$1-\Phi(P, 1/t) = -\int_0^\infty [1-\Omega(x/t)]\, d(1-P(x))$$
$$= -\int_0^\infty [1-\Omega(x/t)]\, d\{L(x)/x^\lambda\}.$$

Abelian results as in [5], Section 2, give

$$1-\Phi(P, 1/t) \sim t^{-\lambda} L(t) \lambda \int_0^\infty [1-\Omega(u)] u^{-\lambda} du/u \quad (t \to \infty)$$

proving (3); the converse holds by Tauberian results as in [5], Section 5.

Proof of the Corollary. By the above, P lies in a domain of attraction if and only if (10) holds for some slowly varying L and $\lambda \in (0, \chi]$. If $0 < \lambda < \chi$, then (10) gives (11), whence (6). For $\lambda = \chi$, recall ([7], VIII, (9.16)) that if

$$V(x) = 1-P^*(x), \qquad U(x) = \int_0^x uP^*(du),$$

then

$$\frac{xV(x)}{U(x)} = -1+\frac{x}{y^2\,U(x)}\int_x^\infty U(y)\,dy.$$

As above, (10) with $\lambda=\chi$ is $U(x)\sim L(x)$. By Karamata's theorem ([7], VIII. 9, Theorem 1, and [13], Theorem 2.1) this holds for some slowly varying L if and only if

$$\frac{x}{t^2\,U(x)}\int_x^\infty U(y)\,dy \to 1 \qquad (x\to\infty),$$

i.e. if and only if $xV(x)/U(x)\to 0$ $(x\to\infty)$, which is (7).

3. Examples.

1. Ordinary convolution. Here the convolution is given by

$$\int_0^\infty f(x)(P\circ Q)(dx) = \int_0^\infty\int_0^\infty f(x+y)\,P(dx)\,Q(dy).$$

Since $\delta_1^n=\delta_n$, the characteristic measure M is δ_1; the characteristic exponent is $\chi=1$, and the kernel is $\Omega(x)=e^{-x}$. The operation arises in the addition of independent non-negative random variables. The Tauberian theorem (due to Karamata, above) gives the equivalence of

$$1-P(x)\sim L(x)/x^\lambda \qquad (x\to\infty,\ 0<\lambda<1)$$

and

$$1-\int_0^\infty e^{-tx}P(dx)\sim \Gamma(1-\lambda)\,t^\lambda L(1/t) \qquad (t\to 0).$$

We can pass from $\exp(-x^\chi)$ to e^{-x} by a change of variable, and the equality

$$\Phi(M,\,vt)=\int_0^\infty \Omega(vtx)\,M(dx)=\exp(-t^\chi) \tag{12}$$

shows that $\exp(-t^\chi)$ lies in the closed convex hull of the functions $\Omega(ct)$, $c>0$, where Ω is the kernel of an arbitrary generalised convolution. We summarise this by saying that ordinary convolution is *subordinate* to an arbitrary (regular) generalised convolution.

2. Cosine transforms. Here the convolution is given by

$$\int_0^\infty f(x)(P\circ Q)(dx)=\tfrac{1}{2}\int_0^\infty\int_0^\infty [f(x+y)+f(|x-y|)]\,P(dx)\,Q(dy).$$

That the characteristic measure M is the truncated normal law

$$M(dx)=\sqrt{2/\pi}\,\exp(-\tfrac{1}{2}x^2)\,dx$$

expresses the central limit theorem for the law $\frac{1}{2}(\delta_1+\delta_{-1})$. The operation arises in the addition of independent symmetric random variables. Here $\chi = 2$, $\Omega(x) = \cos x$, and the Mellin transform is given by

$$\frac{s}{x^{1+s}}\int_0^\infty (1-\cos x)\,dx = \frac{1}{x^s}\int_0^\infty \sin x\,dx = \frac{\pi^{1/2}\Gamma(1-\frac{1}{2}s)}{2^s\Gamma(\frac{1}{2}+\frac{1}{2}s)}.$$

The Tauberian theorem giving the equivalence of

$$1-P(x) \sim L(x)/x^\lambda \quad (x\to\infty,\ 0<\lambda<2),$$

$$1-\int_0^\infty \cos tx P(dx) \sim \frac{\pi^{1/2}\Gamma(1-\frac{1}{2}\lambda)}{2^\lambda\Gamma(\frac{1}{2}+\frac{1}{2}\lambda)}t^\lambda L\left(\frac{1}{t}\right) \quad (t\to 0)$$

is due to Pitman [12].

3. Hankel transforms. The convolution given for $\nu > -\frac{1}{2}$ by

$$\int_0^\infty f(x)(P\circ Q)(dx)$$

$$= \frac{\Gamma(\nu+1)}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_0^\infty\int_0^\infty\int_{-1}^1 f\left((x^2+2uxy+y^2)^{1/2}\right)(1-u^2)^{\nu-1/2}P(dx)Q(dy)$$

was introduced by Kingman [9]. It arises in the addition of independent spherically symmetric random vectors (in Euclidean n-space with $n = 2(\nu+1)$ when this is an integer). The characteristic measure M is a Rayleigh law

$$M(dx) = 2(\nu+1)^{\nu+1}x^{2\nu+1}\exp\left(-(\nu+1)x^2\right)dx/\Gamma(\nu+1)$$

and arises in Kingman's central limit theorem for random walks with spherical symmetry [9]. We have $\chi = 2$ and

$$\Omega(x) = \Lambda_\nu(x) = \Gamma(\nu+1)J_\nu(x)/(\tfrac{1}{2}x)^\nu;$$

the Mellin transform is given by

$$s\int_0^\infty [1-\Lambda_\nu(x)]\,dx/x^{1+s} = \Gamma(1+\nu)\Gamma(1-\tfrac{1}{2}s)/[2^s\Gamma(1+\nu+\tfrac{1}{2}s)]$$

([15], p. 45 and 391). The limiting case $\nu = -\frac{1}{2}$ reduces to the cosine transform above: $\Lambda_\nu(x)\to\cos x$ as $\nu\to-\frac{1}{2}$ (we write $\Lambda_{-1/2}(x) = \cos x$). Then (12) reduces to

$$\int_0^\infty J_\nu(2tx)x^{\nu+1}e^{-x^2}dx = \tfrac{1}{2}t^\nu e^{-t^2}$$

([15], p. 394). The Tauberian theorem here was given by the author [3].

By Sonine's first finite integral for the Bessel function ([15], p. 373), for $\mu > \nu \geqslant -\frac{1}{2}$ we have

$$\Lambda_\mu(x) = \frac{2\Gamma(\mu+1)}{\Gamma(\nu+1)\Gamma(\mu-\nu)} \int_0^1 u^{2\nu+1}(1-u^2)^{\mu-\nu-1} \Lambda_\nu(ux)\,du,$$

and so the Kingman convolution with parameter $\mu > \nu$ is subordinate to that with parameter ν (in particular, ordinary convolution is subordinate to each Kingman convolution, which is subordinate to cosine convolution).

4. Kucharczak-Urbanik convolution. For $n = 1, 2, \ldots$ the function $\Omega(x) = (1-x)_+^n$ is the kernel of a generalised convolution [11]; here $\chi = 1$, $M(dx) = x^{-n-2}\exp(-1/x)\,dx/n!$, and

$$s\int_0^\infty [1-\Omega(x)]\,dx/x^{1+s} = n!\;\Gamma(1-s)/\Gamma(n+1-s).$$

The Tauberian theorem here is treated in [5], Section 6.4.

The case $n = 1$ is relevant to work of Kendall ([8], p. 371) on stationary random closed sets; here the convolution is given by

$$\int_0^\infty f(x)(P\circ Q)(dx) = \int_0^\infty\int_0^\infty [1-\min(u/v, v/u)]\, f(\max(u, v))\, P(du)\,Q(dv) +$$

$$+2\int_0^\infty\int_0^\infty uv \int_{\max(u,v)}^{\infty} x^{-3} f(x)\,dx P(du)\,Q(dv).$$

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