A REMARK
ON A MARCINKIEWICZ–HÖRMANDER MULTIPLIER THEOREM
FOR SOME NON-DIFFERENTIAL CONVOLUTION OPERATORS

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Introduction. Let $A$ be a positive self-adjoint operator on $L^2(\mathfrak{M})$ and let

$$Af = \int_0^\infty \lambda dE(\lambda)f$$

be its spectral resolution. Following a general idea of Stein [8], we say that the
Marcinkiewicz–Hörmander multiplier theorem holds if for a number $\alpha$ and
$m \in C^\alpha(R^+)$

$$\sup \lambda^j|m^{(j)}(\lambda)| < \infty$$

for $j = 0, 1, \ldots, \alpha$ implies that

$$m(A) = \int_0^\infty m(\lambda)dE(\lambda)$$

is of weak type $(1, 1)$.

The Marcinkiewicz–Hörmander multiplier theorem was proved by
A. Hulanicki and E. M. Stein for $A$ being a sublaplacian on a stratified group

P. Głowacki studied a homogeneous convolution operator $P$ on a general
homogeneous group, which proved to be useful especially in the cases where
there is no hypoelliptic homogeneous differential operator (cf. [4]).

E. M. Stein asked whether the Marcinkiewicz–Hörmander multiplier
theorem holds for $P$. The answer to this question is in the affirmative, as is
shown in the present paper.

It is natural to ask about the lower bound for the number $\alpha$ of derivatives
required for the Marcinkiewicz–Hörmander multiplier theorem to hold. In the
case of sublaplacian, Christ showed (cf. [1]) that as in the classical
Marcinkiewicz theorem the bound is $Q/2$, where $Q$ is the homogeneous dimension
of the group under consideration. Recent results by W. Hebisch combined with
the methods we present here yield the same estimate for the number $\alpha$ of
required derivatives.
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Preliminaries. A family of dilations on a nilpotent Lie algebra \( \mathcal{N} \) is a one-parameter group \( \{ \delta_t \}_{t>0} \) of automorphisms of \( \mathcal{N} \) determined by

\[
\delta_t e_j = t^{d_j} e_j,
\]

where \( e_1, \ldots, e_n \) is a linear basis for \( \mathcal{N} \), and \( d_1, \ldots, d_n \) are positive real numbers called exponents of homogeneity. The smallest \( d_j \) is assumed to be 1.

If we regard \( \mathcal{N} \) as a Lie group with multiplication given by the Campbell–Hausdorff formula, then the nilpotent Lie group \( \mathcal{N} \) is said to be a homogeneous group. The homogeneous dimension of \( \mathcal{N} \) is the number \( Q \) defined by

\[
d(\delta_t x) = t^Q dx, \quad t > 0,
\]

where \( dx \) is a right-invariant Haar measure on \( \mathcal{N} \).

A distribution \( T \) on \( \mathcal{N} \) which is regular, i.e., smooth away from the origin and satisfies

\[
\langle T, f \circ \delta_t \rangle = t^r \langle T, f \rangle, \quad f \in C_c^\infty(\mathcal{N}), \ t > 0,
\]

for some \( r \in \mathbb{R} \), is called a kernel of order \( r \).

We choose and fix a homogeneous norm on \( \mathcal{N} \), that is, a continuous, positive, and symmetric function \( x \rightarrow |x| \) which is, moreover, smooth on \( \mathcal{N} \setminus \{0\} \), homogeneous of degree 1, and which vanishes only for \( x = 0 \).

Let

\[
X_j f(x) = \frac{d}{dt} \bigg|_{t=0} f(x \cdot te_j), \quad j = 1, \ldots, n,
\]

be left-invariant basic vector fields. If \( I = (i_1, \ldots, i_n) \) is a multi-index, we set

\[
X^I f = X_1^{i_1} \cdots X_n^{i_n} f, \quad |I| = i_1d_1 + \ldots + i_nd_n.
\]

Let us denote by \( S^\infty(\mathcal{N}) \) the space of functions \( f \) such that

\[
f, \ X^I f \in L^2(\mathcal{N})
\]

for every multi-index \( I \).

Let \( \psi \in C_c^\infty(B_1) \) be positive and equal to 1 in a neighbourhood of the origin, where \( B_\varepsilon = \{ x : |x| < \varepsilon \} \). If \( T \) is a kernel of order \( r > 0 \), then for every \( \varepsilon \) \((0 < \varepsilon \leq 1)\) \( T \) decomposes as

\[
T = T_\varepsilon + k_\varepsilon,
\]

where \( T = (\psi \circ \delta_{1/\varepsilon}) \cdot T \) is supported in the ball \( B_\varepsilon \subseteq B_1 \), and \( k_\varepsilon \) is a smooth function on \( \mathcal{N} \) (cf. [4]). Let us put

\[
k_\varepsilon(x) = k_\varepsilon(x^{-1}).
\]
1. The semi-group generated by $P^N$. On a homogeneous group $\mathcal{N}$ let us consider Głowacki's distribution $P$ defined by

\[
\langle P, f \rangle = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{f(0) - f(x)}{|x|^{q+1}} \Omega(x) dx,
\]

where $\Omega \neq 0$ is a function homogeneous of degree 0, positive, symmetric, and smooth on $\mathcal{N} - \{0\}$. It is well known that $P$ is a generator of a semi-group of symmetric measures $\{\mu_t\}$ on $\mathcal{N}$. It has been shown by Głowacki [4] that this continuous semi-group of measures has the following properties:

\[
d\mu_t(x) = h_t(x) dx, \quad \text{where} \quad h_t \in S^\infty(\mathcal{N});
\]

\[
|X^I h_t(x)| \leq C_t(1 + |x|)^{-q-1-|I|};
\]

\[
|\partial^j X^I h_t(x)| \leq C_{t,j}(t + |x|)^{-q-1-|I|-j}, \quad j = 1, 2, \ldots
\]

Głowacki also proves that for every kernel $T$ of order $r > 0$ and every natural $N, N \geq r$, there is a constant $C > 0$ such that

\[
\| T^r \| \leq C(\| P^N f \| + \| f \|) \quad \text{for} \quad f \in C^\infty_c(\mathcal{N}),
\]

where $Tf(x) = \langle T, \lambda_x f \rangle$, $\lambda_x f(y) = f(xy)$, and $\| \cdot \|$ is the $L^2$-norm on $\mathcal{N}$. Moreover, $P$ is essentially self-adjoint and, for every natural $N$, $S^\infty(\mathcal{N})$ is a core for $(P)^N$ (cf. [4], Proposition (4.13)). Therefore, for each natural $N$ we can investigate the semi-group $\{T_t\}_{t > 0}$ of operators on $L^2(\mathcal{N})$ generated by $R = P^N$. By the spectral theorem, this semi-group can be written as

\[
T_t f = \int_0^\infty \exp(-\lambda^N t) dE(\lambda)f,
\]

where $E$ is the spectral resolution for $P$.

By the spectral theorem and (1.5), we have

\[
\|X^I T_t f\| \leq C(\|P^N T_t f\| + \| T_t f\|)
\]

\[
\leq C \left( \| \int_0^\infty \lambda^{Nk} \exp(-\lambda^N t) dE(\lambda)f\| + \| T_t f\| \right)
\]

\[
\leq C_t \| f\|.
\]

The Sobolev estimates and (1.7) give us

\[
|T_t f(0)| \leq C_t \| f\|.
\]

By (1.8), we obtain

\[
T_t f(x) = f * q_t(x), \quad \text{where} \quad q_t \in L^2(\mathcal{N}).
\]

Formulas (1.7) and (1.9) imply:

\[
q_t = T_{t/2} q_{t/2} \in S^\infty(\mathcal{N}).
\]
Note also that

\[(1.11) \quad R = P^N \text{ is a kernel of order } N.\]

Due to the homogeneity of \(R\) we have

\[(1.12) \quad q_1(x) = t^{-Q/N} q_1(\delta_{1/N} x).\]

The following estimate, which is our main goal in this paper, is an essential step for proving multiplier theorems for \(P\) (see Section 5 below).

\[(1.13) \text{Theorem. For every } N > Q \text{ there exists a constant } C_N \text{ such that} \]

\[|q_1(x)| \leq C_N (1 + |x|)^{-Q+N}.\]

2. Sobolev spaces. In this section we introduce Sobolev spaces associated with the operator \(P\) and recall some inequalities we shall need later (cf. [3] and [4] for details).

For \(m \in \mathbb{N}\) we denote by \(S^m\) the completion of \(S^\infty(\mathcal{N})\) with respect to the norm

\[\|f\|_{(m)} = \|(I + P)^m f\|.\]

One can prove that \(f \in S^m(\mathcal{N})\) for \(m \in \mathbb{N}\) if and only if

\[f \in L^2(\mathcal{N}) \quad \text{and} \quad (I + P)^m f \in L^2(\mathcal{N})\]

in the weak sense.

For \(m \in \mathbb{N}\) let \(S^{-m}(\mathcal{N})\) be the dual space to \(S^m(\mathcal{N})\). It can be seen that, for every integer \(m\), the operator \(I + P\) is an isometrical isomorphism from \(S^{m+1}\) onto \(S^m\), and

\[(S^m)^* = S^{-m}.\]

For \(M = 0, 1, \ldots\) and an integer \(m\) we denote by \(S^{M,m}(\mathcal{N} \times \mathbb{R})\) the completion of \(S^\infty(\mathcal{N} \times \mathbb{R})\) with respect to the norm

\[\|u\|_{(M,m)}^2 = \int_{-\infty}^{\infty} \|(I - \partial_t)^M u(\cdot, t)\|_{(m)}^2 dt.\]

(2.1) Proposition. For every integer \(m\) there is a constant \(C\) such that

\[\|u\|_{(1,m)}^2 + \|u\|_{(0,m+N)}^2 \leq C (\|(R + \partial_t)u\|_{(0,m)}^2 + \|u\|_{(0,m)}^2)\]

for \(u \in S^\infty(\mathcal{N} \times \mathbb{R})\).

(2.2) Lemma. For every kernel \(T\) of order \(r \in \mathbb{N}\) and every integer \(m\) there is a constant \(C\) such that

\[\|Tf\|_{(m)} \leq C \|f\|_{(r+m)} \quad \text{for } f \in S^\infty(\mathcal{N}).\]

Remark. The proof of this lemma presented in [4] is incomplete in the case where \(0 < m < r\). It can be, however, completed, as was communicated to us by P. Glowacki. We will not go into details here.
(2.3) **Lemma.** For every \( \varphi \in C^\infty_c(\mathcal{N}) \) and every integer \( m \) there is a constant \( C \) which depends only on \( \| \varphi \|_{C^m(\mathcal{N}),} \) where \( r = r(R, m) \) is such that

\[
\| [R, M_\varphi] f \|_{(m)} \leq C \| f \|_{(m+N-1)}
\]

for \( f \in S^\infty(\mathcal{N}) \), where \( (M_\varphi f)(x) = \varphi(x)f(x) \).

By Lemma (2.3) and Proposition (2.1), we obtain

(2.4) **Proposition.** For every \( \varphi, \tilde{\varphi} \in C^\infty_c(\mathcal{N} \times \mathbb{R}) \) such that \( \tilde{\varphi} \equiv 1 \) on the support of \( \varphi \) and for every \( \tilde{\varphi} \in C^\infty_c(\mathcal{N} \times \mathbb{R}) \) such that \( \tilde{\varphi}(x, t) = \tilde{\varphi}_0(t), \tilde{\varphi}_0 \in C^\infty_c(\mathbb{R}) \), \( \tilde{\varphi} = 1 \) on the support of \( \tilde{\varphi} \) there are constants \( C, \varepsilon > 0 \) such that

\[
\| \varphi u \|_{(1,m)} + \| \varphi u \|_{(0,m+N)} \leq C (\| \tilde{\varphi} u \|_{(0,m)} + \| \varphi (R + \partial_t) u \|_{(0,m)} + \| \varphi (\tilde{\varphi} u) * k_\varepsilon \|_{(0,m)} + \| \varphi u \|_{(0,m+N-1)}),
\]

where \( R = R_\varepsilon + k_\varepsilon \).

3. **Fundamental solution for** \( \partial_t + R \). Recall that \( R = P^N \), where \( P \) is defined by (1.1), is the generator of the convolution semi-group \( f \to f * q_t \) with \( q_t \in S^\infty(\mathcal{N}) \).

Let

(3.1) \[
\langle H, u \rangle = \int_0^\infty \langle q_t, u(\cdot, t) \rangle dt,
\]

where \( u \in C^\infty_c(\mathcal{N} \times \mathbb{R}) \). It is not hard to see that \( H \) is a homogeneous distribution on \( \mathcal{N} \times \mathbb{R} \), where \( \delta_t(x, t) = (\delta(x, t^N), t) \), and the degree of homogeneity of \( H \) is \( -Q \).

(3.2) **Remark.** In addition, if \( N \) is sufficiently large, \( H \) is square-integrable in every strip \( \mathcal{N} \times (-k, k) \), where \( k > 0 \). In fact, it is sufficient to observe that, by (1.12),

\[
\| q_t \| = \| q_1 \| t^{-Q/2N}, \quad t > 0.
\]

It is also easy to check (cf., e.g., [2], Proposition (1.68)) that \( H \) is the fundamental solution for \( \partial_t + R \), i.e.,

(3.3) \[
(\partial_t + R) H = \delta.
\]

4. **Proof of Theorem (1.13).** The proof goes along the example of [4]. Let \( \varphi, \tilde{\varphi} \) be smooth functions on \( \mathcal{N} \times \mathbb{R} \) with compact support contained in \( \mathcal{N} \times \mathbb{R} \) \( - \{0, 0\} \) and such that \( \tilde{\varphi} = 1 \) on the support of \( \varphi \). Let \( \varphi \) be a smooth function on \( \mathcal{N} \times \mathbb{R} \) such that \( \varphi(x, t) = \tilde{\varphi}_0(t), \tilde{\varphi}_0 \in C^\infty_c(\mathbb{R}), \tilde{\varphi} = 1 \) on the support of \( \tilde{\varphi} \). In virtue of (3.3) and by the choice of \( \varphi \), we get

(4.1) \[
(R + \partial_t) H = 0.
\]

Now, by iterating Proposition (2.4) (since convolution with \( k_\varepsilon \) is a smoothing operator, cf. [4]) and applying (3.2), (4.1), (1.5), and the Sobolev inequality, we obtain

(4.2) \[
\varphi H \in S^{1,\infty}(\mathcal{N} \times \mathbb{R}) = \bigcap_0^{\infty} S^{1,m}(\mathcal{N} \times \mathbb{R}),
\]

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which implies that, for every $I$,

\begin{align}
(4.3) \quad X^I H & \in L^2_{\text{loc}}(U), \\
(4.4) \quad \partial_i X^I H & \in L^2_{\text{loc}}(U),
\end{align}

where $U = \mathcal{N} \times \mathbb{R} - \{(0, 0)\}$.

As a consequence of (4.3), (4.4), and again the Sobolev inequality, we have

\begin{equation}
(4.5) \quad X^I H \in C(U).
\end{equation}

By homogeneity of $X^I H$, (1.10), and (4.5), we get

\begin{equation}
(4.6) \quad |X^I H(x, t)| \leq C_t (t + |x|^{-Q-|I|}) \quad \text{for } x \in \mathcal{N}, \ t > 0.
\end{equation}

Similarly as in [4], by the fact that $f \to q_{s} * f$ is a family of uniformly bounded operators on $L^2(\mathcal{N})$ with respect to $t \in (0, s), s > 0$, we can prove that, for every $I$ and every natural $m$,

\begin{equation}
(4.7) \quad \sup_{t > 0} \|\varphi X^I R^m q_t\| < \infty.
\end{equation}

Since $\partial_i$ and $X^I$ commute, $Rq_t = -\partial_i q_t$ for $t > 0$, by (3.1), (4.5), (4.7) we get

\[|q_t(x)| \leq Ct\]

in $\{(x, t) \in \mathcal{N} \times \mathbb{R} : 1 < |x| < 2, t \in (0, 1]\}$. Now the argument of Folland and Stein (cf. [2], Proposition (8.11)) establishes the theorem.

5. Final remarks.

(5.1) For a natural $N$ let us denote by $E_N(\omega)$ the spectral resolution for $R = P^N$. The resolutions $E_N$ and $E = E_1$ are related by

\[E(\omega) = E_N(\omega^N) \quad \text{for Borel } \omega \subset \mathbb{R}^+\]

and, consequently,

\[\int_0^\infty m(\lambda) dE(\lambda)f = \int_0^\infty m(\lambda^{1/N}) dE_N(\lambda)f\]

for $m \in L^\infty(\mathbb{R}^+)$ and $f \in L^2(\mathcal{N})$.

It is now easy to see that if the Marcinkiewicz–Hörmander multiplier theorem holds for $P^N$ with the bound $\alpha$ for the number of derivatives, then it holds for $P$ with the same bound.

(5.2) Now, if we take $N$ sufficiently large and use Theorem (1.13), the estimates by Hulanicki [7], the method of Hulanicki and Stein ([2], pp. 208–215), and Remark (5.1), we get the Marcinkiewicz–Hörmander multiplier theorem for $P$, but the number of required derivatives $\alpha$ is pretty large.

(5.3) In his recent paper, Hebisch [5] gives precise estimates for the number $\alpha$ of derivatives required in a very general theorem of Marcinkiewicz–Hörmander type, which applied to $R = P^N$ yield

\[\alpha_N > (Q/2 + Q/2N) \cdot 2^{[Q/2N]} + \varepsilon_N,\]
where $\varepsilon_n \leq 1/2$, and $[Q/2N]$ is the integer part of $Q/2N$.

(5.4) Note that in the case where $N = 1$, i.e., $R = P$, we get

$$\alpha_1 > Q \cdot 2^{[Q/2]} + 1/2.$$  

(5.5) It is also proved by Hebisch [5] that $\varepsilon_n \to 0$ when $N$ tends to infinity. Therefore, by letting $N \to \infty$ and by Theorem (1.13), Remarks (5.1) and (5.3), we have the Marcinkiewicz–Hörmander multiplier theorem for $P$ with the critical number of derivatives $\alpha > Q/2$.

REFERENCES


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