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A REMARK ON A MARCINKIEWICZ–HÖRMANDER MULTIPLIER THEOREM FOR SOME NON-DIFFERENTIAL CONVOLUTION OPERATORS

BY

JACEK DZIUBAŃSKI (WROCŁAW)

Introduction. Let A be a positive self-adjoint operator on $L^2(\mathfrak{M})$ and let

$$Af = \int_0^\infty \lambda dE(\lambda)f$$

be its spectral resolution. Following a general idea of Stein [8], we say that the *Marcinkiewicz–Hörmander multiplier theorem holds* if for a number α and $m \in C^\alpha(\boldsymbol{R}^+)$

$$\sup \lambda^j |m^{(j)}(\lambda)| < \infty$$

for $j = 0, 1, \ldots, \alpha$ implies that

$$m(A) = \int_0^\infty m(\lambda) dE(\lambda)$$

is of weak type (1, 1).

The Marcinkiewicz–Hörmander multiplier theorem was proved by A. Hulanicki and E. M. Stein for A being a sublaplacian on a stratified group (cf. [2], pp. 208–215).

P. Głowacki studied a homogeneous convolution operator P on a general homogeneous group, which proved to be useful especially in the cases where there is no hypoelliptic homogeneous differential operator (cf. [4]).

E. M. Stein asked whether the Marcinkiewicz–Hörmander multiplier theorem holds for P. The answer to this question is in the affirmative, as is shown in the present paper.

It is natural to ask about the lower bound for the number α of derivatives required for the Marcinkiewicz–Hörmander multiplier theorem to hold. In the case of sublaplacian, Christ showed (cf. [1]) that as in the classical Marcinkiewicz theorem the bound is $Q/2$, where Q is the homogeneous dimension of the group under consideration. Recent results by W. Hebisch combined with the methods we present here yield the same estimate for the number α of required derivatives.

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Preliminaries. A *family of dilations* on a nilpotent Lie algebra $\mathcal{N}$ is a one-parameter group $\{\delta_t\}_{t>0}$ of automorphisms of $\mathcal{N}$ determined by

$$\delta_t e_j = t^{d_j} e_j,$$

where $e_1,\dots,e_n$ is a linear basis for $\mathcal{N}$, and $d_1,\dots,d_n$ are positive real numbers called *exponents of homogeneity*. The smallest d_j is assumed to be 1.

If we regard $\mathcal{N}$ as a Lie group with multiplication given by the Campbell–Hausdorff formula, then the nilpotent Lie group $\mathcal{N}$ is said to be a *homogeneous group*. The *homogeneous dimension* of $\mathcal{N}$ is the number Q defined by

$$d(\delta_t x) = t^Q dx, \qquad t > 0,$$

where dx is a right-invariant Haar measure on $\mathcal{N}$.

A distribution T on $\mathcal{N}$ which is regular, i.e., smooth away from the origin and satisfies

$$\langle T, f\circ\delta_t\rangle = t^r \langle T, f\rangle, \qquad f\in C_c^\infty(\mathcal{N}),\ t>0,$$

for some $r\in \mathbf{R}$, is called a *kernel of order* r.

We choose and fix a *homogeneous norm* on $\mathcal{N}$, that is, a continuous, positive, and symmetric function $x\to|x|$ which is, moreover, smooth on $\mathcal{N}-\{0\}$, homogeneous of degree 1, and which vanishes only for $x=0$.

Let

$$X_j f(x) = \frac{d}{dt}\bigg|_{t=0} f(x\cdot te_j), \qquad j=1,\dots,n,$$

be left-invariant basic vector fields. If $I=(i_1,\dots,i_n)$ is a multi-index, we set

$$X^I f = X_1^{i_1}\dots X_n^{i_n} f, \qquad |I| = i_1 d_1 + \dots + i_n d_n.$$

Let us denote by $S^\infty(\mathcal{N})$ the space of functions f such that

$$f,\ X^I f\in L^2(\mathcal{N})$$

for every multi-index I.

Let $\psi\in C_c^\infty(B_1)$ be positive and equal to 1 in a neighbourhood of the origin, where $B_\varepsilon = \{x\colon |x|<\varepsilon\}$. If T is a kernel of order $r>0$, then for every ε $(0<\varepsilon\leqslant 1)$ T decomposes as

$$T = T_\varepsilon + k_\varepsilon,$$

where $T = (\psi\circ\delta_{1/\varepsilon})\cdot T$ is supported in the ball $B_\varepsilon\subseteq B_1$, and k_ε is a smooth function on $\mathcal{N}$ (cf. [4]). Let us put

$$\check{k}_\varepsilon(x) = k_\varepsilon(x^{-1}).$$

1. The semi-group generated by P^N. On a homogeneous group $\mathcal{N}$ let us consider Głowacki's distribution P defined by

$$\langle P, f\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{f(0)-f(x)}{|x|^{Q+1}}\Omega(x)dx, \tag{1.1}$$

where $\Omega \not\equiv 0$ is a function homogeneous of degree 0, positive, symmetric, and smooth on $\mathcal{N}-\{0\}$. It is well known that P is a generator of a semi-group of symmetric measures $\{\mu_t\}$ on $\mathcal{N}$. It has been shown by Głowacki [4] that this continuous semi-group of measures has the following properties:

$$d\mu_t(x) = h_t(x)dx, \qquad \text{where } h_t \in S^\infty(\mathcal{N}); \tag{1.2}$$

$$|X^I h_1(x)| \leqslant C_I(1+|x|)^{-Q-1-|I|}; \tag{1.3}$$

$$|\partial_t^j X^I h_t(x)| \leqslant C_{I,j}(t+|x|)^{-Q-|I|-j}, \quad j = 1, 2, \ldots \tag{1.4}$$

Głowacki also proves that for every kernel T of order $r>0$ and every natural N, $N \geqslant r$, there is a constant $C>0$ such that

$$\|Tf\| \leqslant C(\|P^N f\| + \|f\|) \qquad \text{for } f\in C_c^\infty(\mathcal{N}), \tag{1.5}$$

where $Tf(x) = \langle T, \lambda_x f\rangle$, $\lambda_x f(y) = f(xy)$, and $\|\cdot\|$ is the L^2-norm on $\mathcal{N}$. Moreover, P is essentially self-adjoint and, for every natural N, $S^\infty(\mathcal{N})$ is a core for $(\bar{P})^N$ (cf. [4], Proposition (4.13)). Therefore, for each natural N we can investigate the semi-group $\{T_t\}_{t>0}$ of operators on $L^2(\mathcal{N})$ generated by $R = P^N$. By the spectral theorem, this semi-group can be written as

$$T_t f = \int_0^\infty \exp(-\lambda^N t)dE(\lambda)f, \tag{1.6}$$

where E is the spectral resolution for P.

By the spectral theorem and (1.5), we have

$$\begin{aligned}\|X^I T_t f\| &\leqslant C(\|P^{Nk}T_t f\| + \|T_t f\|)\\ &\leqslant C\left(\left\|\int_0^\infty \lambda^{Nk}\exp(-\lambda^N t)dE(\lambda)f\right\| + \|T_t f\|\right)\\ &\leqslant C_t\|f\|.\end{aligned} \tag{1.7}$$

The Sobolev estimates and (1.7) give us

$$|T_t f(0)| \leqslant C_t\|f\|. \tag{1.8}$$

By (1.8), we obtain

$$T_t f(x) = f*q_t(x), \qquad \text{where } q_t\in L^2(\mathcal{N}). \tag{1.9}$$

Formulas (1.7) and (1.9) imply .

$$q_t = T_{t/2}q_{t/2}\in S^\infty(\mathcal{N}). \tag{1.10}$$

Note also that

(1.11) $R = P^N$ is a kernel of order N.

Due to the homogeneity of R we have

$$q_t(x) = t^{-Q/N} q_1(\delta_{t^{-1/N}} x). \tag{1.12}$$

The following estimate, which is our main goal in this paper, is an essential step for proving multiplier theorems for P (see Section 5 below).

(1.13) THEOREM. *For every $N > Q$ there exists a constant C_N such that*

$$|q_1(x)| \leqslant C_N(1+|x|)^{-Q-N}.$$

2. Sobolev spaces. In this section we introduce Sobolev spaces associated with the operator P and recall some inequalities we shall need later (cf. [3] and [4] for details).

For $m \in \boldsymbol{N}$ we denote by S^m the completion of $S^\infty(\mathcal{N})$ with respect to the norm

$$\|f\|_{(m)} = \|(I+P)^m f\|.$$

One can prove that $f \in S^m(\mathcal{N})$ for $m \in \boldsymbol{N}$ if and only if

$$f \in L^2(\mathcal{N}) \quad \text{and} \quad (I+P)^m f \in L^2(\mathcal{N})$$

in the weak sense.

For $m \in \boldsymbol{N}$ let $S^{-m}(\mathcal{N})$ be the dual space to $S^m(\mathcal{N})$. It can be seen that, for every integer m, the operator $I+P$ is an isometrical isomorphism from S^{m+1} onto S^m, and

$$(S^m)^* = S^{-m}.$$

For $M = 0, 1, \ldots$ and an integer m we denote by $S^{M,m}(\mathcal{N} \times \boldsymbol{R})$ the completion of $S^\infty(\mathcal{N} \times \boldsymbol{R})$ with respect to the norm

$$\|u\|^2_{(M,m)} = \int_{-\infty}^{\infty} \|(I-\partial_t)^M u(\cdot, t)\|^2_{(m)} dt.$$

(2.1) PROPOSITION. *For every integer m there is a constant C such that*

$$\|u\|^2_{(1,m)} + \|u\|^2_{(0,m+N)} \leqslant C(\|(R+\partial_t)u\|^2_{(0,m)} + \|u\|^2_{(0,m)})$$

for $u \in S^\infty(\mathcal{N} \times \boldsymbol{R})$.

(2.2) LEMMA. *For every kernel T of order $r \in \boldsymbol{N}$ and every integer m there is a constant C such that*

$$\|Tf\|_{(m)} \leqslant C\|f\|_{(r+m)} \quad \textit{for } f \in S^\infty(\mathcal{N}).$$

Remark. The proof of this lemma presented in [4] is incomplete in the case where $0 < m < r$. It can be, however, completed, as was communicated to us by P. Głowacki. We will not go into details here.

(2.3) LEMMA. *For every* $\varphi \in C_c^\infty(\mathcal{N})$ *and every integer* m *there is a constant* C *which depends only on* $\|\varphi\|_{C^r(\mathcal{N})}$, *where* $r = r(R, m)$ *is such that*

$$\|[R, M_\varphi]f\|_{(m)} \leqslant C\|f\|_{(m+N-1)}$$

for $f \in S^\infty(\mathcal{N})$, *where* $(M_\varphi f)(x) = \varphi(x)f(x)$.

By Lemma (2.3) and Proposition (2.1), we obtain

(2.4) PROPOSITION. *For every* $\varphi, \bar{\varphi} \in C_c^\infty(\mathcal{N} \times \boldsymbol{R})$ *such that* $\bar{\varphi} \equiv 1$ *on the support of* φ *and for every* $\tilde{\varphi} \in C^\infty(\mathcal{N} \times \boldsymbol{R})$ *such that* $\tilde{\varphi}(x, t) = \tilde{\varphi}_0(t)$, $\tilde{\varphi}_0 \in C_c^\infty(\boldsymbol{R})$, $\tilde{\varphi} = 1$ *on the support of* $\bar{\varphi}$ *there are constants* C, $\varepsilon > 0$ *such that*

$$\|\varphi u\|_{(1,m)} + \|\varphi u\|_{(0,m+N)} \leqslant C(\|\bar{\varphi}u\|_{(0,m)} + \|\varphi(R+\partial_t)u\|_{(0,m)}$$
$$+ \|\varphi(\tilde{\varphi}u)*\check{k}_\varepsilon\|_{(0,m)} + \|\bar{\varphi}u\|_{(0,m+N-1)}),$$

where $R = R_\varepsilon + k_\varepsilon$.

3. Fundamental solution for $\partial_t + R$. Recall that $R = P^N$, where P is defined by (1.1), is the generator of the convolution semi-group $f \to f * q_t$ with $q_t \in S^\infty(\mathcal{N})$.

Let

$$\langle H, u\rangle = \int_0^\infty \langle q_t, u(\cdot, t)\rangle dt, \tag{3.1}$$

where $u \in C_c^\infty(\mathcal{N} \times \boldsymbol{R})$. It is not hard to see that H is a homogeneous distribution on $\mathcal{N} \times \boldsymbol{R}$, where $\delta_r(x, t) = (\delta_r x, r^N t)$, and the degree of homogeneity of H is $-Q$.

(3.2) Remark. In addition, if N is sufficiently large, H is square-integrable in every strip $\mathcal{N} \times (-k, k)$, where $k > 0$. In fact, it is sufficient to observe that, by (1.12),

$$\|q_t\| = \|q_1\| t^{-Q/2N}, \qquad t > 0.$$

It is also easy to check (cf., e.g., [2], Proposition (1.68)) that H is the fundamental solution for $\partial_t + R$, i.e.,

$$(\partial_t + R)H = \delta. \tag{3.3}$$

4. Proof of Theorem (1.13). The proof goes along the example of [4]. Let φ, $\bar{\varphi}$ be smooth functions on $\mathcal{N} \times \boldsymbol{R}$ with compact support contained in $\mathcal{N} \times \boldsymbol{R} - \{(0, 0)\}$ and such that $\bar{\varphi} = 1$ on the support of φ. Let $\tilde{\varphi}$ be a smooth function on $\mathcal{N} \times \boldsymbol{R}$ such that $\tilde{\varphi}(x, t) = \tilde{\varphi}_0(t)$, $\tilde{\varphi}_0 \in C_c^\infty(\boldsymbol{R})$, $\tilde{\varphi} = 1$ on the support of $\bar{\varphi}$. In virtue of (3.3) and by the choice of φ, we get

$$(R + \partial_t)H = 0. \tag{4.1}$$

Now, by iterating Proposition (2.4) (since convolution with $\check{k}_\varepsilon$ is a smoothing operator, cf. [4]) and applying (3.2), (4.1), (1.5), and the Sobolev inequality, we obtain

$$\varphi H \in S^{1,\infty}(\mathcal{N} \times \boldsymbol{R}) = \bigcap_m S^{1,m}(\mathcal{N} \times \boldsymbol{R}), \tag{4.2}$$

which implies that, for every I,

$$X^I H \in L^2_{\mathrm{loc}}(U), \tag{4.3}$$

$$\partial_t X^I H \in L^2_{\mathrm{loc}}(U), \tag{4.4}$$

where $U = \mathcal{N} \times \boldsymbol{R} - \{(0, 0)\}$.

As a consequence of (4.3), (4.4), and again the Sobolev inequality, we have

$$X^I H \in C(U). \tag{4.5}$$

By homogeneity of $X^I H$, (1.10), and (4.5), we get

$$|X^I H(x, t)| \leqslant C_I(t+|x|)^{-Q-|I|} \quad \text{for } x \in \mathcal{N},\ t > 0. \tag{4.6}$$

Similarly as in [4], by the fact that $f \to q_t * f$ is a family of uniformly bounded operators on $L^2(\mathcal{N})$ with respect to $t \in (0, s)$, $s > 0$, we can prove that, for every I and every natural m,

$$\sup_{t>0} \|\varphi X^I R^m q_t\| < \infty. \tag{4.7}$$

Since ∂_t and X^I commute, $Rq_t = -\partial_t q_t$ for $t > 0$, by (3.1), (4.5), (4.7) we get

$$|q_t(x)| \leqslant Ct$$

in $\{(x, t) \in \mathcal{N} \times \boldsymbol{R}\colon 1 < |x| < 2,\ t \in (0, 1]\}$. Now the argument of Folland and Stein (cf. [2], Proposition (8.11)) establishes the theorem.

5. Final remarks.

(5.1) For a natural N let us denote by $E_N(\omega)$ the spectral resolution for $R = P^N$. The resolutions E_N and $E = E_1$ are related by

$$E(\omega) = E_N(\omega^N) \quad \text{for Borel } \omega \subset \boldsymbol{R}^+$$

and, consequently,

$$\int_0^\infty m(\lambda)dE(\lambda)f = \int_0^\infty m(\lambda^{1/N})dE_N(\lambda)f$$

for $m \in L^\infty(\boldsymbol{R}^+)$ and $f \in L^2(\mathcal{N})$.

It is now easy to see that if the Marcinkiewicz–Hörmander multiplier theorem holds for P^N with the bound α for the number of derivatives, then it holds for P with the same bound.

(5.2) Now, if we take N sufficiently large and use Theorem (1.13), the estimates by Hulanicki [7], the method of Hulanicki and Stein ([2], pp. 208–215), and Remark (5.1), we get the Marcinkiewicz–Hörmander multiplier theorem for P, but the number of required derivatives α is pretty large.

(5.3) In his recent paper, Hebisch [5] gives precise estimates for the number α of derivatives required in a very general theorem of Marcinkiewicz–Hörmander type, which applied to $R = P^N$ yield

$$\alpha_N > (Q/2 + Q/2N)\cdot 2^{[Q/2N]} + \varepsilon_N,$$

where $\varepsilon_N \leqslant 1/2$, and $[Q/2N]$ is the integer part of $Q/2N$.

(5.4) Note that in the case where $N = 1$, i.e., $R = P$, we get

$$\alpha_1 > Q \cdot 2^{[Q/2]} + 1/2.$$

(5.5) It is also proved by Hebisch [5] that $\varepsilon_N \to 0$ when N tends to infinity. Therefore, by letting $N \to \infty$ and by Theorem (1.13), Remarks (5.1) and (5.3), we have the Marcinkiewicz–Hörmander multiplier theorem for P with the critical number of derivatives $\alpha > Q/2$.

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, WROCŁAW

INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW

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