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ON SEMI-INNER PRODUCT SPACES, II

BY

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1. Introduction. Ambrose [1] studied the structure theorems for a special class of Banach algebras A which is also a Hilbert space under the same norm and has the property that, for every $x \in A$, there corresponds an $x^* \in A$ such that $(xy, z) = (y, x^*z) = (x, zy^*)$. Such algebras were called by Ambrose *H^*-algebras.* The purpose of the present paper is to replace the Hilbert space structure in H^*-algebras by a more general structure called a *semi-inner product space* (henceforth abbreviated to *s. i. p. space*) defined by Lumer [7], and the algebra thus obtained will be called a *semi-inner product algebra* (henceforth abbreviated to *s. i. p. algebra*). Some interesting properties of the s. i. p. spaces have been studied by Giles [3] and present authors [4]. As follows from the definitions, s. i. p. spaces are more general than inner product spaces, but it seems worthwhile to examine the conditions under which one can obtain theorems analogous to that of H^*-algebras. We prove here a number of results under these weakened axioms and also give some analogues of the theorems obtained by Kaplansky [5] for H^*-algebras. Our approach is similar to Ambrose [1] but where we have adopted a different type of argument the proof has been given.

2. In this section we give definitions and an example of s. i. p. algebra. Other definitions would be given at appropriate places in the present paper.

Definition 2.1. A complex (real) vector space X is called an *s. i. p. space* if corresponding to every pair of elements $x, y \in X$, there corresponds a complex (real) number written as $[x, y]$ with the following properties:

(i) $$[x+y, z] = [x, z]+[y, z],$$
$$[\lambda x, y] = \lambda[x, y], \quad x, y, z \in X \text{ and } \lambda \text{ is scalar},$$

(ii) $$[x, x] > 0 \quad \text{for } x \neq 0,$$

(iii) $$|[x, y]|^2 \leqslant [x, x][y, y].$$

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We do not assume that s. i. p. spaces need satisfy the following properties:

(iv) $$[x, \lambda y] = \bar{\lambda}[x, y],$$

(v) $$[x, y+z] = [x, y]+[x, z].$$

It is to be noted [7] that, with $\|x\| = [x, x]^{1/2}$, an s. i. p. space becomes a normed space. Further, if the norm in a Banach space (in particular, a complete s. i. p. space) satisfies the parallelogram equality, then it becomes a Hilbert space.

Remark. It is clear from (i) and (ii) that $[x, y] = 0$ for all $y \in A$ if and only if $x = 0$. Moreover, if either of x, y is zero, then $[x, y] = 0$.

Definition 2.2. An s. i. p. space is said to satisfy the *continuity property* (or to be *continuous*) if $\mathrm{Re}[y, x+\lambda y] \to \mathrm{Re}[y, x]$ for all real $\lambda \to 0$, where $\mathrm{Re}[y, x]$ means the real part of $[y, x]$.

Definition 2.3. For x, y in the s. i. p. space X, x is said to be *orthogonal* to y if $[y, x] = 0$.

It is to be noted that if x is orthogonal to y, then y is not necessarily orthogonal to x. The following lemma will be useful in our future discussions:

LEMMA 2.1. *Let X be a complete and continuous s. i. p. space which satisfies the inequality* $\|x+y\|^2+\mu^2\|x-y\|^2 \leqslant 2\|x\|^2+2\|y\|^2$, $0 < \mu < 1$. *Then to any closed proper subspace Y of X there is a non-zero vector orthogonal to Y and any $x \in X$ can be expressed in the form $x = y+z$, where $y \in Y$ and z is orthogonal to Y. Moreover, this representation is unique.*

The proof is contained in [4].

Throughout the present paper μ will be taken to be positive and less than one.

Definition 2.4. A non-empty set A is called an *s. i. p. algebra* if it satisfies the following conditions:

(1) A is a Banach algebra;

(2) A is an s. i. p. space with the same norm as that of the Banach algebra A;

(3) corresponding to any $x \in A$, there corresponds an element $x^* \in A$ (called *involution*) satisfying one of the conditions (a) $[xy, z] = [y, x^*z] = [x, zy^*]$ or (b) $[z, xy] = [x^*z, y] = [zy^*, x]$.

Note. When the underlying s. i. p. space is continuous, we say that the s. i. p. algebra is continuous.

Remark. It is easy to see that (3) implies the following: $x^{**} = x$ and $(xy)^* = y^*x^*$, where x and y are the elements of s. i. p. algebra.

Example. Let G be a compact topological group and let $L^p(G)$ $(1 < p < \infty)$ be the space of measurable functions whose p-th power is integrable with respect to the Haar measure of G. Then $L^p(G)$ becomes an s. i. p. algebra if

$$(f+g)\sigma = f(\sigma)+g(\sigma),$$

$$(fg)\sigma = \int_G f(\sigma\tau^{-1})g(\tau)d\tau,$$

$$(\lambda f)\sigma = \lambda f(\sigma),$$

$$[f, g] = \frac{1}{\|g\|_p^{p-2}} \int_G f|g|^{p-1} \operatorname{sgn} g d\sigma,$$

$$(f^*)\sigma = \bar{f}(\sigma^{-1}),$$

where $f(\sigma)$ and $g(\sigma)$ are functions in $L^p(G)$ [10].

This is an example of an s. i. p. algebra which is not an H^*-algebra in the sense of Ambrose [1].

3. Proper s. i. p. algebras.

LEMMA 3.1. *If x is an element of an s. i. p. algebra A, then $xA = \{0\}$ is equivalent to $Ax = \{0\}$.*

Proof. Let $x, y, z \in A$ and let x^*, y^*, z^* be their respective adjoints. Then, by hypothesis, we have $xy = 0$ for any $y \in A$. Hence $0 = [xy, z] = [x, zy^*] = [z^*x, y^*]$ for all $y \in A$. Therefore, $z^*x = 0$ or $Ax = \{0\}$.

Lemma 3.1 leads us to the formulation of the following definition:

Definition 3.1. An s. i. p. algebra A is called *proper* if it satisfies either of the following equivalent conditions:

(1) $xA = \{0\} \Rightarrow x = 0$,

(2) $Ax = \{0\} \Rightarrow x = 0$.

In the other words, an s. i. p. algebra is called *proper* if it has no non-zero annihilators. The significance of proper s. i. p. algebras is exhibited in the following

THEOREM 3.1. *An s. i. p. algebra A is proper iff every element has a unique adjoint.*

Proof. Let $x, y, z \in A$ and suppose if possible x has two adjoints x_1^* and x_2^*. Then $[z, xy] = [x_1^*z, y] = [x_2^*z, y]$ or $[(x_1^*-x_2^*)z, y] = 0$ for all $y, z \in A$. Putting $y = (x_1^*-x_2^*)z$, we obtain $\|(x_1^*-x_1^*)z\| = 0$ for all $z \in A$. The last equality implies $x_1^* = x_2^*$ since A is assumed to be proper.

Conversely, suppose that A is not proper, that is, $x_0A = Ax_0 = \{0\} \Rightarrow x_0 \neq 0$; then $[z, (x^*+x_0)y] = [z, x^*y+x_0y] = [z, x^*y] = [xz, y]$. Hence x_0+x^* is another adjoint of x.

LEMMA 3.2. *If $x \neq 0$ is an element of a proper s. i. p. algebra, then $xx^* \neq 0$, $x^*x \neq 0$ and $x^* \neq 0$.*

Proof. Let $x^*x = 0$; then $\|xy\|^2 = [xy, xy] = [x^*xy, y] = 0$ for all y. Hence $xy = 0 \Rightarrow xA = \{0\} \Rightarrow x = 0$, a contradiction. Other implications can be proved similarly.

Definition 3.2. An ideal I in the s. i. p. algebra A is said to be *two-sided* if it is a left as well as a right ideal. An ideal is called *minimal* if it does not contain properly any ideal other than $\{0\}$.

Notation. By E^p we shall mean the orthogonal complement of the set E, that is, the set of elements $\{x: [y, x] = 0, y \in E\}$. All the ideals in A are taken to be closed with respect to the norm topology induced by the s. i. p. space.

LEMMA 3.3. *If R is a right ideal in a complete continuous proper s. i. p. algebra A satisfying the property $\|x+y\|^2+\mu^2\|x-y\|^2 \leqslant 2\|x\|^2+2\|y\|^2$ for any $x, y \in A$, then $xA \subset R \Rightarrow x \in R$.*

Proof. For any $x \in A$, let $xA \subset R$. By Lemma 2.1, we have $x = x_1 + x_2$, where $x_1 \in R$ and $x_2 \in R^p$. For any $z \in A$, we have $xz = x_1z + x_2z$. Since R is a right ideal, $xz \in R$ and $x_1z \in R$; thus, from the last equation, we have $x_2z \in R$ and $x_2 \in R^p$. Therefore, $[x_2z, x_2] = 0$ or $[z, x_2x_2^*] = 0$. But this is true for all z, therefore $x_2x_2^* = 0$. By Lemma 3.2, $x_2 = 0$, and so $x = x_1 \in R$.

LEMMA 3.4. *Every two-sided ideal in a complete continuous proper s. i. p. algebra A which satisfies the inequality $\|x+y\|^2+\mu^2\|x-y\|^2 \leqslant 2\|x\|^2 + 2\|y\|^2$ is self-adjoint, where $x, y \in A$.*

Proof. Let I be a two-sided ideal in A. Let $x_1 \in I$ and $x_2 \in I^p$. Now $\|x_1x_2\|^2 = [x_1x_2, x_1x_2] = [x_1^*x_1x_2, x_2] = 0$, since I being a two-sided ideal, $x_1^*x_1x_2 \in I$ and $x_2 \in I^p$. Therefore, $x_1x_2 = 0$. For any $z \in A$, we have $[z, x_1x_2] = [x_1^*z, x_2] = 0 \Rightarrow x_1^*z \in I$ which, by Lemma 3.3, implies that $I^* \subset I$, but also $I^{**} = I \subset I^*$ giving $I = I^*$.

LEMMA 3.5. *If R is a right ideal in an s. i. p. algebra in which* 2.1 (iv) *and* (v) *hold, then R^p is also a right ideal.*

Proof. Observe that R^p is a subspace in view of 2.1 (iv) and (v). Let R be a right ideal; then $x \in R \Rightarrow xz \in R$ for any $z \in A$. Let $y \in R^p$; then $[xz, y] = 0$ or $[x, yz^*] = 0$. The last relation is true for any $x \in R$, hence $yz^* \in R^p$ for any $z \in A$, which shows that R^p is a right ideal.

Definition 3.3. On the s. i. p. algebra we can define the transformations $l_x: y \to xy$ and $r_x: y \to yx$. The l_x and r_x are called, respectively, the *left* and *right* transformations. The greater of the two bounds $|||l_x|||$ and $|||r_x|||$, where $|||l_x|||$ and $|||r_x|||$ are bounds of the respective transformations, is called the *uniform norm* of x and is denoted by $|||x|||$.

LEMMA 3.6. *Every element x of an s. i. p. algebra, in which $(ax+\beta y)^* = \bar{a}x^*+\bar{\beta}y^*$ holds, can be expressed uniquely in the form $x = x_1+ix_2$, where x_1 and x_2 are self-adjoint.*

The proof is obvious.

LEMMA 3.7. *In an s. i. p. algebra A, a necessary and sufficient condition that $|||x||| \leqslant 1$ for a self-adjoint element $x \in A$ is that $x^2-x^4 \geqslant 0$.*

The proof follows as in Rajagopalan [9] by using axiom (iii) in the definition of the s. i. p. space.

LEMMA 3.8. *If R is a right ideal in a proper s. i. p. algebra, then the right ideal generated by R^n is R, where R^n stands for the set of elements of the form $x_1x_2x_3\ldots x_n$, where $x_1, x_2, \ldots, x_n \in R$.*

The proof follows as in [1].

Definition 3.4. Let A be an s. i. p. algebra and $\{A_a\}$ a family of subalgebras which spans A. If A_a's are mutually orthogonal, then A is the *direct sum* of the subalgebras A_a and we write $A = \sum A_a$.

THEOREM 3.2. *Every s. i. p. algebra A is the direct sum of an ideal,*

$$A_1 = \{y\colon y \neq 0, Ay = (0)\} = \{y\colon y \neq 0, yA = (0)\}$$

and another proper two-sided ideal, provided A in addition satisfies conditions 2.1 (iv) *and* (v).

Proof. Let $A = A_1+A_1^p$. According to Lemma 3.5, A_1^p is a two-sided ideal since A_1 is so. Also $A_1A_1^p = A_1 \cap A_1^p = (0)$. Now, since A_1 contains all non-zero annihilators, hence A_1^p is proper.

LEMMA 3.9. *In a proper complete, continuous s. i. p. algebra A with $\|x+y\|^2+\mu^2\|x-y\|^2 \leqslant 2(\|x\|^2+\|y\|^2)$ the set E of all elements of the form $x_1y_1+x_2y_2+\ldots+x_ny_n$ is dense in A.*

Proof. If E is not dense in A, then there exists $y_0 \in A$, $y_0 \neq 0$, such that

$$[x_1y_1+x_2y_2+\ldots+x_ny_n, y_0] = 0$$

or

$$[x_1y_1, y_0]+[x_2y_2, y_0]+\ldots+[x_ny_n, y_0] = 0$$

or

$$[y_1, x_1^*y_0]+[y_2, x_2^*y_0+\ldots+[y_n, x_n^*y_0] = 0.$$

Since $y_1, y_2, \ldots, y_n$ are arbitrary, we have $x_1^*y_0 = x_2^*y_0 = \ldots = x_n^*y_0 = 0$. Let us choose $y_0 = x_i$, $i = 1, 2, \ldots, n$; then $x_i^*x_i = 0$ for some i, which contradicts the assumption that an s. i. p. algebra is proper.

LEMMA 3.10. *The set of all bounded operators defined on an s. i. p. space with the property that $(aS+\beta T)^* = \bar{a}S^*+\bar{\beta}T^*$ is an algebra with involution.*

The proof is obvious, and so is the proof of

LEMMA 3.11. *In an s. i. p. space X, a bounded operator T satisfies the relation* $\|T^*T\| = \|T\|^2$.

Definition 3.5. An algebra of operators with involution satisfying the property $\|T^*T\| = \|T\|^2$ is called a *completely regular algebra.*

Definition 3.6. An s. i. p. algebra is said to be *semi-simple* if its radical consists of the only zero element. For various equivalent definitions of radical, see Naĭmark [8], p. 162.

LEMMA 3.12. *Every completely regular algebra of bounded operators on an s. i. p. space X with the property $(aS+\beta T)^* = \bar{a}S^*+\bar{\beta}T^*$, where S and T are bounded operators on X and a, β are scalars, is semi-simple.*

The proof follows as in [8], p. 309.

THEOREM 3.3. *Every proper s. i. p. algebra A, which satisfies the condition $(ax+\beta y)^* = \bar{a}x^*+\bar{\beta}y^*$ (a, β are scalars), is semi-simple.*

Proof. To each element $x \in A$, there corresponds an operator A_x in the s. i. p. space, namely the operator of a left regular representation defined by $A_x y = xy$ and, consequently, $A_{x^*}y = x^*y$. We have $[xy, z] = [A_x y, z] = [y, (A_x)^* z]$, therefore $(A_x)^* = A_{x^*}$. Now, $A_x = 0 \Rightarrow xy = 0$ for all $y \in A \Rightarrow xA = (0) \Rightarrow x = 0$, since A is proper. Therefore, the left regular representation $x \to A_x$ is an isomorphism (preserving the involution also) of the s. i. p. algebra into the algebra of bounded operators on A satisfying the condition $(aS+\beta T)^* = \bar{a}S^*+\bar{\beta}T^*$, but the latter is completely regular, hence, by Lemma 3.12, semi-simple. Thus, A is semi-simple.

Definition 3.7. Let A be an s. i. p. algebra and let $L(I) = \{x\colon xI = (0)\}$ and $R(I) = \{x\colon Ix = (0)\}$ denote, respectively, the left and right annihilators of any subset I of A. An s. i. p. algebra is said to be *dual* if for any ideal I in A the following conditions are satisfied: $L(R(I)) = I$ and $R(L(I)) = I$.

The following lemma is analogous to a result obtained by Kaplansky [5] pertaining to H^*-algebras:

LEMMA 3.13. *If I is a right ideal in a proper continuous s. i. p. algebra A, with $\|x+y\|^2+\mu^2\|x-y\|^2 \leqslant 2\|x\|^2+2\|y\|^2$, then $L(I) = \{x\colon xI = (0)\}$ is the orthogonal complement of I^* in A.*

Proof. First we prove that AI^* is dense in I^*. If not, then there exists $y^* \in I^*$, $y^* \neq 0$, such that $[AI^*, y^*] = 0$ or $[I^*, Ay^*] = 0$. The latter implies that $Ay^* = 0$ or $Ay^* \perp I^*$. In the first case, in particular, $yy^* = 0$, contradicting the properness of A. In the second case, $yy^* \in I^{*p}$. But $yy^* \in I$ because I is a right ideal, $(yy^*)^* = yy^* \in I^* \cap (I^*)^p = \{0\}$, again contradicting properness. Now, $x \in L(I) \Leftrightarrow [A, xI] = 0 \Leftrightarrow [AI^*, x] = 0 \Leftrightarrow [I^*, x] = 0 \Leftrightarrow x \in I^{*p}$, and therefore $A = I^*+L(I)$.

COROLLARY. *Every proper s. i. p. algebra satisfying conditions on A, as stated in Lemma 3.13, is dual.*

LEMMA 3.14. *The sum of all minimal left (or right) ideals of a proper complete, continuous s. i. p. algebra A having an identity element and satisfying $\|x+y\|^2+\mu^2\|x-y\|^2 \leqslant 2\|x\|^2+2\|y\|^2$, and with the property that $(\alpha x+\beta y)^* = \bar{\alpha}x^*+\bar{\beta}y^*$ (α, β are scalars), is dense in A.*

Proof. Let $\{I_a\}$ denote the left minimal ideals in A. Then, by the sum S of $\{I_a\}$, we mean the set of all finite sums of elements $x_a \in I_a$. Let $\bar{S}$ denote the closure of S. Suppose $\bar{S}$ is not equal to A; then there exists an element yz, $y \neq 0$, $z \neq 0$, such that $[\bar{S}, yz] = 0$ or $[y^*\bar{S}, z] = 0 \Rightarrow y^* \in L(\bar{S}) = \{x\colon x\bar{S} = (0)\}$. Hence y^* is a left annihilator of each I_a. Since I_a is a minimal left ideal, y^* is in the maximal right ideal for each a. Hence y^* is in the radical of A. By Theorem 3.3, our algebra is semi-simple and as such the radical consists of the zero element alone. So $y^* = 0 \Rightarrow y = 0$, since the algebra is proper. But this gives us a contradiction and the lemma is proved.

Definition 3.8. A normed vector space X is *strictly convex* if whenever $\|x\|+\|y\| = \|x+y\|$, where $0 \neq x, y \in X$, then $y = \lambda x$ for some real $\lambda > 0$.

Definition 3.9. A sequence $\{x_n\}$ of elements of the s. i. p. space X is said to *converge weakly* in the second coordinate to an element $x \in X$ if $[y, x_n] \to [y, x]$ for every $y \in X$.

THEOREM 3.4. *Let A be a proper, strictly convex, complete, continuous, s. i. p. algebra with $\|x+y\|^2+\mu^2\|x-y\|^2 \leqslant 2\|x\|^2+2\|y\|^2$ and in which the weak convergence with respect to the second coordinate is finer than the norm topology. Further assume that the relation $[x, y] = [y^*, x^*]$ holds for $x \in A$ and $y \in E$ (E defined as in Lemma 3.9); then $\|x\| = \|x^*\|$ and the transformation $x \to x^*$ is continuous.*

Proof. By Lemma 3.9, we can choose a sequence $\{x_n\}$ of elements in E such that x_n converges to some element $x \in A$. Now $x_n - x_m \in E$, hence, by the hypothesis $[x, y] = [y^*, x^*]$, we have $\|x_n^* - x_m^*\| = \|x_n - x_m\| \to 0$, as $m, n \to \infty$, giving us $x_n^* \to x'$, say. The weak convergence with respect to the second argument gives

$$\begin{aligned}[xy, z] &= \lim_{n\to\infty} [x_n y, z] \\ &= \lim_{n\to\infty} [y, x_n^* z] = [y, x' z] = [y, x^* z].\end{aligned}$$

Using the last two relations in this equality and the strict convexity (see [4]), we have $x' = x^*$. By hypothesis, $x_n^* \to x^*$. Therefore, by $x_n \to x$, $x_n^* \to x^*$ and $\|x_n\| = \|x_n^*\|$, we have $\|x\| = \|x^*\|$, as is easy to verify using the axioms of the s. i. p. space.

4. Existence of idempotents.

Definition 4.1. An element e in an s. i. p. algebra is called *idempotent* if $0 \neq e = e^2$. The element e is called *self-adjoint* if $e = e^*$.

Note. Henceforth self-adjoint idempotent would be abbreviated to sa-idempotent. Throughout α, β are scalars.

LEMMA 4.1. *Let A be a proper s. i. p. algebra satisfying the conditions $(\alpha x+\beta y)^* = \bar{\alpha}x^*+\bar{\beta}y^*$ and $[x, y] = [y^*, x^*]$. Let x be a self-adjoint element of A whose norm as a left multiplication operator is 1. Then the sequence x^{2n} converges to a non-zero sa-idempotent.*

Proof. Following Loomis (see [6], p. 101) very closely, with suitable changes, we get, for $m > n$ and both even, $1 \leqslant [x^m, x^m] \leqslant [x^m, x^n] \leqslant \leqslant [x^n, x^n] \leqslant \ldots \leqslant [x^2, x^2]$ and $[x^m, x^n]$ has a limit l as $m, n \to \infty$ through even integers. We have

$$\lim \|x^m - x^n\|^2 = \lim [x^m - x^n, x^m - x^n]$$
$$= \lim [x^m, x^m - x^n] - \lim [x^n, x^m - x^n] = \lim [x^m - x^n, x^m] - \lim [x^m - x^n, x^n]$$
$$= \lim [x^m, x^m] - \lim [x^n, x^m] - \lim [x^m, x^n] + \lim [x^n, x^n],$$

which tends to zero as $m, n \to \infty$.

Arguing again as in Loomis (see [6], p. 101), it follows that x^{2n} converges to a non-zero sa-idempotent. The properness of the s. i. p. algebra is needed to ensure the existence of unique adjoints.

COROLLARY. *Any left (or right) ideal in proper s. i. p. algebra satisfying the conditions $(\alpha x+\beta y)^* = \bar{\alpha}x^*+\bar{\beta}y^*$ and $[x, y] = [y^*, x^*]$ contains a non--zero sa-idempotent.*

The proof follows as in [6], p. 101.

Notation. Henceforth we shall assume that our s. i. p. algebra is proper and satisfies the conditions $[x, y] = [y^*, x^*]$ and $(\alpha x+\beta y)^* = \bar{\alpha}x^* + + \bar{\beta}y^*$.

Definition 4.2. The idempotents e, f of the s. i. p. algebra are called *doubly orthogonal* if $[e, f] = 0$ and $ef = fe = 0$.

Definition 4.3. An idempotent is said to be *primitive* if it can not be expressed as the sum of doubly orthogonal idempotents.

The following lemmas hold and their proofs are the same as in Ambrose [1]:

LEMMA 4.2. *Let A be an s. i. p. algebra, e an idempotent and R the right ideal defined by $R = eA$. If R is the direct sum of a finite number of right ideals,*

$$R = R_1 + R_2 + \ldots + R_n,$$

and if we write

$$e = e_1 + e_2 + \ldots + e_n \quad (e_i \in R_i),$$

then e_i are doubly orthogonal idempotents and $R_i = e_i A$. If e is a sa-idempotent, then each e_i is a sa-idempotent.

LEMMA 4.3. *Let A be an s. i. p. algebra, e an idempotent and R the right ideal defined by $R = eA$. If e can be expressed as a finite sum of doubly orthogonal sa-idempotents,*

$$e = e_1 + e_2 + \dots + e_n,$$

and if we define R_i by $R_i = e_i A$, then R is the direct sum of right ideals R_i.

LEMMA 4.4. *If R is a right ideal in an s. i. p. algebra A of the form $R = eA$, where e is an idempotent, then R is minimal if and only if e is primitive.*

Proof. One can either argue as in Ambrose [1], by using Lemma 4.2 of the present paper, or else one can proceed as follows. If e is not primitive, then $e = e_1 + e_2$, where $e_1 e_2 = e_2 e_1 = 0$. Now $R = eA = e_1 A + e_2 A$. So $e_1 A = ee_1 A \subset eA = R$, which contradicts the minimality of R. Conversely, if R is not minimal, then $0 \subset R_1 \subset eA$, where R_1 is a right ideal. Now, by corollary of Lemma 4.1 of the present paper, R_1 contains a sa-idempotent. Let us denote this idempotent by f. Then there is an element $x \in A$ such that $f = ex$. We have $f = ex = e^2 x = eex = ef$. Writing $e = f + e - f$, we have, $[f, e-f] = [f^2, e-f] = [f, (e-f)f^*] = [f, ef - f^2] = [f, 0] = 0$. Therefore, $e-f$ is orthogonal to f. Let us write $g = e - f$; then $e = f + g$. It is easily seen that $fg = gf = 0$, $g = g^2$ and $g^* = g$. So g is sa-idempotent. Thus e is not primitive. Here e has been taken to be the sa-idempotent of the right ideal defined by $R = eA$, whose existence is guaranteed by the corollary of Lemma 4.1.

THEOREM 4.1. *If e is a sa-idempotent in an s. i. p. algebra, then e is the sum of a finite number of doubly orthogonal primitive sa-idempotents.*

Proof. Following Ambrose [1], we can write $e = e_1 + e_2 + \dots + e_n$, where $e_1, e_2, \dots, e_n$ are sa-idempotents. We have

$$\begin{aligned}
\|e\|^2 &= [e_1 + e_2 + \dots + e_n, e_1 + e_2 + \dots + e_n] \\
&= [e_1^2, e_1 + e_2 + \dots + e_n] + [e_2^2, e_1 + e_2 + \dots + e_n] + \\
&\qquad + \dots + [e_n^2, e_1 + e_2 + \dots + e_n] \\
&= [e_1, e_1(e_1 + e_2 + \dots + e_n)] + [e_2, e_2(e_1 + e_2 + \dots + e_n)] + \\
&\qquad + \dots + [e_n, e_n(e_1 + e_2 + \dots + e_n)] \\
&= [e_1, e_1] + [e_2, e_2] + \dots + [e_n, e_n] \\
&= \|e_1\|^2 + \|e_2\|^2 + \dots + \|e_n\|^2 \geqslant n,
\end{aligned}$$

since in an s. i. p. algebra an idempotent e satisfies the relation $\|e\|^2 = [e, e] = [e^2, e] \leqslant \|e^2\| \, \|e\| \leqslant \|e\|^3$. This shows that the process of splitting e must terminate at some finite stage.

Now, the following theorem is almost obvious by using Zorn's lemma:

THEOREM 4.2. *Every s. i. p. algebra contains a maximal family of doubly orthogonal primitive sa-idempotents.*

5. Structure theorems. We now come to structure theorems. As in the previous section, an s. i. p. algebra would be taken to fulfil the following conditions: $(ax+\beta y)^* = \bar{a}x^* + \bar{\beta}y^*$ and $[x, y] = [y^*, x^*]$.

THEOREM 5.1. *Let $\{e_i\}$ be a maximal family of doubly orthogonal primitive sa-idempotents in a complete s. i. p. algebra satisfying* 2.1 (iv) *and* (v). *Then*

$$A = \sum_i e_i A = \sum_i A e_i,$$

hat is, A is the direct sum of the minimal right ideals $e_i A$ or of the minimal eft ideals Ae_i.

Proof. If $A \neq \sum e_i A$, we have

$$A = \sum_i e_i A + \Big(\sum_i e_i A\Big)^p.$$

Now $(\sum_i e_i A)^p$ is a minimal right ideal and as such it should contain a sa-idempotent. But this would contradict the maximality of $\{e_i\}$, hence $(\sum e_i A)^p = (0)$, and the theorem is proved.

Definition 5.1. An s. i. p. algebra is called *simple* if it contains no proper two-sided ideals.

THEOREM 5.2. *Every complete continuous s. i. p. algebra A is the direct sum of simple s. i. p. algebras, each of which is a minimal two-sided ideal in A, provided in A the inequality $\|x+y\|^2 + \mu^2 \|x-y\|^2 \leqslant 2\|x\|^2 + 2\|y\|^2$ holds for $x, y \in A$.*

Proof. The proof follows as in Ambrose [1] with suitable changes. Certain portions of the proof can be based upon Loomis (see [6], p. 102, Theorem 27C).

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