COLLOQUIUM MATHEMATICUM

VOL. LIII 1987 FASC. 1

SUMS OF ADDITIVE FUNCTIONS ON ARITHMETICAL SEMI-GROUPS(1)

BY

Š. PORUBSKÝ (BRATISLAVA) AND W. SCHWARZ (FRANKFURT AM MAIN)

1. Introduction. Let f be a real-valued, additive, and prime-independent(2) arithmetical function. If f satisfies some growth conditions, then the asymptotic formula

$$\sum_{n \leqslant x} f(x) = f(2) \cdot x \log \log x + A \cdot x + O\left(\frac{x}{\log x}\right)$$

holds, as was shown by S. L. Segal [11] (see also Duncan [2])(3).

In the special case $f = \omega$, B. Saffari [9] deduced an asymptotic expansion for $\sum_{n \leqslant x} \omega(n)$, namely(4)

$$\sum_{n \leqslant x} \omega(n) = x \cdot \log \log x + B_\omega \cdot x + \sum_{r=1}^{m} C_r \cdot \frac{x}{(\log x)^r} + O\left(\frac{x}{(\log x)^{m+1}}\right),$$

where the constants C_r are given by

$$C_r = \int_1^\infty \frac{\{t\}}{t^2} (\log t)^{r-1}\, dt = \frac{(-1)^{r-1}}{r} \cdot \frac{d^r}{ds^r} \left(\frac{(s-1)\zeta(s)}{s}\right)_{s=1}.$$

In this formula, $\{t\}$ denotes the fractional part of t.

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(1) This note was written, when the first-named author was staying at the Johann Wolfgang Goethe–University Frankfurt as a research fellow of the Alexander von Humboldt–Foundation.

(2) f is called *prime-independent*, if the map $(p, k) \to f(p^k)$ is constant with respect to p.

(3) These results extend similar, well-known results for special arithmetical functions, e.g. for $f(n) = \omega(n)$ (the number of different prime divisors of n), for $f(n) = \Omega(n)$ (the total number of prime divisors of n) and for $f(n) = \Omega_k(n)\colon = \alpha_1^k + \ldots + \alpha_r^k$, if $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ (see Duncan [1]).

(4) In fact he proved a result with sharper error terms.

In this note we prove a result along these lines in a more abstract setting. We are concerned with additive prime-independent functions defined on arithmetical semigroups; our aim is to obtain an asymptotic expansion with a "good" remainder term.

An *arithmetical semigroup* $\mathcal{G}$ is an abelian semigroup (written multiplicatively) with identity element 1 and a countable set $\boldsymbol{P} = \{p_1, p_2, \ldots\}$ of generators (which are called the "primes" of $\mathcal{G}$) such that every element $a \neq 1$ in $\mathcal{G}$ has a unique factorization of the form $a = p_1^{a_1} \ldots p_r^{a_r}$, and equipped with a norm $\|\cdot\|\colon \mathcal{G} \to R$ satisfying:

(i) $\|a \cdot b\| = \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{G}$, $\|1\| = 1$, and $\|p\| > 1$ for any prime $p \in \boldsymbol{P}$.

(ii) The number

$$N_{\mathcal{G}}(x) = \# \{a \in \mathcal{G}, \|a\| \leqslant x\}$$

of elements $a \in \mathcal{G}$ with norm $\|a\| \leqslant x$ is finite for any real x.

Moreover, we assume that all arithmetical semi-groups under consideration satisfy the so-called

AXIOM A. There exist positive constants A, δ, β_1 and a constant η, $0 \leqslant \eta < \delta$, so that the asymptotic formula

$$|N_{\mathcal{G}}(x) - A \cdot x^{\delta}| \leqslant \beta_1 \cdot x^{\eta} \tag{1.1}$$

holds, where (without loss of generality) $\beta_1 \geqslant 1$.

For further details and background we refer to J. Knopfmacher's book [4]. Put

$$w = \min\{\|p\|, p \in \boldsymbol{P}\}.$$

THEOREM. *Let g be a real-valued additive arithmetical function defined on an arithmetical semi-group $\mathcal{G}$ satisfying Axiom* A. *Assume there is a "prime-number theorem for g"*, i.e., *an asymptotic formula*

$$\Big| \sum_{p \in \boldsymbol{P}, \|p\| \leqslant x} g(p) - G \cdot \mathrm{li}(x^{\delta})\Big| \leqslant \beta_2 \cdot x^{\delta} \exp(-\beta_3 \sqrt{\log x}); \tag{1.2}$$

furthermore assume that for $\alpha \geqslant 2$

$$|g(p^{\alpha})| \leqslant \beta_4 \cdot \theta^{\alpha}, \quad \text{where } 1 \leqslant \theta < w^{\delta-\eta},\ \beta_4 \geqslant 1. \tag{1.3}$$

Finally, suppose

$$\sum_{\|p\| \leqslant x} |g(p)| \leqslant \beta_5 \cdot x^{\delta}. \tag{1.4}$$

Then

$$\sum_{\|n\| \leqslant x} g(n) = G \cdot A \cdot x^{\delta} \log\log\left(\frac{x}{w}\right) + B_{\mathcal{G},g} \cdot x^{\delta} - \tag{1.5}$$

$$- G x^{\delta} J(x) + R(x),$$

where

(1.6) $$J(x) = \int_{1}^{\sqrt{x}} \frac{A\cdot t^{\delta} - N_{g}(t)}{t^{1+\delta}(\log x - \log t)}\,dt,$$

and where, with an absolute O-constant, the remainder term $R(x)$ satisfies[1]

(1.7) $$R(x) \ll \beta_1(\beta_1 + A)\cdot H^3 \cdot L \times \\ \times \{x^{\delta/2+\eta/2} + x^{\delta-\varepsilon} + x^{\delta}\cdot\exp(-\tfrac{1}{2}\beta_3\sqrt{\log x})\}\cdot \log x,$$

with the abbreviation

$$H = \max\left\{\beta_2, \frac{1}{\beta_3}, \frac{1}{\delta-\eta}, \delta, |G|, A, \beta_4, \beta_5, \theta, w, \frac{1}{\log w}\right\}.$$

ε is defined by $\theta = w^{\delta-\eta-2\varepsilon}$, and so $0 < \varepsilon \leqslant \frac{1}{2}\delta$. Finally

$$L = \left(1 - \frac{\theta}{w^{\delta}}\right)^{-1}\cdot w\cdot(\alpha_0\theta^{\alpha_0}+1)\cdot\begin{cases}\dfrac{1}{\delta-2\eta}, & \text{if } \delta \neq 2\eta,\\ 1, & \text{if } \delta = 2\eta,\end{cases}$$

where

$$\alpha_0 = \left[\frac{\delta}{\varepsilon}\right] + 1.$$

Remark. There is an asymptotic expansion of $J(x)$ in terms of negative powers of $\log x$,

(1.8) $$J(x) = \sum_{k=1}^{K} C_k^{*}\cdot\frac{1}{\log^k x} + O\left(\frac{K!}{(\delta-\eta)^{K+1}}\left\{\frac{1}{\log^{K+1} x} + \frac{1}{x^{(\delta-\eta)/2}}\right\}\right)$$

with an O-constant not depending on K. The coefficients C_k^{*} are given by

$$C_k^{*} = \int_{w}^{\infty} \frac{At^{\delta} - N_{g}(t)}{t^{\delta}}\cdot\frac{\log^{k-1} t}{t}\,dt.$$

2. Proof of Theorem. We decompose $\sum_{\|n\|\leqslant x} g(n)$ in the following way[2]

(2.1) $$\sum_{\|n\|\leqslant x} g(n) = \sum_{\|n\|\leqslant x}\sum_{p^{\alpha}\|n} g(p^{\alpha}) \\ = \sum_{\|p^{\alpha}\|\leqslant x} g(p^{\alpha})\cdot\left\{N_{g}\left(\frac{x}{\|p^{\alpha}\|}\right) - N_{g}\left(\frac{x}{\|p^{\alpha+1}\|}\right)\right\} \\ = \Sigma_1 - \Sigma_2 + \Sigma_3,$$

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(1) It is possible to state explicitly the dependence of $R(x)$ on $\beta_2, \beta_3, \delta, \ldots$, etc. For the application in Corollary 2 we only stress the dependence of $R(x)$ on β_1.

(2) $p^{\alpha}\| n$ means: p^{α} divides n, but $p^{\alpha+1}$ does not.

where

$$\Sigma_1 = \sum_{\|p\| \leqslant x} g(p)\cdot N_{\mathscr{G}}\left(\frac{x}{\|p\|}\right), \tag{2.1$_1$}$$

$$\Sigma_2 = \sum_{\|p\| \leqslant x} g(p)\cdot N_{\mathscr{G}}\left(\frac{x}{\|p\|^2}\right) = \sum_{\|p\| \leqslant \sqrt{x}} g(p)\cdot N_{\mathscr{G}}\left(\frac{x}{\|p\|^2}\right), \tag{2.1$_2$}$$

and where finally

$$\Sigma_3 = \sum_{\|p^\alpha\| \leqslant x, \alpha \geqslant 2} g(p^\alpha)\cdot\left\{N_{\mathscr{G}}\left(\frac{x}{\|p^\alpha\|}\right) - N_{\mathscr{G}}\left(\frac{x}{\|p^{\alpha+1}\|}\right)\right\} \tag{2.1$_3$}$$

contains the remaining terms with exponents $\alpha \geqslant 2$.

To estimate the sum Σ_1 we split it into three sums in the following manner, canonical in divisor problems (see for example Tull [12], [13]).

$$\Sigma_1 = \sum_{\|m\| < \sqrt{x}} \Big(\sum_{\|p\| \leqslant x/\|m\|} g(p)\Big) + \sum_{\|p\| \leqslant \sqrt{x}} g(p)\cdot \sum_{\|m\| \leqslant x/\|p\|} 1 - \sum_{\|p\| \leqslant \sqrt{x}} g(p)\cdot \sum_{\|m\| \leqslant \sqrt{x}} 1 = \Sigma_{11} + \Sigma_{12} - \Sigma_{13}. \tag{2.2}$$

Suppressing the details of the calculation(¹), as there is a similar manipulation in the proof of Satz 1 of [10], we find

$$\Sigma_{11} = G\cdot A\cdot x^{\delta/2}\,\mathrm{li}(x^{\delta/2}) + G\cdot A\cdot x^{\delta}(\log\log x - \log\log x^{1/2}) - G\cdot x^{\delta}\cdot J(x) + R_{11}, \tag{2.3$_1$}$$

$$\Sigma_{12} = G\cdot A\cdot x^{\delta}\cdot \log\log x + B_1 x^{\delta} + R_{12}, \tag{2.3$_2$}$$

and

$$\Sigma_{13} = G\cdot A\cdot x^{\delta/2}\,\mathrm{li}(x^{\delta/2}) + R_{13}. \tag{2.3$_3$}$$

For the remainder terms R_{11}, R_{12} and R_{13}, the following estimates are obtained:

$$R_{11} \ll |G|\,\beta_1\cdot x^{\delta/2+\eta/2} + (A+\beta_1)\,\delta\beta_2\, x^{\delta}\log x\cdot \exp(-\tfrac{1}{2}\beta_3\sqrt{\log x}), \tag{2.4$_1$}$$

$$R_{12} \ll \beta_1\beta_5\left(1+\frac{\eta}{\delta-\eta}\right)\cdot x^{\delta/2+\eta/2} + GAw^{\delta} + \beta_2 A\cdot\{1+\delta\log x\}\cdot x^{\delta}\cdot \exp(-\tfrac{1}{2}\beta_3\sqrt{\log x}), \tag{2.4$_2$}$$

(¹) Essentially done by use of (1.1), (1.2), and partial summation.

and finally

$$R_{13} \ll \beta_1(|G|+\beta_2)x^{(\delta+\eta)/2}+A\beta_2 x^{\delta}\cdot\exp(-\tfrac{1}{2}\beta_3\sqrt{\log x}), \tag{2.4$_3$}$$

where all the implied O-constants are absolute, i.e., independent of β_1, β_2, β_3, A, G, δ, η.

It is not difficult to prove that

$$\Sigma_2 = A\cdot B_2\cdot x^{\delta}+R_2, \tag{2.5}$$

where

$$B_2 = \sum_p \frac{g(p)}{\|p\|^{2\delta}}$$

and where

$$R_2 \ll \beta_1\beta_5\cdot x^{\delta/2}\cdot\begin{cases} A+1+\dfrac{\delta}{|\delta-2\eta|}, & \text{if } \delta\neq 2\eta,\\ A+1+\delta\log x, & \text{if } \delta = 2\eta,\end{cases} \tag{2.6}$$

and the implied O-constant is absolute.

Finally the sum Σ_3 will be estimated in the following way:

$$\begin{aligned}\Sigma_3 &= \sum_{\|p^{\alpha}\|\leqslant x,\alpha\geqslant 2} g(p^{\alpha})\cdot\left\{N_{\mathscr{G}}\left(\frac{x}{\|p^{\alpha}\|}\right)-N_{\mathscr{G}}\left(\frac{x}{\|p^{\alpha+1}\|}\right)\right\}\\ &= A\cdot x^{\delta}\cdot\sum_{\|p^{\alpha}\|\leqslant x,\alpha\geqslant 2} g(p^{\alpha})\cdot\left\{\frac{1}{\|p^{\alpha}\|^{\delta}}-\frac{1}{\|p^{\alpha+1}\|^{\delta}}\right\}+R_3',\end{aligned}$$

using Axiom A. Extending the summation over all "primes" of $\mathscr{G}$ we obtain

$$\Sigma_3 = A\cdot D\cdot x^{\delta}-R_3''+R_3', \tag{2.7}$$

where

$$D = \sum_p\sum_{\alpha\geqslant 2}\frac{g(p^{\alpha})}{\|p^{\alpha}\|^{\delta}}\cdot\left\{1-\frac{1}{\|p\|^{\delta}}\right\}$$

is a convergent series by (1.3) and

$$\sum\|p\|^{-2\delta}\leqslant\sum_{m\in\mathscr{G}}\|m\|^{-2\delta}<\infty,$$

and where

$$R_3' = \sum_{\|p^{\alpha}\|\leqslant x,\alpha\geqslant 2} g(p^{\alpha})\cdot\left\{N_{\mathscr{G}}\left(\frac{x}{\|p^{\alpha}\|}\right)-N_{\mathscr{G}}\left(\frac{x}{\|p^{\alpha+1}\|}\right)-\frac{Ax^{\delta}}{\|p^{\alpha}\|^{\delta}}+\frac{Ax^{\delta}}{\|p^{\alpha+1}\|^{\delta}}\right\}, \tag{2.8$'$}$$

and

$$R_3'' = Ax^{\delta} \cdot \sum_{\substack{p, \alpha \geqslant 2, \\ \|p^{\alpha}\| > x}} g(p^{\alpha}) \cdot \left\{ \frac{1}{\|p^{\alpha}\|^{\delta}} - \frac{1}{\|p^{\alpha+1}\|^{\delta}} \right\}. \tag{2.8''}$$

By (1.3) it is possible to estimate $|g(p^{\alpha})|$ by $\beta_4 \cdot \theta^{\alpha}$, where $1 \leqslant \theta < w^{\delta-\eta}$. Put

$$\theta = w^{\delta-\eta-2\varepsilon}, \tag{2.9}$$

then

$$0 < \varepsilon \leqslant \tfrac{1}{2}(\delta - \eta).$$

Fix the integer

$$\alpha_0 = \left[\frac{1}{\varepsilon} \cdot \delta \right] + 1 > \frac{\delta}{\varepsilon}$$

and abbreviate

$$K = \left[\frac{\log x}{\log w} \right].$$

Then (with absolute O-constants)

$$\begin{aligned} R_3' &\ll \sum_{\|p^{\alpha}\| \leqslant x, 2 \leqslant \alpha \leqslant K} \beta_4 \theta^{\alpha} \cdot \beta_1 \cdot \frac{x^{\eta}}{\|p^{\alpha}\|^{\eta}} \\ &\leqslant \beta_4 \beta_1 \cdot x^{\eta} \cdot \left\{ \sum_{2 \leqslant \alpha \leqslant \alpha_0} \sum_{\|p\| \leqslant x^{1/2}} \frac{\theta^{\alpha}}{\|p\|^{2\eta}} + \sum_{\|p\| \leqslant x^{1/\alpha_0}} \sum_{\alpha_0 < \alpha \leqslant K} \theta^{\alpha} \right\}. \end{aligned}$$

Since, by partial summation from Axiom A, for any $\tau \geqslant 0$,

$$\sum_{\|p\| \leqslant y} \frac{1}{\|p\|^{\tau}} \leqslant \sum_{m \in \mathscr{G}, \|m\| \leqslant y} \frac{1}{\|m\|^{\tau}} \leqslant (A + \beta_1)(y^{\delta-\tau} + 1) \cdot U_{\delta,\tau}(y),$$

where

$$U_{\delta,\tau}(y) = \begin{cases} 1 + \dfrac{\tau}{|\delta - \tau|}, & \text{if } \delta \neq \tau, \\ 1 + \tau \log y, & \text{if } \delta = \tau, \end{cases}$$

we get, noticing that

$$\theta^{K} \leqslant w^{(\delta-\eta-2\varepsilon) \cdot \log x / \log w} = x^{\delta-\eta-2\varepsilon},$$

the estimate

$$\begin{aligned} R_3' &\ll \beta_4 \beta_1 x^{\eta} \cdot \{ \alpha_0 \cdot \theta^{\alpha_0} \cdot (A + \beta_1) \cdot (x^{\delta/2-\eta} + 1) \cdot U_{\delta,2\eta}(x) \\ &\qquad + (A + \beta_1) x^{\delta/\alpha_0} \cdot \log x \cdot x^{\delta-\eta-2\varepsilon} \} \\ &\ll \beta_1 \beta_4 (A + \beta_1) \cdot \{ \alpha_0 \cdot \theta^{\alpha_0} \cdot (x^{\delta/2} + x^{\eta}) \cdot U_{\delta,2\eta}(x) + x^{\delta-\varepsilon} \log x \}. \end{aligned} \tag{2.10}$$

Before dealing with R_3'', we notice that for $\alpha \geqslant 2$

$$(2.11) \qquad \sum_{\|p\| > y^{1/\alpha}} \frac{1}{\|p\|^{\alpha\delta}} \leqslant \sum_{m \in \mathscr{G}, \|m\| > y^{1/\alpha}} \frac{1}{\|m\|^{\alpha\delta}} = \alpha \cdot \delta \cdot \int_{y^{1/\alpha}}^{\infty} \frac{N_{\mathscr{G}}(u)\,du}{u^{\alpha\delta+1}} \leqslant 2(A+\beta_1) \cdot y^{-\delta+\delta/\alpha}.$$

Now we split

$$R_3'' \leqslant \sum_{p, \alpha \geqslant 2, \|p^\alpha\| > x} \beta_4 \cdot \theta^\alpha \cdot \frac{1}{\|p\|^{\alpha\delta}} \cdot A \cdot x^\delta$$
$$= \Bigg\{ \sum_{2 \leqslant \alpha \leqslant \alpha_0} \theta^\alpha \cdot \sum_{\|p^\alpha\| > x} \frac{1}{\|p\|^{\alpha\delta}} + \sum_{\alpha_0 < \alpha \leqslant K} \theta^\alpha \cdot \sum_{\|p^\alpha\| > x} \frac{1}{\|p\|^{\alpha\delta}}$$
$$+ \sum_{\|p\| \geqslant w} \sum_{\alpha > K} \left(\frac{\theta}{\|p\|^\delta} \right)^\alpha \Bigg\} \cdot \beta_4 \cdot A \cdot x^\delta$$
$$= \beta_4 \cdot A \cdot x^\delta \cdot \{R_{31} + R_{32} + R_{33}\}.$$

At first, using (2.11),

$$(2.12_1) \qquad R_{31} \leqslant \alpha_0 \cdot \theta^{\alpha_0} \cdot x^{-\delta+\delta/2} \cdot 2(A+\beta_1).$$

Next, remembering that $\theta = w^{\delta-\eta-2\varepsilon}$, and $K \leqslant \log x/\log w$, we obtain

$$(2.12_2) \qquad R_{32} \leqslant \sum_{\alpha_0 < \alpha \leqslant K} \theta^\alpha \cdot 2(A+\beta_1) \cdot x^{-\delta+\delta/\alpha}$$
$$\leqslant 2(A+\beta_1)\, x^{-\delta+\delta/\alpha_0} \cdot \theta^K \cdot K$$
$$\leqslant 2(A+\beta_1) \cdot K \cdot x^{\delta/\alpha_0 - \eta - 2\varepsilon}$$
$$\leqslant 2(A+\beta_1) \cdot x^{-\eta-\varepsilon} \cdot \frac{\log x}{\log w}.$$

And finally, first summing the geometric series, we get

$$(2.12_3) \qquad R_{33} \leqslant \left(1 - \frac{\theta}{w^\delta}\right)^{-1} \cdot \theta^{\log x/\log w + 1} \cdot \sum_p \|p\|^{-\delta(K+1)}$$
$$\leqslant \theta \left(1 - \frac{\theta}{w^\delta}\right)^{-1} \cdot x^{\delta-\eta-2\varepsilon} \cdot 2(A+\beta_1) \cdot w^{-\delta K}$$
$$\leqslant \theta w \left(1 - \frac{\theta}{w^\delta}\right)^{-1} \cdot 2(A+\beta_1) \cdot x^{-\eta-2\varepsilon}.$$

Thus the remainder terms R'_3 and R''_3 are less than

$$\mathrm{const}(\beta_1, \beta_4, A, \delta, \eta, \theta, w)\cdot x^{\delta-\varepsilon}\cdot \log x,$$

and the dependence on the parameters β_1, β_4, etc. is explicit. Collecting the estimates (2.3) and (2.4), (2.5) and (2.6), (2.7) and (2.10) and (2.12) we get the Theorem.

3. Corollaries. The following result embodies some implications for classical arithmetical functions, for example an improvement of Theorem 6.2.4 of [4] (p. 165).

COROLLARY 1. *Let g be an additive prime-independent function on an arithmetical semi-group $\mathscr{G}$ as in the Theorem, satisfying* (1.4) *and*

$$|g(p^\alpha)| \leqslant \beta_4\cdot\theta^\alpha \quad \textit{for } \alpha \geqslant 2,$$

where $1 \leqslant \theta < w^{\delta-\eta}$. *Then*

$$\sum_{\|n\|\leqslant x} g(n) = A\cdot g(p)\cdot x^\delta \log\log(x^\delta) + B\cdot g(p)\cdot x^\delta -$$
$$-g(p)\cdot x^\delta J(x) + O\left(x^\delta\cdot\exp\left(-\tfrac{1}{2}\beta_3\sqrt{\log x}\right)\right).$$

The proof follows immediately from the Theorem and from the abstract prime number theorem[1] in a form proved by H. Müller [8]. If $\mathscr{G}$ satisfies Axiom A with $\delta = 1$, then

$$\pi_{\mathscr{G}}(x) := \sum_{p\in P,\|p\|\leqslant x} 1 = \mathrm{li}\ x + O\left(x\cdot\exp\left(-\beta_3^*\sqrt{\log x}\right)\right)$$

according to H. Müller. Using the "normalization principle" from Knopfmacher's book ([4], p. 78) we obtain

$$\pi_{\mathscr{G}}(x) = \mathrm{li}(x^\delta) + O\left(x^\delta\exp\left(-\beta_3\sqrt{\log x}\right)\right)$$

with a suitable constant $\beta_3 > 0$, if $\mathscr{G}$ satisfies Axiom A in the form (1.1).

The next result improves the remainder term of the Theorem from A. Mercier's paper [7], if stronger assumptions on $\sum_{\|p\|\leqslant x} g(p)$ are taken for granted than in [7][2]. For the formulation of Corollary 2 we need the Euler-type function[3] φ_r, defined for $a\in\mathscr{G}$ and real r by

$$\varphi_r(a) = \sum_{d|a}\mu(d)\cdot\left\|\frac{a}{d}\right\|^r = \|a\|^r\cdot\prod_{p|a}\left(1-\frac{1}{\|p\|^r}\right)$$

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[1] For further references concerning the "abstract" prime number theorem see Knopfmacher [4], Chapter 6. See also H. G. Diamond, Illinois Journal of Mathematics 14 (1970), 12–28 and 29–34.

[2] A. Mercier only assumes $\sum_p \frac{1}{p}\cdot|g(p)-a| < \infty$.

[3] See [4], p. 108.

and the function $2^{\omega(a)}$, counting the number of square-free divisors of a in $\mathscr{G}$.

COROLLARY 2. *Let g be an additive arithmetical function satisfying the hypotheses of the Theorem. Then for given $k \in \mathscr{G}$ the asymptotic formula*

$$\sum_{\|n\| \leqslant x, (n,k)=1} g(n) = A \cdot \frac{\varphi_\delta(k)}{\|k\|^\delta} \cdot G \cdot x^\delta \cdot \log\log(x^\delta) + B(k) \cdot x^\delta - C(k) \cdot x^\delta J(x) + O\left(2^{2\omega(k)} \cdot x^\delta \cdot \exp\left(-\tfrac{1}{2}\beta_3 \sqrt{\log x}\right)\right)$$

holds, where the O-constant does not depend on k.

Proposition 1.3 of [4], Chapter IV, p. 77, shows that the semi-group $\mathscr{G}\langle k\rangle$ of all those elements of $\mathscr{G}$, which are coprime to k, also satisfies Axiom A, provided $\mathscr{G}$ does; more precisely

$$\left| N_{\mathscr{G}\langle k\rangle}(x) - A \cdot \frac{\varphi_\delta(k)}{\|k\|^\delta} \cdot x^\delta \right| \leqslant \beta_1 \cdot 2^{\omega(k)} \cdot x^\eta,$$

if $\mathscr{G}$ satisfies (1.1). Taking into account that $\varphi_\delta(k) \cdot \|k\|^{-\delta} \leqslant 1$, an application of the Theorem gives the result.

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DEPARTMENT OF MATHEMATICS
SLOVAKIAN ACADEMY OF SCIENCES
BRATISLAVA, CZECHOSLOVAKIA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF FRANKFURT
FRANKFURT AM MAIN, GFR

Reçu par la Rédaction le 03. 09. 1982