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A non-m-convex algebra on which operate all entire functions*

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Abstract. We construct a locally convex complete algebra with the property announced in the title. This solves in negative a problem posed in [2].

1. Introduction. All algebras considered in this paper are commutative complex algebras possessing unit elements. A topological algebra is a topological linear space together with an associative jointly continuous multiplication. A locally convex algebra is a topological algebra which as a topological linear space is a locally convex space. The topology of such an algebra A can be given by means of a family $(\|x\|_\alpha)$, $\alpha \in \mathfrak{A}$, of seminorms such that for each index α there is an index β with

$$\|xy\|_\alpha \leqslant \|x\|_\beta \|y\|_\beta \tag{1}$$

for all x and y in A. Moreover, for each finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathfrak{A}$ there is an index β in $\mathfrak{A}$ such that

$$\|x\|_{\alpha_i} \leqslant \|x\|_\beta \tag{2}$$

for all x in A and $i = 1, 2, \ldots, n$. Relations (2) make of $\mathfrak{A}$ a directed set if we put $\alpha \leqslant \beta$ whenever $\|x\|_\alpha \leqslant \|x\|_\beta$ for all x in A. Under condition (2) a seminorm $\|x\|$ on A is continuous if and only if there is an index $\alpha \in \mathfrak{A}$ and a positive constant C such that

$$\|x\| \leqslant C\|x\|_\alpha \tag{3}$$

for all elements x in A. A system $(|x|_\beta)$, $\beta \in \mathfrak{B}$, of seminorms on A gives the same topology as the previous system $(\|x\|_\alpha)$, $\alpha \in \mathfrak{A}$, if and only if for each α in $\mathfrak{A}$ there is a β in $\mathfrak{B}$, a γ in $\mathfrak{A}$ and two constants $p, q > 0$ such that

$$p\|x\|_\alpha \leqslant |x|_\beta \leqslant q\|x\|_\gamma \tag{4}$$

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for all x in A. This means that all seminorms of one system are continuous with respect to the other one. A non-void subset S of a topological linear space, or a topological algebra A is said to be *bounded* if for each neighbourhood U of the origin in A there is a positive constant C_U such that $C_U S \subset U$. If A is locally convex, then a subset $S \subset A$ is bounded if and only if for each index $\alpha \in \mathfrak{A}$ we have

$$\sup_{x \in S} \|x\|_\alpha = C_\alpha < \infty. \tag{5}$$

A locally convex space or algebra is said to be *complete* if each its Cauchy net is convergent to some element. A net (x_i), $i \in I$, of elements of A is a *Cauchy net* if for each $\alpha \in \mathfrak{A}$ and each positive ε there is an index $i_0 \in I$ such that

$$\|x_{i_1} - x_{i_2}\|_\alpha < \varepsilon$$

for all $i_1, i_2 \geqslant i_0$. Each Cauchy net must be almost bounded in the sense that for each $\alpha \in \mathfrak{A}$ there is an index $i(\alpha) \in I$ and a constant $M_\alpha > 0$ such that

$$\|x_i\|_\alpha \leqslant M_\alpha \tag{6}$$

for all $i \geqslant i(\alpha)$. A completely metrizable locally convex space is called a B_0-*space*. Its topology can be given by means of a sequence of seminorms satisfying the relations

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \ldots$$

for all its elements x. If A is a B_0-algebra (i.e., a topological algebra which is a B_0-space), then the seminorms can be choosen so that

$$\|xy\|_i \leqslant \|x\|_{i+1} \|y\|_{i+1} \tag{7}$$

for all x and y in A and $i = 1, 2, \ldots$

A locally convex algebra (in particular a B_0-algebra) is said to be *locally multiplicatively convex* (shortly *m-convex*) *algebra* if relations (1) or (7) can be replaced by relations of the form

$$\|xy\|_\alpha \leqslant \|x\|_\alpha \|y\|_\alpha \tag{8}$$

for all α in $\mathfrak{A}$ and all $x, y \in A$. Let $F(z) = \sum a_n z^n$ be an entire function of the complex variable z. We say that the function F operates on a topological algebra A if for each its element x the series $\sum a_n z^n$ converges in A. Clearly, all entire functions operate in complete, or sequencially complete m-convex algebras. In [1] it was shown that for commutative B_0-algebras the converse also holds true, so that a B_0-algebra is locally multiplicatively convex if and only if all entire functions operate on it. The problem whether this result is also true for non-commutative algebras is still open. In [2], Problem 16.8, we asked whether this result can be extended onto complete locally convex algebras. The purpose of the present note is to disprove this conjecture. We shall construct a complete commutative locally convex algebra on which operate all entire functions and which is not locally multiplicatively convex. We give also some remarks concerning multiplicative linear functionals in the constructed algebra.

2. Construction of the example. Denote by Φ the set of all complex-valued continuous functions $\varphi(t)$ defined on

$$\boldsymbol{R}^+ = \{t \in \boldsymbol{R}:\ t \geqslant 0\}$$

satisfying the following conditions

$$0 < \varphi(t) \leqslant 1 \tag{9}$$

for all $t \in \boldsymbol{R}^+$ and

$$\lim_{t=\infty} \varphi(t) = 0. \tag{10}$$

We define A to be the set of all continuous complex-valued functions defined on $\boldsymbol{R}^+$ such that for every $\varphi \in \Phi$ we have

$$\|x\|_\varphi = \sup_{t \in \boldsymbol{R}^+} |x(t)\varphi(t)| < \infty. \tag{11}$$

Clearly, A endowed with the system of seminorms $(\|x\|_\varphi)$, $\varphi \in \Phi$, is a locally convex space. It is also a locally convex algebra under pointwise defined algebra operations. This follows immediately from the fact that for any φ in Φ the function $\psi(t) = \varphi(t)^{1/2}$ is also in Φ and we have

$$\|xy\|_\varphi \leqslant \|x\|_\psi \|y\|_\psi$$

for all x, y in A so that conditions (1) are satisfied. Relations (2) are also satisfied with $\beta = \psi$ if for $\varphi_1, \varphi_2, \ldots, \varphi_n \in \Phi$ we put

$$\psi(t) = \max(\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)).$$

PROPOSITION 1. *The algebra A coincides with the algebra of all bounded continuous functions on* $\boldsymbol{R}^+$ *and, moreover,*

$$\|x\|_\varphi \leqslant \|x\|_\infty \tag{12}$$

for all x in A and all φ *in* Φ. *Here* $\|x\|_\infty = \sup_{t \in \boldsymbol{R}^+} |x(t)|$.

Proof. Clearly all bounded continuos functions on $\boldsymbol{R}^+$ are in A. If some unbounded function x_0 is in A, then there is a sequence $(t_n) \subset \boldsymbol{R}^+$, $t_n \to \infty$, with $|x_0(t_n)| > n$. We can easily construct a function φ in Φ satisfying $\varphi(t_n) = n^{-1/2}$. We have $|\varphi(t_n)x_0(t_n)| > n^{1/2}$ and so $\|x_0\|_\varphi = \infty$. The contradiction proves our assertion. Relation (12) follows immediately from formulas (9) and (11).

PROPOSITION 2. *A non-void subset $S \subset A$ is bounded in A if and only if it is bounded with respect to the seminorm $\|x\|_\infty$, i.e., if*

$$\sup\{\|x\|_\infty : x \in S\} < \infty. \tag{13}$$

Proof. If relation (13) is satisfied, then by (12) and (5) the set S is bounded. We have to show that if relation (13) fails, then relation (5) fails too for some φ in Φ. But if (13) fails, then there is a sequence $(x_n) \subset S$ with $\|x_n\|_\infty > n$, and so we can choose points $t_n \in \boldsymbol{R}^+$ so that $|x_n(t_n)| > n$. If the sequence (t_n) is bounded, say $t_n < M$ for all n, then we choose φ in Φ in such a way that $\varphi(t) = 1$ for $0 \leqslant t \leqslant M$. We have $\|x_n\|_\varphi > n^{1/2}$ and so relation (5) fails in this case. If the sequence (t_n) is unbounded, then passing if necessary, to a subsequence we can assume that $t_n \to \infty$. We choose now φ in Φ in such a way that $\varphi(t_n) = n^{-1/2}$. Again we have $\|x_n\|_\varphi \geqslant |x_n(t_n)\varphi(t_n)| > n^{1/2}$ so that relation (5) fails too. The conclusion follows.

We shall show now that the algebra A is complete. Let (x_α) be a Cauchy net in A. We have to show that the net (x_α) converges to some element x_0 in A. Choose first elements φ_n in Φ so that $\varphi_n(t) = 1$ for $0 \leqslant t \leqslant n$. Since the net (x_α) is a Cauchy net with respect to each seminorm $\|x\|_{\varphi_n}$ we infer that it is also a Cauchy net with respect to the almost uniform convergence on $\boldsymbol{R}^+$ (uniform convergence on each compact subset of $\boldsymbol{R}^+$). This implies that the net (x_α) tends almost uniformly to a continuous function x_0 on $\boldsymbol{R}^+$. We shall show that the function x_0 is in A, i.e., it is bounded, and that $\lim_\alpha \|x_\alpha - x_0\|_\varphi = 0$ for all φ in Φ. But if x_0 is an unbounded function, there is a sequence $t_n \to \infty$ with $|x_0(t_n)| > n$. Since x_α tends to x_0 almost uniformly, it implies that $|x_\alpha(t_n)| \geqslant n$ for large α, say for $\alpha \geqslant \alpha(n)$. Choosing φ in Φ so that $\varphi(t_n) = n^{-1/2}$ we see that $\|x_\alpha\|_\varphi \geqslant n^{1/2}$ for $\alpha > \alpha(n)$. This is in contradiction with formula (6). Thus the function x_0 is in A. It remains to be shown that $\lim_\alpha x_\alpha = x_0$. To this end choose a φ in Φ and a positive ε. We have to show that there is an index $\alpha(\varphi, \varepsilon)$ such that

$$\|x_\alpha - x_0\|_\varphi < \varepsilon \tag{14}$$

for all $\alpha > \alpha(\varphi, \varepsilon)$. Put $\psi(t) = \varphi(t)^{1/2}$. We have $\psi \in \Phi$ and formula (6) implies that there is an index $\alpha(\psi)$ and a positive constant M_ψ such that

$$|x_\alpha(t)\psi(t)| < M_\psi$$

for all t in $\boldsymbol{R}^+$ provided $\alpha \geqslant \alpha(\psi)$. This implies

$$|x_\alpha(t)\varphi(t)| = |x_\alpha(t)\psi(t)|\,\psi(t) < M_\psi\,\psi(t)$$

for all t in $\boldsymbol{R}^+$ and all $\alpha > \alpha(\psi)$. Since $\lim \psi(t) = 0$, there is a $t_\psi \in \boldsymbol{R}^+$ such that

$$|x_\alpha(t)\varphi(t)| < \varepsilon/2 \tag{15}$$

for all $\alpha > \alpha(\psi)$ and all $t \geqslant t_\psi$.

Since x_α tends to x_0 almost uniformly, relation (15) implies

$$|x_0(t)\varphi(t)| \leqslant \varepsilon/2 \tag{16}$$

for all $t \geqslant t_\psi$. The almost uniform convergence of the net (x_α) implies also that there is an index α_0 such that

$$|x_\alpha(t) - x_0(t)| < \varepsilon \tag{17}$$

for all $\alpha > \alpha_0$ and all t satisfying $0 \leqslant t \leqslant t_\psi$. We put now $\alpha(\varphi, \varepsilon) = \max(\alpha(\psi), \alpha_0)$. By (9) and (17) we have

$$\sup_{0 \leqslant t \leqslant t_\psi} \left|\left(x_\alpha(t) - x_0(t)\right)\varphi(t)\right| < \varepsilon \tag{18}$$

for all $\alpha \geqslant \alpha(\varphi, \varepsilon)$. Relations (15) and (16) imply

$$\sup_{t \geqslant t_\psi} \left|\left(x_\alpha(t) - x_0(t)\right)\varphi(t)\right| \leqslant \sup_{t \geqslant t_\psi} |x_\alpha(t)\varphi(t)| + \sup_{t \geqslant t_\psi} |x_0(t)\varphi(t)| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \tag{19}$$

for all $\alpha \geqslant \alpha(\varphi, \varepsilon)$. Formulas (18) and (19) imply now (14) and we are done.

PROPOSITION 3. *All entire functions operate on the algebra A.*

Proof. This follows immediately from relation (12). For, if $F(z) = \sum a_n z^n$ is an entire function and $\varphi \in \Phi$, then for each x in A we have

$$\sum \|a_n x^n\|_\varphi = \sum |a_n| \, \|x^n\|_\varphi \leqslant \sum |a_n| \, \|x^n\|_\infty = \sum |a_n| \, \|x\|_\infty^n < \infty$$

and so the series $\sum a_n x^n$ converges absolutely in A.

PROPOSITION 4. *The algebra A is a non-m-convex algebra.*

Proof. Suppose conversely that A is m-convex, so that there exists an equivalent system of seminorms satisfying relations (8). By formula (4) for a given φ in Φ we can find a submultiplicative seminorm $\|x\|$ (i.e., $\|xy\| \leqslant \|x\| \, \|y\|$) and positive constants p and q so that for a certain ψ in Φ we have

$$p\|x\|_\varphi \leqslant \|x\| \leqslant q\|x\|_\psi$$

for all elements x in A. Thus whenever $q\|x\|_\psi < 1$, then $\|x\| < 1$, what implies $\|x^n\| < 1$ and so $p\|x^n\|_\varphi < 1$ for all n. Since $\lim \psi(t) = 0$ we can find a t_0 in $\boldsymbol{R}^+$ so that $q\psi(t) < 1/2$ for $t > t_0$. Let x be a non-negative function in A with support in $[t_0, \infty)$ such that $x(t) \leqslant 2$ and $x(t_1) = 2$ for some $t_1 > t_0$. We have $q\|x\|_\psi < 1$. On the other hand $p\|x^n\|_\varphi \geqslant p2^n\varphi(t_1)$ and the right-hand expression tends to an infinity as $n \to \infty$. The contradiction shows that A is a non-m-convex algebra.

The above construction gives the result announced in the abstract.

Remarks. The constructed above algebra has the following property. All its multiplicative-linear functionals are the evaluations at the points of $\beta\boldsymbol{R}^+$ (the Čech compactification of $\boldsymbol{R}^+$). All functionals corresponding to the points of $\boldsymbol{R}^+$

are continuous with respect to all seminorms $\|x\|_\varphi$, for if $f(x) = x(t_0)$, $t_0 \in \boldsymbol{R}^+$, then

$$|f(x)| < \varphi(t_0)^{-1} \|x\|_\varphi$$

for all φ in Φ. On the other hand all functionals corresponding to the points in $\beta\boldsymbol{R}^+ \setminus \boldsymbol{R}^+$ are discontinuous (and so A possesses more discontinuous multiplicative linear functionals then continuous ones). In fact, suppose that such a functional F is continuous so that $|F(x)|$ is a continuous seminorm on A. By formula (3) there is a φ in Φ and a constant $C > 0$ such that

$$|F(x)| \leqslant C\|x\|_\varphi \tag{20}$$

for all x in A. Choose $t_0 \in \boldsymbol{R}^+$ so that $C\varphi(t) \leqslant 1/2$ for all $t \geqslant t_0$. Take an element x in A so that $0 \leqslant x(t) \leqslant 1$ for all $t \in \boldsymbol{R}^+$, $x(t) = 0$ for $t \leqslant t_0$, and $x(t) = 1$ for all $t \geqslant t_1$, where $t_1 > t_0$. Evidently we have $F(x) = 1$ and $C\|x\|_\varphi \leqslant 1/2$. This is in contradiction with formula (20), what proves our assertion.

However, by Proposition 2, all multiplicative-linear functionals on A are bounded.

Let us also remark that we could perform our construction taking instead of $\boldsymbol{R}^+$ the discrete space N of all natural numbers. In this case we would obtain the algebra of all bounded sequences and the set of all countinuous multiplicative linear functionals would be countable, while the set of all mutiplicative linear functionals would be of cardinality $2^{\mathfrak{c}}$, where $\mathfrak{c}$ is the cardinality of continuum.

References

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