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Inverse problems connected with periods of oscillations described by $\ddot{x}+g(x) = 0$

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Abstract. The problem of the existence and construction of a system of differential equations with prescribed properties is called the *inverse problem*. We are investigating the problem of the existence and construction of a function g such that the periodic solutions of the differential equation $\ddot{x}+g(x) = 0$, have prescribed half-periods and periods provided half-periods and periods are Lipschitz continuous function of half-amplitudes and amplitude, respectively.

1. Introduction. Let g be a real-valued function defined and continuous on the interval $[b_0, a_0]$, $-\infty < b_0 < 0 < a_0 < +\infty$. Assume g satisfies the following hypothesis

$$xg(x) > 0, \quad x \neq 0. \tag{1.1}$$

We define

$$G(x) = \int_0^x g(u)\,du. \tag{1.2}$$

In the theory of the differential equations of the form

$$\ddot{x}+g(x) = 0 \tag{1.3}$$

there appear the integrals

$$T_g^+(a) = \sqrt{2}\int_0^a \frac{dx}{\sqrt{G(a)-G(x)}} \tag{1.4}$$

and

$$T_g^-(b) = \sqrt{2}\int_b^0 \frac{dx}{\sqrt{G(b)-G(x)}}. \tag{1.5}$$

They determine the half-periods of the periodic solution $x = x(t)$, $\max\{x(t)\colon -\infty < t < +\infty\} = a$, $\min\{x(t)\colon -\infty < t < +\infty\} = b$, of equation (1.3).

The numbers $a > 0$, $b < 0$ are called the *positive* and *negative half-amplitudes* of $x = x(t)$, respectively.

If g is a continuous function satisfying (1.1), then there exists a family of periodic solutions of (1.3). Formulae (1.4), (1.5) define the positive half-period function $T_g^+ = T_g^+(a)$ and the negative half-period function $T_g^- = T_g^-(b)$, respectively (see [10], p. 244–246, [7]).

Assume $G(a_0) = G(b_0) < +\infty$. By (1.1), the function G is strictly increasing on the interval $[0, a_0]$ and strictly decreasing on the interval $[b_0, 0]$. Thus, for any value $a \in (0, a_0]$ there exists exactly one value $b = \varphi(a) \in [b_0, 0)$ such that $G(a) = G(b)$. The number

$$A = \tfrac{1}{2}[a - \varphi(a)] \tag{1.6}$$

is an amplitude of the periodic solution with half-amplitudes a and $\varphi(a)$. The number

$$T_g(A) = T_g^+(a) + T_g^-[\varphi(a)] \tag{1.7}$$

is a period of that solution. Equality (1.7) defines the period function $T_g = T_g(A)$, $A \in \left(0, \tfrac{1}{2}(a_0 - b_0)\right]$.

The present paper concerns the problem of the existence and construction of a function g with a prescribed half-period functions or period function. This is called the *inverse problem*. For constant half-period functions and a period function this problem has been called a *problem of isochronism* (see [4], p. 208–210, [8], [9], [10], p. 244–279) or the *problem of tautochronism* (see [1], vol. 1, p. 340–342, [4], [7]).

Opial showed in [7] that, given functions T^+, T^-, there exists at most one function g for which $T_g^+ = T^+$ and $T_g^- = T^-$. On the other hand, Urabe proved in [8], [9] and [10] that given positive functions T^+, T^- with Lipschitz continuous derivatives, there exists a unique continuous function $g = g(x)$ differentiable at the point $x = 0$, such that $T_g^+ = T^+$ and $T_g^- = T^-$; but for a given positive function T with Lipschitz continuous derivative there exist infinitely many continuous functions $g = g(x)$ differentiable at the point $x = 0$ such that $T_g = T$. A method of construction of continuous functions g with a prescribed constant period function has been given by Erhmann (see [2], compare [3]), without a full formalization.

Our aim is to obtain Urabe's results under weaker assumptions. We assume that the half-period functions and the period function are Lipschitz continuous (see Theorem 3.2, 4.1). A full formalization of Erhmann's considerations is given by Corollary 2.2.

In Section 2 we solve the inverse problem for the half-period functions associated with energy $\tilde{T}_g^+ = \tilde{T}_g^+(E)$, $\tilde{T}_g^- = \tilde{T}_g^-(E)$. We reduce this problem to the conditions of regularity of convolution of the half-period function with the function $k(E) = \sqrt{2/E}$. In Section 3, we solve the inverse problem for the half-period functions associated with half-amplitudes. We reduce this problem to the problem of the existence of sufficiently regular solution of a certain non-linear integral equation. Section 4 deals with the inverse problem for the period function associated with the amplitude.

2. Inverse problem for the half-period functions associated with energy. Problem of isochronism. In the subsequent discussion k will be a function defined for $E > 0$ by the formula

$$k(E) = \sqrt{2/E}. \tag{2.1}$$

Let $f*h$ denote the convolution of the summable functions f, h defined almost everywhere in the interval $[0, E_0]$, given by the formula

$$f*h(E) = \int_0^E f(E-t)h(t)\,dt,$$

for any value $E \in [0; E_0]$ for which the above integral exists.

As we have mentioned before, by (1.1), the function $G = G(x)$ is strictly increasing on the interval $[0, a_0]$ and strictly decreasing on the interval $[b_0, 0]$. Let $x = x(G)$ and $y = y(G)$ be the inverse functions of G on $[0, a_0]$ and $[b_0, 0]$, respectively. Let $G(a_0) = G(b_0) = E_0$. On the interval $(0, E_0]$ we define functions $\tilde{T}_g^+ = \tilde{T}_g^+(E)$, $\tilde{T}_g^- = \tilde{T}_g^-(E)$, $\tilde{T}_g = \tilde{T}_g(E)$ by the formulae

$$\tilde{T}_g^+(E) = T_g^+[x(E)], \tag{2.2}$$

$$\tilde{T}_g^-(E) = T_g^-[y(E)], \tag{2.3}$$

$$\tilde{T}_g(E) = \tilde{T}_g^+(E) + \tilde{T}_g^-(E). \tag{2.4}$$

If a, b are half-amplitudes of any periodic solution $x = x(t)$ of (1.3), then $G(a) = G(b)$ (see [10], p. 244). Set $G(a) = E$. By (1.4), (2.2) and (1.5), (2.3), it follows that

$$\tilde{T}_g^+(E) = \sqrt{2}\int_0^E \frac{dx(G)}{\sqrt{E-G}}, \tag{2.5}$$

$$\tilde{T}_g^-(E) = -\sqrt{2}\int_0^E \frac{dy(G)}{\sqrt{E-G}}. \tag{2.6}$$

Since $x = x(G)$ and $y = y(G)$ are absolutely continuous functions on the interval $[0, E_0]$, in the Lebesgue–Stieltjes integrals (2.5), (2.6) we have $dx(G) = x'(G)dG$, $dy(G) = y'(G)dG$ and we can rewrite (2.5), (2.6) in the form

$$\tilde{T}_g^+(E) = k*x'(E), \tag{2.7}$$

$$\tilde{T}_g^-(E) = -k*y'(E). \tag{2.8}$$

THEOREM 2.1. *Given positive functions $\tilde{T}^+$, $\tilde{T}^-$, summable over the interval $[0, E_0]$, let us define functions $x = x(E)$, $y = y(E)$ on the interval $[0, E_0]$ by formulae*

$$x(E) = \frac{1}{2\pi} k * \tilde{T}^+(E), \tag{2.9}$$

$$y(E) = -\frac{1}{2\pi} k * \tilde{T}^-(E). \tag{2.10}$$

(i) *If x is an absolutely continuous function with the following properties:*

(2.11) $x' = x'(E)$ *is continuous and positive on the interval* $(0, E_0]$,

$$\lim_{E \to 0^+} x'(E) = +\infty, \tag{2.12}$$

then there exists a unique function g continuous on the interval $[0, x(E_0)]$ and positive on the interval $(0, x(E_0)]$ such that $\tilde{T}_g^+ = \tilde{T}^+$. The function $\tilde{T}_g^+$ is continuous on the interval $(0, E_0]$.

(ii) *If y is an absolutely continuous function with the following properties:*

(2.13) $y' = y'(E)$ *is continuous and negative on the interval* $(0, E_0]$,

$$\lim_{E \to 0^+} y'(E) = -\infty, \tag{2.14}$$

then there exists a unique function g continuous on the interval $[y(E_0), 0]$ and negative on the interval $[y(E_0), 0)$ such that $\tilde{T}_g^- = \tilde{T}^-$. The function $\tilde{T}_g^-$ is continuous on the interval $(0, E_0]$.

Proof. Let x be a function satisfying our assumptions and let us note that by definition of the convolution $*$, condition (2.9) implies $x(0) = 0$. Hence we can define a function g on the interval $(0, x(E_0)]$ by the formula

$$x = x(E), \quad g(x) = \frac{1}{x'(E)}. \tag{2.15}$$

Set $g(0) = 0$. It is easy to see that g is continuous on $[0, x(E_0)]$ and positive on $(0, x(E_0)]$.

Now we show the equality $\tilde{T}_g^+ = \tilde{T}^+$ and the continuity of $\tilde{T}_g^+$. Let us consider the convolution with k of the both sides of (2.9). By the identity

$$k * k = 2\pi, \tag{2.16}$$

we obtain the equality

$$1 * \tilde{T}^+ = k * x.$$

Since $k * x$ is an absolutely continuous function, we can differentiate it (see [6], Theorem 6), and since $x(0) = 0$, we get

$$\tilde{T}^+ = k * x'. \tag{2.17}$$

Now the equality $\tilde{T}_g^+ = \tilde{T}^+$ follows from (2.7) and (2.17). Since k and x' are continuous on $(0, E_0]$, we see that (2.17) implies continuity of $\tilde{T}_g^+$ (see [6], Theorem 3).

It remains to show the uniqueness of the function g. Let g_1 be any function continuous on $[0, x(E_0)]$, positive on $(0, x(E_0)]$ such that

$$\tilde{T}_{g_1}^+(E) = \tilde{T}^+(E), \qquad E \in (0, E_0]. \tag{2.18}$$

Let G_1 be the function defined by (1.2) with $g = g_1$ and let x_1 be its inverse function. Since x_1 is absolutely continuous on $[0, E_1]$, $E_1 = G_1[x(E_0)]$, we have

$$\tilde{T}_{g_1}^+(E) = k * x_1'(E), \qquad E \in (0, E_1]. \tag{2.19}$$

By (2.18) $E_0 \leqslant E_1$. Now, by (2.16), equalities (2.17)–(2.19) imply

$$2\pi * x_1'(E) = 2\pi * x'(E), \qquad E \in (0, E_0].$$

Since $x_1(0) = 0 = x(0)$, this proves that $x_1(E) = x(E)$ for $E \in [0, E_0]$. Now, it is easy to see that $G_1(x) = G(x)$, $x \in [0, x(E_0)]$, whence $g_1(x) = g(x)$ on $[0, x(E_0)]$.

Similarly one can prove part (ii). The function g is defined on $[b_0, 0)$ by the formula

$$x = y(E), \qquad g(x) = \frac{1}{y'(E)}. \tag{2.20}$$

Theorem 2.1 is a base of our method of solving the inverse problem for the half-period functions associated with half-amplitudes. To illustrate some other applications of Theorem 2.1, we solve the well-known problem of isochronism (compare [8]–[10], [5] and [7]).

COROLLARY 2.1. *Given continuous function g satisfying* (1.1). *Functions* $T_g^+(a) = \text{const} = T_0$, $a \in (0, a_0]$ *and* $T_g^-(b) = \text{const} = T_1$, $b \in [b_0, 0)$ *if and only if*

$$g(x) = \begin{cases} \left(\dfrac{\pi}{T_0}\right)^2 x, & x \in [0, a_0], \\ -\left(\dfrac{\pi}{T_1}\right)^2 x, & x \in [b_0, 0). \end{cases} \tag{2.21}$$

Proof. If the function g is of the form (2.21), then the equalities $T_g^+(a) = T_0$ and $T_g^-(b) = T_1$ follow from (1.4), (1.5).

To prove the converse, let us note that $T_g^+(a) = T_0$ holds if and only if $\tilde{T}_g^+(E) = T_0$. Now, from (2.9) for $\tilde{T}^+(E) = T_0$, we obtain the equality

$$x(E) = \frac{\sqrt{2}\, T_0}{\pi} \sqrt{E}. \tag{2.22}$$

Now, (2.22) implies (2.21) for $x \in [0, a_0]$ by Theorem 2.1 and formula (2.15). Similarly, by (2.10) and (2.20), the constant function $T_g^- = T_g^-(b)$ determines g on $[b_0, 0)$.

COROLLARY 2.2. *Let g be a continuous function on the interval $[b_0, a_0]$. Assume* (1.1) *and that $G(a_0) = G(b_0) = E_0$. Let $x = x(E)$, $y = y(E)$ be the inverse functions of G on $[0, a_0]$ and $[b_0, 0]$, respectively. The period function $T_g(A) = \text{const} = T_0$ on the interval $(0, A_0]$, $A_0 = \frac{1}{2}(a_0 - b_0)$, if and only if*

$$x(E) = y(E) + \frac{\sqrt{2}\,T_0}{\pi}\sqrt{E} \tag{2.23}$$

on the interval $[0, E_0]$.

Proof. Equalities (2.4), (2.6), (2.7) imply

$$\tilde{T}_g(E) = k*(x'-y')(E). \tag{2.24}$$

Assume (2.23). Then

$$x'(E) - y'(E) = \frac{\sqrt{2}\,T_0}{2\pi\sqrt{E}} = \frac{T_0}{2\pi}k(E)$$

and by (2.16), condition (2.24) imply equality $\tilde{T}_g(E) = T_0$, $E \in (0, E_0]$, that holds if and only if $T_g(A) = T_0$ on $(0, A_0]$.

To prove the converse, assume that $T_g(A) = T_0$ on $(0, A_0]$. Then $\tilde{T}_g(E) = T_0$ on $(0, E_0]$. Set $\tilde{T}_g(E) = T_0$ in (2.24). By (2.16) and the equality $x(0) = 0 = y(0)$, the convolution with k of the both sides of (2.24) imply (2.23).

By Corollary 2.1, it is easy to see that there exist infinitely many continuous functions g such that $T_g(A) = \text{const} = T_0$. For example, for any number $s \in (0, 1)$ there exists a unique function g_s such that $T_g^+(a) = sT_0$ and $T_g^-(b) = (1-s)T_0$, whence by (1.7), $T_g(A) = T_0$.

3. Inverse problem for the half-period functions associated with half-amplitudes. In the subsequent discussion T will be a positive continuous function defined on the interval $[0, r]$. Let us consider the following integral equation

$$x = \frac{1}{2\pi}k*(T\circ x), \tag{3.1}$$

where k is defined by (2.1) and $T\circ x$ is a superposition of the given function T and an unknown function x.

Let g be a function continuous on the interval $[b_0, a_0]$ and satisfying (1.1). If we put $T = T_g^+$ and $r = a_0$, then there exists an absolutely continuous solution x of (3.1) with the properties (2.11), (2.12). The function x that satisfies (3.1) is the inverse function of G on $[0, a_0]$ (see (2.2) and (2.9)). The aim of Theorem 3.1 below is to prove the converse:

THEOREM 3.1. *Let T be a positive continuous function defined on the interval $[0, r]$. Assume that there exists an absolutely continuous solution x of* (3.1), *defined on the interval $[0, E_0]$ and satisfying* (2.11), (2.12). *Then there exists a function g continuous on the interval $[0, a_0]$, $a_0 = x(E_0)$, positive on the interval $(0, a_0]$ and such that $T_g^+(a) = T(a)$ for $a \in (0, a_0]$.*

Proof. Let x be a solution of (3.1) with the required properties. If we put $\tilde{T}^+ = T \circ x$ in equality (2.9), then by Theorem 2.1 there exist a function g continuous on $[0, a_0]$, positive on $(0, a_0]$ such that $\tilde{T}_g^+ = \tilde{T}^+$ on $(0, E_0]$. Since $\tilde{T}^+ = T \circ x$, (2.2) implies equality $T_g^+ = T$.

Now, our aim is to show that under certain assumptions on the function T there exists a solution of (3.1) with the required properties.

Let us consider the Banach space $C[0, E_1]$ with the supremum norm $\|\cdot\|$. We define a mapping S on the set X of all non-negative functions from the closed ball $B \subset C[0, E_1]$ of radius r with a centre at the point $x = 0$, by the formula

$$Sx = \frac{1}{2\pi} k * (T \circ x).$$

LEMMA 3.1. *Let a function T be positive on the interval $[0, r]$ and satisfies the Lipschitz condition with a constant L and*

$$L \leqslant \frac{\pi}{2\sqrt{2E_1} + T(0)} \tag{3.2}$$

and such that

$$\|T\| \leqslant \frac{\pi r}{\sqrt{2E_1}}. \tag{3.3}$$

Then the mapping S is a contraction on the set X.

Proof. For any $x \in X$, the function Sx is continuous since T is bounded on $[0, r]$ (see [6], Theorem 3), and non-negative since k and T are positive. By inequality (3.3), we have as estimation

$$\|Sx\| \leqslant \frac{1}{2\pi} \|k * (T \circ x)\| \leqslant \frac{\sqrt{2E_1}}{\pi} \|T\| \leqslant r,$$

hence $S(X) \subset X$. Making use of the Lipschitz condition and inequality (3.2), for any $x_1, x_2 \in X$, we have consequently

$$\|Sx_1 - Sx_2\| \leqslant \frac{L}{2\pi} \|k * |x_1 - x_2|\| \leqslant \frac{L\sqrt{2E_1}}{\pi} \|x_1 - x_2\| \leqslant \tfrac{1}{2} \|x_1 - x_2\|,$$

that completes the proof.

LEMMA 3.2. *Let f be a non-negative function, summable over the interval*

$[0, E_1]$. *Let X be a set of all functions x which are absolutely continuous on the interval $[0, E_1]$ and such that*

$$(3.4) \qquad |x'(t)| \leqslant f(t) \quad \text{a.e. in } [0, E_1].$$

Then X is a closed subset of the Banach space $C[0, E_1]$.

Proof. Let $|Y|$ denote the Lebesgue measure of a set Y. Then

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \big(|Y| < \delta \Rightarrow \int_Y f(t)\,dt < \varepsilon\big).$$

For any numbers $0 \leqslant s_1 < t_1 \leqslant s_2 < t_2 \leqslant \ldots \leqslant s_n < t_n \leqslant E_1$ and for any function $x \in X$ we infer by (3.4) that the following inequalities hold

$$\sum_{i=1}^{n} |x(t_i) - x(s_i)| \leqslant \sum_{i=1}^{n} \int_{s_i}^{t_i} |x'(t)|\,dt \leqslant \int_{\bigcup_{i=1}^{n} [s_i, t_i]} f(t)\,dt.$$

Now, it is easy to see that

$$(3.5) \qquad \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ \Big(\sum_{i=1}^{n} (t_i - s_i) < \delta \Rightarrow \sum_{i=1}^{n} |x(t_i) - x(s_i)| < \varepsilon\Big).$$

Let $x_j \in X$, $j = 1, 2, \ldots$, and let $\lim_{j \to \infty} \|x_j - x\| = 0$. By (3.5), for given $\varepsilon > 0$ and $0 \leqslant s_1 < t_1 \leqslant s_2 < t_2 \leqslant \ldots \leqslant s_n < t_n \leqslant E_1$ if $\sum_{i=1}^{n} (t_i - s_i) < \delta$ we have

$$\sum_{i=1}^{n} |x(t_i) - x(s_i)| = \lim_{j \to \infty} \sum_{i=1}^{n} |x_j(t_i) - x_j(s_i)| \leqslant \varepsilon,$$

hence x is an absolutely continuous function. It remains to prove that x satisfies (3.4). Let $\bigvee_s^t x$ denote variation of a function x in the interval $[s, t]$. For any absolutely continuous function x there holds the equality

$$(3.6) \qquad \bigvee_s^t x = \int_s^t |x'(u)|\,du.$$

By (3.4) and (3.6), for $j = 1, 2, \ldots$, holds

$$\bigvee_s^t x_j \leqslant \int_s^t f(t)\,dt.$$

Since

$$\bigvee_s^t x \leqslant \liminf_{j \to \infty} \bigvee_s^t x_j,$$

for any numbers $s, t \in [0, E_1]$, $s < t$, we then have the inequality

$$\int_s^t |x'(t)|\,dt \leqslant \int_s^t f(t)\,dt,$$

which proves (3.4).

LEMMA 3.3. *If function T satisfies assumptions of Lemma 3.1, then there exists a unique absolutely continuous solution x of (3.1) satisfying (2.11) and such that*

$$x' = \frac{T(0)}{2\pi} k + O(1). \tag{3.7}$$

Proof. Let X_1 be the set of all non-negative absolutely continuous functions x from the closed ball $B \subset C[0, E_1]$ of radius r with a centre at the point $x = 0$ satisfying inequality

$$|x'(t)| \leqslant 1 + \frac{T(0)}{2\pi} k(t) \quad \text{a.e. in } [0, E_1]. \tag{3.8}$$

We shall show that $S(X_1) \subset X_1$. Since T is Lipschitz continuous, for any function $x \in X_1$, $T \circ x$ is an absolutely continuous function. Hence the function Sx is absolutely continuous too, and the following equality holds

$$(Sx)' = \frac{1}{2\pi} k * (T \circ x)' + \frac{T(0)}{2\pi} k \tag{3.9}$$

(see [6], Theorem 6). Moreover, by inequality (3.4) and identify (2.16) for any function $x \in X_1$ the following inequalities hold

$$\|k * (T \circ x)'\| \leqslant Lk * |x'| \leqslant L\big(1 * k + T(0)\big) \leqslant L\big(2\sqrt{2E_1} + T(0)\big). \tag{3.10}$$

Inequalities (3.2), (3.10) imply

$$\|k * (T \circ x)'\| \leqslant \pi, \tag{3.11}$$

which, together with (3.9), proves that Sx satisfies (3.8). This completes the proof of the inclusion $S(X_1) \subset X_1$.

By Lemma 3.2, X_1 is a closed subset of the Banach space $C[0, E_1]$ and by Lemma 3.1, S is a contraction on X_1. Hence there exists a unique solution $x \in X_1$ of (3.1). By (3.9), equality $Sx = x$ implies the formula

$$x' = \frac{1}{2\pi} k * (T \circ x)' + \frac{T(0)}{2\pi} k. \tag{3.12}$$

Since k is continuous on $(0, E_1]$, property (2.11) follows from (3.8), (3.11), (3.12) (see [6], Theorem 3), while property (3.7) follows immediately from (3.9) and (3.11).

LEMMA 3.4. *Given a positive number c, the following equality holds*

$$cT_g^+ = T_{c^{-2}g}^+.$$

Proof. This equality is an immediate consequence of the formula (1.3).

Now we are ready to prove our main theorem.

THEOREM 3.2. *Given a positive function T^+ satisfying the Lipschitz condition on the interval $[0, r]$, there exists a unique continuous function g defined on the interval $[0, a_0]$, $a_0 \leqslant r$, positive on the interval $(0, a_0]$ and such that $T_g^+(a) = T^+(a)$ for $a \in (0, a_0]$. Moreover,*

$$g(x) = \left(\frac{\pi}{T^+(0)}\right)^2 x + o(x), \quad as\ x \to 0. \tag{3.13}$$

Proof. By Theorem 3.1 it is sufficient to prove that equation (3.1) with $T = T^+$ has an absolutely continuous solution x satisfying (2.11) and (2.12). By Lemma 3.4 we can assume that the function $T = T^+$ satisfies (3.2), (3.3). Now by Lemma 3.3, equation (3.1) has an absolutely continuous on the interval $[0, E_0]$ solution x satisfying (2.11) and (3.7). Since (2.12) is an immediate consequence of (3.7), the first part of Theorem 3.2 is proved.

Now we prove equality (3.13). Since $x(0) = 0$, the integration of both sides of (3.7) over the interval $[0, E]$ gives us

$$x(E) = \frac{\sqrt{2}T^+(0)}{\pi}\sqrt{E} + O(1)E. \tag{3.14}$$

By (2.15) and (3.7) we have

$$g[x(E)] = \frac{1}{\dfrac{T^+(0)}{\pi\sqrt{2E}} + O(1)} = \frac{\pi\sqrt{2E}}{T^+(0) + O(1)\sqrt{E}}. \tag{3.15}$$

Since $x \to 0$ if and only if $E \to 0$, from (3.14) and (3.15) we obtain

$$\lim_{x \to 0} \frac{x}{g(x)} = \lim_{E \to 0} \frac{\left(\dfrac{\sqrt{2}T^+(0)}{\pi}\sqrt{E} + O(1)E\right)\left(T^+(0) + O(1)\sqrt{E}\right)}{\pi\sqrt{2E}} = \left(\frac{T^+(0)}{\pi}\right)^2.$$

This proves (3.13) and completes the proof of Theorem 3.2.

Remark. The class of half-period functions contains that of positive Lipschitz continuous functions as a proper subset. For example, if $g(x) = c\operatorname{sgn} x$, c being a positive constant, then $T_g^+(a) = 2\sqrt{2a/c}$ vanishes for $a = 0$ and is not Lipschitz continuous.

Similar conclusions are true for the negative half-period function. We state them without proofs.

THEOREM 3.3. *Given a positive function T^- satisfying the Lipschitz condition on the interval $[r_1, 0]$, there exists a unique continuous function g defined on the interval $[b_0, 0]$, $r_1 \leqslant b_0$, negative on $[b_0, 0)$ and such that $T_g^-(b) = T^-(b)$ for $b \in [b_0, 0)$. Moreover,*

$$g(x) = -\left(\frac{\pi}{T^-(0)}\right)^2 x + o(x), \qquad x \to 0. \tag{3.16}$$

4. Inverse problem for the period function associated with amplitude. Now we solve the inverse problem for the positive and Lipschitz continuous period function $T_g = T_g(A)$. The proof of Theorem 4.1 below is based on results of Section 2 and Theorems 3.2, 3.3.

THEOREM 4.1. *Given a function T, positive and Lipschitz continuous on the interval $[0, r]$, there exist infinitely many continuous functions g defined on intervals $[b_0, a_0]$, where $b_0 < 0 < a_0$, satisfying (1.1) and such that $T_g(A) = T(A)$ for $A \in (0, A_0]$, $A_0 = \frac{1}{2}(b_0 - a_0)$. Moreover, there exist positive constants c_1, c_2 satisfying the equality $c_1 + c_2 = T(0)$ and such that*

$$g(x) = (\pi/c_1)^2 x + o(x), \qquad x \to 0,\ x \geqslant 0, \tag{4.1}$$

$$g(x) = -(\pi/c_2)^2 x + o(x), \qquad x \to 0,\ x \leqslant 0. \tag{4.2}$$

Proof. Let φ be any continuously differentiable function defined on $[0, r]$ such that $\varphi' < -c$, c being a positive constant. Then the function φ maps the interval $[0, r]$ onto a certain interval $[r_1, 0]$, $r_1 < 0$ and formula (1.6) defines a continuously differentiable function $A = A(a)$. Hence, for any positive Lipschitz continuous function $T^+ = T^+(a)$ defined on $[0, r]$ and satisfying the inequality

$$T^+(a) < T[A(a)]$$

the function $T^- = T^-(b)$ defined on the interval $[r_1, 0]$ by the formula

$$T^-(b) = T\big(A(\varphi^{-1}(b))\big) - T^+(\varphi^{-1}(b))$$

is both Lipschitz continuous and positive. By Theorem 3.2, there exists a unique function g_1 continuous a certain interval $[0, a_1]$, $a_1 > 0$, positive on $(0, a_1]$, $g_1(0) = 0$, and such that $T_{g_1}^+(a) = T^+(a)$ for $a \in (0, a_1]$. By (3.13), condition (4.1) holds with $g = g_1$, $c_1 = T^+(0)$. By Theorem 3.3 there exists a unique function g_2 continuous on a certain interval $[b_1, 0]$, $b_1 < 0$, negative on $[b_1, 0)$, $g_2(0) = 0$ and $T_{g_2}^-(b) = T^-(b)$ for $b \in [b_1, 0)$. Formula (4.2) holds for $g = g_2$, $c_2 = T^-(0)$. If we set $g = g_1$ on $[0, a_1]$, $g = g_2$ on $[b_1, 0)$ and choose $a_0 > 0$, $b_0 < 0$ such that $G(a_0) = G(b_0)$, then the function $g = g(x)$, $x \in [b_0, a_0]$ has all the required properties and $T_g(A) = T(A)$ for $A \in (0, A_0]$, $A_0 = \frac{1}{2}(b_0 - a_0)$.

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