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CONTRACTIVE OPERATORS OF CERTAIN SPACES

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Introduction. Let X be a measure (topological) space and $A(X)$ a normed algebra of functions on X. An important problem in analysis is to characterize homomorphisms of $A(X)$. Saeki [5] considered the contractive homomorphisms of the tensor algebra $C(X) \hat{\otimes} C(X)$, where $C(X)$ is the space of continuous functions on the compact space X. Cohen [1], Wood [7], Greenleaf [2], and Rigelhof [4] studied norm decreasing homomorphisms of the group algebra $L^1(G)$ and the measure algebra $M(G)$ for a locally compact abelian group G.

In this paper* we study norm decreasing operators on $L^2(I, m) \hat{\otimes} L^2(I, m)$ which are of the form $U(\varphi) = \varphi \circ F$ for all trace class functions φ (F is a measurable map on $I \times I$). We prove that if

$$\|U(\varphi)\|_{\mathrm{Tr}} = \|\varphi \circ F\|_{\mathrm{Tr}} \leqslant \|\varphi\|_{\mathrm{Tr}},$$

then F is essentially of the form (F_1, F_2), where F_1 and F_2 are measure-preserving maps on I; that is

$$F(x, y) = \big(F_1(x), F_2(y)\big) \quad \text{or} \quad F(x, y) = \big(F_1(y), F_2(x)\big).$$

Contractive maps of the trace class operators. Let I denote the unit interval with the usual Lebesgue measure m and let $L^2(I, m) \hat{\otimes} L^2(I, m)$ be the complete projective tensor product of $L^2(I, m)$ with itself. It is well known [6] that $L^2(I, m) \hat{\otimes} L^2(I, m)$ is isometrically isomorphic to the space of the trace class operators. If $\psi \in L^2(I, m) \hat{\otimes} L^2(I, m)$, then $\|\psi\|_{\mathrm{Tr}}$ denotes the trace class norm of ψ, and $\|\psi\|_{\mathrm{HS}}$ its Hilbert-Schmidt norm.

THEOREM. *Let $F\colon I \times I \to I \times I$ be a measurable map and let*

$$U\colon L^2(I, m) \hat{\otimes} L^2(I, m) \to L^2(I, m) \hat{\otimes} L^2(I, m)$$

be a contractive operator defined by $U(\varphi) = \varphi \circ F$. Then F is essentially of the form (F_1, F_2), where F_1 and F_2 are measure-preserving maps on I.

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One has to remark here that if $F=(F_1, F_2)$ and F_1, F_2 are measure-preserving maps on I, then the operator U on the trace class functions defined by $U(\varphi)=\varphi\circ(F_1\otimes F_2)$ is contractive.

Proof of the Theorem. Since the proof is long, we prove the claim in four steps.

Step I. *The mapping F is measure preserving.*

Let X_1 and X_2 be any two disjoint sets in I such that $I=X_1\cup X_2$. If Y_1 and Y_2 is a similar pair of sets in I, then we put $1_{X_i\times Y_j}$ to denote the characteristic function of $X_i\times Y_j$, $1\leqslant i, j\leqslant 2$. From the definition of the Hilbert-Schmidt norm we obtain

$$[m(F^{-1}(X_i\times Y_j))]^{1/2}=\|1_{X_i\times Y_j}\circ F\|_{\mathrm{HS}}\leqslant\|1_{X_i\times Y_j}\circ F\|_{\mathrm{Tr}}\leqslant|m(X_i\times Y_j)|^{1/2}.$$

Now we have

$$1=\sum_{i,j=1}^{2} m(F^{-1}(X_i\times Y_j))\leqslant\sum_{i,j=1}^{2} m(X_i\times Y_j)=1.$$

Hence $m(F^{-1}(X_i\times Y_j))=m(X_i\times Y_j)$, $1\leqslant i, j\leqslant 2$. Consequently, F preserves the measure of rectangles. Since the σ-algebra of Lebesgue measurable sets is the completion of the smallest σ-algebra containing the rectangles, F is a measure-preserving map on $(I\times I,\ m\times m)$.

Step II. *The operator U preserves atoms in $L^2(I, m)\hat{\otimes}L^2(I, m)$.*

Let $f\otimes g\in L^2(I, m)\hat{\otimes}L^2(I, m)$. If $\|f\otimes g\|_2$ denotes the norm of $f\otimes g$ as an element of $L^2(I\times I, m\otimes m)$, then $\|f\otimes g\|_2=\|f\otimes g\|_{\mathrm{Tr}}=\|f\|_2\|g\|_2$. Since F is measure preserving (step I), one can prove that U is an isometry on $L^2(I\times I, m\otimes m)$. Hence

$$\|f\otimes g\|_{\mathrm{HS}}=\|(f\otimes g)\circ F\|_{\mathrm{HS}}\leqslant\|(f\otimes g)\circ F\|_{\mathrm{Tr}}\leqslant\|f\otimes g\|_{\mathrm{Tr}}=\|f\otimes g\|_2.$$

Therefore, $\|(f\otimes g)\circ F\|_{\mathrm{HS}}=\|(f\otimes g)\circ F\|_{\mathrm{Tr}}$, which is possible only if $(f\otimes g)\circ F$ is of rank one. That is, $(f\otimes g)\circ F=u\otimes v$ for some atom $u\otimes v$ in $L^2(I, m)\hat{\otimes}L^2(I, m)$.

Step III. *Construction of F_1 and F_2.*

Let $i\colon I\to I$ be the identity map $i(x)=x$, and let $\pi_1, \pi_2\colon I\times I\to I$ be the first and the second projections, respectively. Set $F_1=\pi_1\circ F$ and $F_2=\pi_2\circ F$. Then $F(x, y)=(F_1(x, y), F_2(x, y))$. Consider the map

$$i\otimes 1\colon I\times I\to I\times I,$$

where 1 is the constant function with range $\{1\}$. Step II implies that $(i\otimes 1)\circ F=\alpha_1\otimes\alpha_2$ for some $\alpha_1\otimes\alpha_2$ in $L^2(I, m)\hat{\otimes}L^2(I, m)$. Hence $F_1(x, y)=\alpha_1(x)\cdot\alpha_2(y)$. Similarly, $F_2(x, y)=\beta_1(x)\cdot\beta_2(y)$ for some $\beta_1\otimes\beta_2$ in $L^2(I, m)\hat{\otimes}L^2(I, m)$.

Step IV. *Each of the functions F_1 and F_2 depends on one of the variables x and y, but not on both of them.*

For any function $\psi \in L^2(I, m) \hat{\otimes} L^2(I, m)$, set

$$m(\psi) = \int\int_{I \times I} \psi(x, y)\, dx dy.$$

From step I it follows that

$$m(\psi) = m(U(\psi)). \tag{1}$$

Now, if φ is an atom in $L^2(I, m) \hat{\otimes} L^2(I, m)$, we put

$$m_1(\varphi) = \int_I \varphi(x, y)\, dy \quad \text{and} \quad m_2(\varphi) = \int_I \varphi(x, y)\, dx.$$

Hence we can write

$$m(\varphi) \cdot \varphi = m_1(\varphi) \otimes m_2(\varphi). \tag{2}$$

Step II together with (1) implies

$$m(\varphi) \cdot U(\varphi) = m_1(U(\varphi)) \otimes m_2(U(\varphi)).$$

Therefore, if $m(\varphi) = 0$, then either $m_1(U(\varphi)) = 0$ or $m_2(U(\varphi)) = 0$. Now, take $\varphi = f \otimes 1$ and write $U(f)$ instead of $U(f \otimes 1)$. Set

$$V_j = \{f \mid m_j(U(f - m(f) \cdot 1)) = 0\},$$

where $m(f)$ denotes $m(f \otimes 1)$. Since for any $f \in L^2(I, m)$ we have

$$m(f - m(f) \cdot 1) = m(f) - m(f) = 0,$$

it follows that for any $f \in L^2(I, m)$ either $m_1(U(f - m(f) \cdot 1)) = 0$ or $m_2(U(f - m(f) \cdot 1)) = 0$. Hence V_1 and V_2 are closed subspaces of $L^2(I, m)$ such that $L^2(I, m) = V_1 \cup V_2$. In this case, as well known, either $V_1 = L^2(I, m)$ or $V_2 = L^2(I, m)$, That is, there exists $j = 1$ or $j = 2$ such that

$$m_j(U(f - m(f) \cdot 1)) = 0$$

for all $f \in L^2(I, m)$. Without loss of generality, we can assume that $j = 1$. Thus $m_1(U(f)) = m(f) \cdot 1$. Relations (1) and (2) then imply

$$m(f) \cdot U(f) = m(f) \cdot (1 \otimes m_2(U(f))).$$

Hence, if $m(f) \neq 0$, we obtain

$$U(f) = U(f \otimes 1) = 1 \otimes m_2(U(f)).$$

On the other hand, we get

$$U(f \otimes 1) = (f \otimes 1) \circ (F_1, F_2) = (f \circ F_1) \otimes 1.$$

Consequently, using step III, for any $(x, y) \in I \times I$ we have

$$m_2(U(f))(y) = ((f \otimes 1) \circ F)(x, y) = (f \circ F_1)(x, y) = f(\alpha_1(x) \alpha_2(y)).$$

Hence α_1 is a constant. In case $j = 2$, we can see that α_2 is a constant. This shows that the function F_1 depends either on the first coordinate or on the second coordinate but not on both of them. One can prove the same thing for F_2 considering the atom $1 \otimes f$. We conclude that F takes one of the following forms:

$$F = (\alpha_1, \beta_1), \quad F = (\alpha_1, \beta_2),$$
$$F = (\alpha_2, \beta_1), \quad F = (\alpha_2, \beta_2).$$

Finally, we have to show that α_1, α_2, β_1, and β_2 are all measure-preserving maps on I. We prove the claim only for α_1. Now, let E be any set in I. Put $f = 1_E$ and consider $f \otimes 1 = 1_E \otimes 1$. Since F is measure preserving, we have

$$[m(E)]^{1/2} = \|1_E \otimes 1\|_2 = \|(1_E \otimes 1) \circ F\|_2 = \|1_E(\alpha_1)\|_2 = [m(\alpha_1^{-1}(E))]^{1/2}.$$

(Here we are considering F to be of the form $F = (\alpha_1, \beta_1)$.) This completes the proof of the Theorem.

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