COLLOQUIUM MATHEMATICUM

VOL. XLIX 1984 FASC. 1

ON SOME INTEGRAL INEQUALITIES OF WEYL TYPE

BY

B. FLORKIEWICZ (WROCŁAW)

The paper is a continuation of [2]. We derive and study some integral inequalities of Weyl type (see [6]), i.e., integral inequalities of the form

$$(1)\qquad \int_I u|h|^p dt \leqslant p\Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I s|h|^p dt\Big)^{1/q},$$

where $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$, $\dot{h} \equiv dh/dt$, and $p > 1$. The inequalities of the form (1) were investigated by Redheffer [6], Benson [1], Florkiewicz and Rybarski [3], and others. The multidimensional case was studied by Redheffer (see [7]). In the second part of the paper some integral inequalities of the form

$$(2)\qquad \sum_{i=1}^{n} v_j(x_i)|h(x_i)|^p \leqslant p\Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I s|h|^p dt\Big)^{1/q} + \int_I u|h|^p dt$$

are obtained. The inequalities of the form (2) were considered by Redheffer (see [6] and [7]).

We denote by $abs\,C$ the class of real functions which are defined and absolutely continuous on the open interval $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$. Let p be any real number such that $p > 1$ and let $q = p/(p-1)$. Let $r \in abs\,C$ and $w \in abs\,C$ be functions such that

$$r > 0,\ w \not\equiv 0 \text{ in } I \quad \text{and} \quad r|w|^{p-1}\operatorname{sgn} w \in abs\,C.$$

Let us put

$$s \equiv r|w|^p, \quad u \equiv (r|w|^{p-1}\operatorname{sgn} w)^{\cdot}, \quad \text{and} \quad v \equiv r|w|^{p-1}\operatorname{sgn} w.$$

We denote by W the class of functions $h \in abs\,C$ satisfying the following integral conditions:

$$(3)\qquad \int_I r|\dot{h}|^p dt < \infty, \qquad \int_I s|h|^p dt < \infty.$$

LEMMA 1. *For every function $h \in W$ the function $v(|h|^p)^{\cdot}$ is summable in the interval I.*

Proof. By Hölder's inequality for $h \in W$ we obtain

$$\int_I |v(|h|^p)^\cdot|\,dt = \int_I r|w|^{p-1}|h|^{p-1}|\dot{h}|\,dt \leqslant \Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I s|h|^p dt\Big)^{1/q}.$$

We denote by $\tilde{W}$ the class of functions $h \in W$ satisfying the following integral and limit conditions:

$$\int_I u|h|^p dt > -\infty; \tag{4}$$

$$\limsup_{t\to\alpha} v|h|^p > -\infty, \qquad \liminf_{t\to\beta} v|h|^p < \infty. \tag{5}$$

LEMMA 2. *Let h belong to $\tilde{W}$. Then*
(i) *the function $u|h|^p$ is summable in I;*
(ii) *there exist finite limits $\lim_{t\to\alpha} v|h|^p$ and $\lim_{t\to\beta} v|h|^p$.*

Proof. (i) Let $h \in \tilde{W}$ and let $\langle a, b\rangle \subset I$ be an arbitrary closed interval. Then the function $(v|h|^p)^\cdot$ is summable in $\langle a, b\rangle$, since $v|h|^p \in abs\,C$ and the function $v(|h|^p)^\cdot$ is summable in I by Lemma 1. Hence we get the equality

$$\int_a^b u|h|^p dt = v|h|^p\Big|_a^b - \int_a^b v(|h|^p)^\cdot dt \tag{6}$$

and, using (5), in a similar way as in the proof of Theorem 1 in [2] one can show that the function $u|h|^p$ is summable in I.

(ii) By Lemmas 1 and 2 (i) it follows from (6) that for $h \in \tilde{W}$ the finite limits $\lim_{t\to\alpha} v|h|^p$ and $\lim_{t\to\beta} v|h|^p$ exist.

Remark 1. By Lemma 2 (ii), conditions (5) can be written as

$$\lim_{t\to\alpha} v|h|^p > -\infty, \qquad \lim_{t\to\beta} v|h|^p < \infty. \tag{5'}$$

Remark 2. From the proof of Lemma 2 it follows that conditions (4) and (5) in the definition of $\tilde{W}$ are equivalent to one of the following three conditions:

(a)
$$\int_I u|h|^p dt < \infty,$$
$$\liminf_{t\to\alpha} v|h|^p < \infty, \qquad \limsup_{t\to\beta} v|h|^p > -\infty;$$

(b) the function $u|h|^p$ is summable in I;

(c) there exist finite limits $\lim_{t\to\alpha} v|h|^p$ and $\lim_{t\to\beta} v|h|^p$.

THEOREM 1. *For an arbitrary function* $h \in \tilde{W}$ *the inequality*

$$(7) \qquad \int_I u|h|^p dt - \lim_{t\to\beta} v|h|^p + \lim_{t\to\alpha} v|h|^p \leqslant p\Big(\int_I r|\dot{h}|^p dt\Big)^{1/p} \Big(\int_I s|h|^p dt\Big)^{1/q}$$

is valid. If $h \not\equiv 0$, *then we have an equality in* (7) *if and only if*

$$(*) \qquad h = c\exp\Big(\lambda \int_{t_0}^{t} w\,dt\Big),$$

where t_0 *is an arbitrary fixed point in* I *and* $c = \text{const} \neq 0$, $\lambda = \text{const} \neq 0$, *and* λ *satisfies the conditions*

(A) $\int_I r|w|^p \exp\Big(p\lambda \int_{t_0}^{t} w\,dt\Big)dt < \infty$;

(B) *there exist finite limits of the expression*

$$r|w|^{p-1}(\operatorname{sgn} w)\exp\Big(p\lambda \int_{t_0}^{t} w\,dt\Big)$$

as $t \to \alpha$ *and* $t \to \beta$.

Proof. Let $\varphi \in abs\,C$ be an arbitrary function such that $\varphi > 0$ in I and $r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi} \in abs\,C$. Further, let $h \in abs\,C$ and

$$(8) \qquad g = r|\dot{h}|^p + (r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi})^{\cdot}\varphi^{1-p}|h|^p - (r|\dot{\varphi}|^{p-1}(\operatorname{sgn}\dot{\varphi})\varphi^{1-p}|h|^p)^{\cdot}.$$

Then, from Lemma 1 in [2] it follows that $g \geqslant 0$ in I and $g = 0$ in I if and only if $h = c\varphi$, where $c = \text{const}$. Putting

$$\varphi = \exp\Big(\lambda \int_{t_0}^{t} w\,dt\Big)$$

in (8), where $t_0 \in I$ and $\lambda = \text{const} \neq 0$, we obtain

$$(9) \qquad g_\lambda = r|\dot{h}|^p + (p-1)|\lambda|^p s|h|^p + |\lambda|^{p-1}\operatorname{sgn}\lambda\,[u|h|^p - (v|h|^p)^{\cdot}],$$

where $g_\lambda \geqslant 0$ in I and $g_\lambda = 0$ in I if and only if $(*)$ holds and $c = \text{const}$. By Lemma 1, for $h \in W$ the function g_λ is summable in I.

Let $h \in \tilde{W}$. Then, by Lemma 2, all the functions in (9) are summable in I, and since $g_\lambda \geqslant 0$ in I, we obtain

$$(10) \qquad -\operatorname{sgn}\lambda\Big(\int_I u|h|^p dt - \lim_{t\to\beta} v|h|^p + \lim_{t\to\alpha} v|h|^p\Big) \leqslant |\lambda|^{1-p}\int_I r|\dot{h}|^p dt + (p-1)|\lambda|\int_I s|h|^p dt$$

for an arbitrary $\lambda \neq 0$ and the equality in (10) appears if and only if $\int_I g_\lambda\,dt = 0$, e.g., $g_\lambda = 0$ in I. If $h \not\equiv 0$, then

$$\int_I s|h|^p dt > 0$$

because $w \not\equiv 0$ in I. The right-hand side of (10) attains the minimal value with respect to λ for λ_h such that

$$(11) \qquad |\lambda_h| = \Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I s|h|^p dt\Big)^{-1/p}.$$

Hence we obtain immediately inequality (7).

If for some $h \in \tilde{W}$ and $h \not\equiv 0$ inequality (7) becomes an equality and if

$$(12) \qquad \int_I u|h|^p dt - \lim_{t\to\beta} v|h|^p + \lim_{t\to\alpha} v|h|^p \geq 0,$$

then for $\lambda_h < 0$, where λ_h satisfies (11), we obtain an equality in (10) when $\lambda = \lambda_h$, and hence

$$(**) \qquad h = c\exp\Big(\lambda_h \int_{t_0}^{t} w\,dt\Big).$$

At the same time we easily check that for h as in (**) condition (12) holds for $\lambda_h < 0$. Similarly, we consider the case where in (12) the inverse inequality takes place (then $\lambda_h > 0$). Thus, if we have an equality in (7) for some $h \not\equiv 0$, then (**) holds for some $\lambda_h \neq 0$, where $c \neq 0$ is an arbitrary constant. The function h in (**) satisfies (11) as an identity, and therefore, finally, if we have an equality in (7), then (*) holds for an arbitrary constant $\lambda \neq 0$. The function h defined in (*) must belong to the class $\tilde{W}$, and hence by Remark 2 (c) we obtain immediately conditions (A) and (B). On the other hand, we easily check that for the function h defined in (*), where $\lambda \neq 0$, and $h \in \tilde{W}$ inequality (7) becomes an equality, which completes the proof.

In the sequel, we use the following lemmas for the description of the class $\tilde{W}$.

LEMMA 3. *Let $h \in abs\,C$ and $\int_I s|h|^p dt < \infty$.*

(i) *If there exists $\lim_{t\to\alpha} v < 0$ (resp. $\lim_{t\to\alpha} v > 0$) and*

$$\int_{\alpha}^{t} w\,dt = -\infty \quad (\textit{resp. } \int_{\alpha}^{t} w\,dt = \infty) \quad \textit{for some } t \in I,$$

then

$$\limsup_{t\to\alpha} v|h|^p = 0 \quad (\textit{resp. } \liminf_{t\to\alpha} v|h|^p = 0).$$

(ii) *If there exists $\lim_{t\to\beta} v > 0$ (resp. $\lim_{t\to\beta} v < 0$) and*

$$\int_{t}^{\beta} w\,dt = \infty \quad (\textit{resp. } \int_{t}^{\beta} w\,dt = -\infty) \quad \textit{for some } t \in I,$$

then

$$\liminf_{t\to\beta} v|h|^p = 0 \quad (\textit{resp.}\ \limsup_{t\to\beta} v|h|^p = 0).$$

Proof. We prove Lemma 3 only in one case. The remaining cases can be proved similarly. Assume that there exists $\lim_{t\to\alpha} v < 0$ and

$$\int_\alpha^t w\,dt = -\infty.$$

Then there exists a neighbourhood U of the point α such that $v < 0$ and $w < 0$ in U, since $\operatorname{sgn} w = \operatorname{sgn} v$. For arbitrary points $a \in U$ and $t \in U$ such that $\alpha < a < t < \beta$ we have

$$\int_a^t s|h|^p dt = \int_a^t v|h|^p w\,dt \geqslant \sup_{(a,t)} v|h|^p \int_a^t w\,dt.$$

Since

$$\int_\alpha^t s|h|^p dt < \infty \quad \text{and} \quad \int_\alpha^t w\,dt = -\infty$$

by assumptions and, simultaneously, $v|h|^p \leqslant 0$ in U, we get

$$\sup_{(\alpha,t)} v|h|^p = 0 \quad \text{for } t \in U.$$

Hence

$$\limsup_{t\to\alpha} v|h|^p = 0.$$

LEMMA 4. *Let* $h \in abs\,C$ *and* $\int_I r|\dot{h}|^p dt < \infty$.

(i) *If*

$$\int_\alpha^t r^{-q/p} dt < \infty \quad (\textit{resp.}\ \int_t^\beta r^{-q/p} dt < \infty)$$

for some $t \in I$, *then there exists a finite limit value*

$$h(\alpha) = \lim_{t\to\alpha} h \quad (\textit{resp.}\ h(\beta) = \lim_{t\to\beta} h).$$

(ii) *If*

$$v\Big(\int_\alpha^t r^{-q/p} dt\Big)^{p/q} = O(1) \quad \textit{as } t \to \alpha$$

$$(\textit{resp.}\ v\Big(\int_t^\beta r^{-q/p} dt\Big)^{p/q} = O(1) \quad \textit{as } t \to \beta)$$

and

$$\liminf_{t\to\alpha} |h| = 0 \quad (\textit{resp.}\ \liminf_{t\to\beta} |h| = 0),$$

then

$$\lim_{t\to\alpha} v|h|^p = 0 \quad (\textit{resp.}\ \lim_{t\to\beta} v|h|^p = 0).$$

Lemma 4 (i) is identical to Lemma 3 in [2], and Lemma 4 (ii) follows from the proof of Theorem 3 in [2].

Example 1. Let $I = (\alpha, \beta)$, where $0 \leqslant \alpha < \beta < \infty$, and let $r = t^{pa}$ and $w = t^{(a+b-pa)/(p-1)}$ in I with arbitrary constants a and b. In that case we obtain $s = t^{qb}$, $u = (a+b)t^{a+b-1}$, and $v = t^{a+b}$ in I.

Let $0 < \alpha < \beta < \infty$. Then $\int_I r^{-q/p}dt < \infty$ and from Lemma 4 (i) it follows that for $h \in W$ there exist finite values $h(\alpha)$ and $h(\beta)$. Hence, for $h \in W$ there exist finite limits

$$\lim_{t\to\alpha} v|h|^p = \alpha^{a+b}|h(\alpha)|^p \quad \text{and} \quad \lim_{t\to\beta} v|h|^p = \beta^{a+b}|h(\beta)|^p.$$

Thus, by Remark 2 (c), $\tilde{W} = W$. Now, using Theorem 1 we deduce the following:

If a function $h \in \operatorname{abs} C$ satisfies the conditions

$$\int_\alpha^\beta t^{pa}|\dot{h}|^p dt < \infty, \qquad \int_\alpha^\beta t^{qb}|h|^p dt < \infty,$$

then there exist finite limit values $h(\alpha)$ and $h(\beta)$ and the inequality

$$(13) \quad \Big|(a+b)\int_\alpha^\beta t^{a+b-1}|h|^p dt + \alpha^{a+b}|h(\alpha)|^p - \beta^{a+b}|h(\beta)|^p\Big| \leqslant p\Big(\int_\alpha^\beta t^{pa}|\dot{h}|^p dt\Big)^{1/p}\Big(\int_\alpha^\beta t^{qb}|h|^p dt\Big)^{1/q}$$

is valid.

Inequality (13) becomes an equality if and only if

$$h = c\exp\{\lambda t^{[(p-1)(1-a)+b]/(p-1)}\} \quad \text{as } (p-1)(1-a)+b \neq 0$$

or

$$h = ct^\lambda \quad \text{as } (p-1)(1-a)+b = 0,$$

where $c = \text{const}$ and $\lambda = \text{const} \neq 0$.

Let $0 = \alpha < \beta < \infty$. In a similar way as above we show that for $h \in W$ there exist a finite value $h(\beta)$ and a finite limit

$$\lim_{t\to\beta} v|h|^p = \beta^{a+b}|h(\beta)|^p.$$

If $a+b \geqslant 0$, then it can be easily seen that for $h \in abs\,C$ condition (4) and the first of conditions (5) are satisfied. Therefore $\tilde{W} = W$ in that case. If $p(a-1)+1 < 0$, then

$$\int_0^t r^{-q/p}\,dt < \infty \quad \text{for some } t \in I,$$

and using Lemma 4 (i) we infer that for $h \in W$ there exists a finite value $h(0)$. Hence, we deduce that if, in addition, $a+b > 0$, then

$$\lim_{t\to 0} v|h|^p = 0$$

for an arbitrary $h \in W$ since $v(0) = 0$. If $a+b < 0$ and $(p-1)(1-a)+b \leqslant 0$, then $v(0) = \infty$, $\int_0^t w\,dt = \infty$ for $t \in I$, and from Lemma 3 (i) it follows that if $h \in W$, then

$$\liminf_{t\to 0} v|h|^p = 0.$$

Thus the conditions of Remark 2 (a) are satisfied for $h \in W$ and $\tilde{W} = W$. Now, applying Theorem 1 we get the following:

Assume that the conditions $a+b > 0$ and $p(a-1)+1 < 0$ or $a+b < 0$ and $(p-1)(1-a)+b \leqslant 0$ are satisfied. Then if $h \in abs\,C$ satisfies

$$\int_0^\beta t^{pa}|\dot{h}|^p\,dt < \infty \quad \text{and} \quad \int_0^\beta t^{qb}|h|^p\,dt < \infty,$$

then there exists a finite limit value $h(\beta)$ and the inequality

$$(14)\quad \left|(a+b)\int_0^\beta t^{a+b-1}|h|^p\,dt - \beta^{a+b}|h(\beta)|^p\right| \leqslant p\Big(\int_0^\beta t^{pa}|\dot{h}|^p\,dt\Big)^{1/p}\Big(\int_0^\beta t^{qb}|h|^p\,dt\Big)^{1/q}$$

is valid.

If $h \not\equiv 0$, then in the cases $a+b > 0$ and $p(a-1)+1 < 0$ or $a+b < 0$ and $(p-1)(1-a)+b < 0$ inequality (14) becomes an equality if and only if

$$h = c\exp\{-\lambda t^{[(p-1)(1-a)+b]/(p-1)}\},$$

where $c = \text{const} \neq 0$, $\lambda = \text{const}$ and $\lambda \neq 0$ provided $a+b > 0$ and $p(a-1)+1 < 0$ or $\lambda > 0$ provided $a+b < 0$ and $(p-1)(1-a)+b < 0$. In the case $a+b < 0$ and $(p-1)(1-a)+b = 0$ inequality (14) becomes an equality if and only if $h = ct^\lambda$, where $c = \text{const} \neq 0$, $\lambda = \text{const}$, and $\lambda > -[p(a-1)+1]/p > 0$.

In a particular case where $p = 4$, $a = 0$, and $b = 3$ we obtain the inequality otherwise deduced in [1].

Example 2. Let $w = r^{-q/p}$ in I, where $r \in abs\,C$ and $r > 0$ in I. Then $s = r^{-q/p}$, $u = 0$, and $v = 1$ in I. It follows from Remark 2 (b) that in the considered case $\tilde{W} = W$. Thus, for $h \in W$ there is a finite limit

$$\lim_{t \to \alpha} v|h|^p = \lim_{t \to \alpha} |h|^p$$

(Lemma 2 (ii)) and the finite value $h(\alpha)$ exists because $p > 1$ and h is a continuous function on I. If

$$\int_\alpha^t r^{-q/p}\,dt = \infty \quad \text{for some } t \in I,$$

then it follows from Lemma 3 (i) that $h(\alpha) = 0$. Similarly, for $h \in W$ there is a finite value $h(\beta)$, and if

$$\int_t^\beta r^{-q/p}\,dt = \infty \quad \text{for some } t \in I,$$

then $h(\beta) = 0$.

Applying Theorem 1 we infer the following:

(i) *If* $\int_I r^{-q/p}\,dt < \infty$ *and a function* $h \in abs\,C$ *satisfies the conditions*

$$\int_I r^{-q/p}|h|^p\,dt < \infty, \quad \int_I r|\dot{h}|^p\,dt < \infty, \tag{15}$$

then there exist finite values $h(\alpha)$ *and* $h(\beta)$ *and the inequality*

$$\big||h(\beta)|^p - |h(\alpha)|^p\big| \leqslant p\Big(\int_I r|\dot{h}|^p\,dt\Big)^{1/p}\Big(\int_I r^{-q/p}|h|^p\,dt\Big)^{1/q} \tag{16}$$

is valid.

(ii) *If* $\int_\alpha^t r^{-q/p}\,dt = \infty$ *and* $\int_t^\beta r^{-q/p}\,dt < \infty$ *for some* $t \in I$ *and a function* $h \in abs\,C$ *satisfies* (15), *then there exists a finite value* $h(\beta)$ *and the inequality*

$$|h(\beta)|^p \leqslant p\Big(\int_I r|\dot{h}|^p\,dt\Big)^{1/p}\Big(\int_I r^{-q/p}|h|^p\,dt\Big)^{1/q} \tag{17}$$

is valid.

(iii) *If* $\int_\alpha^t r^{-q/p}\,dt < \infty$ *and* $\int_t^\beta r^{-q/p}\,dt = \infty$ *for some* $t \in I$ *and a function* $h \in abs\,C$ *satisfies* (15), *then there exists a finite value* $h(\alpha)$ *and the inequality*

$$|h(\alpha)|^p \leqslant p\Big(\int_I r|\dot{h}|^p\,dt\Big)^{1/p}\Big(\int_I r^{-q/p}|h|^p\,dt\Big)^{1/q} \tag{18}$$

is valid.

Inequalities (16), (17), and (18) become equalities only for the function

$$h = c\exp\Big(\lambda \int_{t_0}^t r^{-q/p}\,dt\Big),$$

where $t_0 \in I$, $c = \text{const}$, $\lambda = \text{const}$, and $\lambda \neq 0$ in the case (i), $\lambda > 0$ in (ii), and $\lambda < 0$ in (iii).

Assuming $\alpha = 0$, $\beta = \infty$ and $r = 1$, we infer from (iii) that for an arbitrary function h absolutely continuous on $(0, \infty)$ and satisfying the integral conditions

$$\int_0^\infty |h|^p dt < \infty, \qquad \int_0^\infty |\dot{h}|^p dt < \infty$$

there is a finite value $h(0)$ and the inequality

$$|h(0)|^p \leqslant p\Big(\int_0^\infty |\dot{h}|^p dt\Big)^{1/p}\Big(\int_0^\infty |h|^p dt\Big)^{1/q} \tag{19}$$

holds. This inequality becomes an equality only for $h = c\exp(-\lambda t)$, where $c = \text{const}$ and $\lambda = \text{const} > 0$ (for the case $p = 2$ see [4], Theorem 263, and [1]).

Now, we consider the case where in (7) no limit conditions appear. We study the case where $u > 0$ in I. The case where $u < 0$ in I can be reduced to the previous case by assuming $-w$ instead of w. These cases occur most frequently. Further, we assume that $u > 0$ almost everywhere in I. In that case the integral condition (4) in the definition of $\tilde{W}$ is trivially satisfied.

We denote by $\hat{W}$ the class of functions $h \in \tilde{W}$ satisfying the following limit condition:

$$\limsup_{t\to\alpha} v|h|^p \geqslant \liminf_{t\to\beta} v|h|^p. \tag{20}$$

By Remark 1, condition (20) can be written in the form

$$\lim_{t\to\alpha} v|h|^p \geqslant \lim_{t\to\beta} v|h|^p. \tag{20'}$$

From Theorem 1 we easily obtain (see Theorem 2 in [2])

THEOREM 2. *Let $u > 0$ almost everywhere in the interval I. Then for an arbitrary function $h \in \hat{W}$ the inequality*

$$\int_I u|h|^p dt \leqslant p\Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I s|h|^p dt\Big)^{1/q} \tag{21}$$

holds. If $h \not\equiv 0$, then (21) becomes an equality if and only if

$$h = c\exp\Big(-\lambda \int_{t_0}^t w\,dt\Big),$$

where $t_0 \in I$, $c = \text{const} \neq 0$, $\lambda = \text{const} > 0$, and, simultaneously, the following conditions are satisfied:

$$\int_I r|w|^p \exp\Big(-p\lambda \int_{t_0}^t w\,dt\Big)dt < \infty, \tag{A}$$

(B)
$$-\infty < \lim_{t\to\alpha} r|w|^{p-1}(\operatorname{sgn} w)\exp\left(-p\lambda \int_{t_0}^{t} w\,dt\right) = \lim_{t\to\beta} r|w|^{p-1}(\operatorname{sgn} w)\exp\left(-p\lambda \int_{t_0}^{t} w\,dt\right) < \infty .$$

Inequalities of the form (21) are usually said to be of *Weyl type* (see [6]).

We describe the class $\hat{W}$. Denote by W_0 (resp. W^0) the class of functions $h \in W$ satisfying the following limit condition:

(22)
$$\liminf_{t\to\alpha} |h| = 0 \quad (\text{resp. } \liminf_{t\to\beta} |h| = 0).$$

In the cases considered in the sequel the condition (22) is equivalent to

(22′)
$$h(\alpha) = 0 \quad (\text{resp. } h(\beta) = 0).$$

The function v is increasing in I because $\dot{v} = u > 0$ in I. Thus, there are limits

$$\lim_{t\to\alpha} v = v(a) \quad \text{and} \quad \lim_{t\to\beta} v = v(\beta)$$

and $v(\alpha) < v(\beta)$.

We introduce the following terminology:

a boundary point α (resp. β) of the interval I is of the *I type* if $v(\alpha) \geqslant 0$ (resp. $v(\beta) \leqslant 0$);

a boundary point α (resp. β) of the interval I is of the *II type* if $v(\alpha) < 0$ (resp. $v(\beta) > 0$) and $\int_{\alpha}^{t} w\,dt = -\infty$ (resp. $\int_{t}^{\beta} w\,dt = \infty$) for some $t \in I$;

a boundary point α (resp. β) of the interval I is of the *III type* if $-\infty < v(\alpha) < 0$ (resp. $0 < v(\beta) < \infty$) and $\int_{\alpha}^{t} w\,dt > -\infty$ (resp. $\int_{t}^{\beta} w\,dt < \infty$) for some $t \in I$;

a boundary point α (resp. β) of the interval I is of the *IV type* if $v(\alpha) = -\infty$ (resp. $v(\beta) = \infty$), $v\left(\int_{\alpha}^{t} r^{-q/p}\,dt\right)^{p/q} = O(1)$ as $t \to \alpha$ (resp. $v\left(\int_{t}^{\beta} r^{-q/p}\,dt\right)^{p/q} = O(1)$ as $t \to \beta$), and $\int_{\alpha}^{t} w\,dt > -\infty$ (resp. $\int_{t}^{\beta} w\,dt < \infty$) for some $t \in I$.

We observe that the boundary points α and β cannot be both of the I type simultaneously because $v(\alpha) < v(\beta)$.

THEOREM 3. *Let $u > 0$ almost everywhere in the interval I.*

(i) *If the point α is of the I type and the point β is of the II type or α is of the II type and β is of the I or II type, then $\hat{W} = W$.*

(ii) *If the point α is of the III type and the point β is of the II type or α is of the IV type and β is of the I or II type, then $\hat{W} = W_0$.*

(iii) *If the point* α *is of the III type and the point* β *is of the I type, then* $\hat{W} \supset W_0$.

(iv) *If the point* α *is of the I type and the point* β *is of the IV type or* α *is of the II type and* β *is of the III or IV type, then* $\hat{W} = W_0$.

(v) *If the point* α *is of the I type and the point* β *is of the III type, then* $\hat{W} \supset W^0$.

(vi) *If both points* α *and* β *are of the III or IV type, then* $\hat{W} = W_0 \cap W^0$

Proof. If α is of the I type and $h \in abs\,C$, then $v|h|^p \geqslant 0$ in some neighbourhood of α because v is increasing in I. Hence

$$\limsup_{t \to \alpha} v|h|^p \geqslant 0.$$

If α is of the II type and $h \in W$, then it follows from Lemma 3 (i) that

$$\limsup_{t \to \alpha} v|h|^p = 0.$$

If α is of the III type and $h \in W_0$, then $v|h|^p \leqslant 0$ in some neighbourhood of α, and hence $\limsup_{t \to \alpha} v|h|^p = 0$ because $-\infty < v(\alpha) < 0$ and $\liminf_{t \to \alpha} |h| = 0$. If α is of the III type and β is of the II or III or IV type and $h \in \hat{W}$, then $\lim_{t \to \alpha} v|h|^p \leqslant 0$ and $\lim_{t \to \beta} v|h|^p \geqslant 0$ and it follows from (20′) that

$$\lim_{t \to \alpha} v|h|^p = 0.$$

Since $-\infty < v(\alpha) < 0$, the finite value $h(\alpha)$ exists and $h(\alpha) = 0$, e.g., $h \in W_0$. If α is of the IV type and $h \in W_0$, then it follows from Lemma 4 (ii) that $\lim_{t \to \alpha} v|h|^p = 0$. If α is of the IV type and $h \in \hat{W}$, then

$$\int_{\alpha}^{t} r^{-q/p}\,dt < \infty \qquad \text{for some } t \in I$$

and by Lemma 4 (i) there exists a finite value $h(\alpha)$. From Lemma 2 (ii) we infer that there exists a finite limit $\lim_{t \to \alpha} v|h|^p$ for $h \in \hat{W}$, and hence $h(\alpha) = 0$ because $v(\alpha) = -\infty$. Thus $h \in W_0$. Similar symmetric conclusions are valid if $\alpha(\beta)$ is replaced by $\beta(\alpha)$ and the class W_0 by W^0. Based on these considerations the theorem can be easily derived.

Corollary. (a) *Under the assumptions of Theorem* 3 (i), *inequality* (21) *becomes an equality for* $h \not\equiv 0$ *if and only if*

$$h = c\exp\left(-\lambda \int_{t_0}^{t} w\,dt\right),$$

where $t_0 \in I$, $c = \text{const} \neq 0$, $\lambda = \text{const} > 0$, *and* λ *satisfies condition* (A) *of Theorem* 2.

(b) *Under the assumptions of Theorem* 3 (ii) *or* 3 (iv) *or* 3 (vi), *inequality* (21) *is strict for* $h \not\equiv 0$.

Proof. From Theorem 2 it follows that (21) becomes an equality only for the function $h = c\exp(-\lambda \int_{t_0}^{t} w\,dt)$, where $\lambda > 0$ and $h \in \hat{W}$. If the assumptions of Theorem 3 (i) are satisfied, then $\hat{W} = W$ and, consequently, condition (A) of Theorem 2 implies condition (B), which proves (a).

Now, let the assumptions of Theorem 3 (ii) or 3 (vi) be satisfied. Then $\hat{W} \subset W_0$ and, simultaneously,

$$\int_{\alpha}^{t} w\,dt > -\infty \qquad \text{for } t \in I$$

since α is of the III or IV type. Hence $\lim_{t \to \alpha} \exp(-\lambda \int_{t_0}^{t} w\,dt) > 0$, and therefore

$$\exp(-\lambda \int_{t_0}^{t} w\,dt) \notin \hat{W} \qquad \text{for } \lambda > 0.$$

Similarly we show that if the assumptions of Theorem 3 (iv) are satisfied, then $\hat{W} = W^0$ and

$$\exp(-\lambda \int_{t_0}^{t} w\,dt) \notin W^0 \qquad \text{for } \lambda > 0,$$

which completes the proof of (b).

From the proofs of Theorems 2 and 3 as well as from the Corollary we easily obtain

THEOREM 4. *Let* $u > 0$ *almost everywhere in the interval* I.

(i) *If* $v(\beta) \leqslant 0$, *then for an arbitrary function* $h \in W$ *with the point* α *of the II type or for an arbitrary function* $h \in W_0$ *with the point* α *of the III or IV type the inequality*

$$\int_I u|h|^p dt - \lim_{t \to \beta} v|h|^p \leqslant p\Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I s|h|^p dt\Big)^{1/q} \tag{23}$$

holds. If α *is of the II type and* $h \not\equiv 0$, *then* (23) *becomes an equality if and only if* $h = c\exp(-\lambda \int_{t_0}^{t} w\,dt)$, *where* $t_0 \in I$, $c = \text{const} \neq 0$, $\lambda = \text{const} > 0$, *and the condition*

$$\int_I r|w|^p \exp(-p\lambda \int_{t_0}^{t} w\,dt)\,dt < \infty \tag{24}$$

is satisfied. If α *is of the III or IV type and* $h \not\equiv 0$, *then* (23) *is a strict inequality.*

(ii) *If $v(\alpha) \geqslant 0$, then for an arbitrary function $h \in W$ with the point β of the II type or for an arbitrary function $h \in W^0$ with the point β of the III or IV type the inequality*

$$\int_I u|h|^p dt + \lim_{t\to\alpha} v|h|^p \leqslant p\Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I s|h|^p dt\Big)^{1/q} \tag{25}$$

holds. If β is of the II type and $h \not\equiv 0$, then (25) *becomes an equality if and only if $h = c\exp(-\lambda \int_{t_0}^{t} w\,dt)$, where $t_0 \in I$, $c = \text{const} \neq 0$, $\lambda = \text{const} > 0$, and condition* (24) *is satisfied. If β is of III or IV type and $h \not\equiv 0$, then* (25) *is a strict inequality.*

Example 3. We take $I = (0, \infty)$ and $r = t^{pa}$. We put $w = t^{(a+b-pa)/(p-1)}$ if $a+b > 0$, and $w = -t^{(a+b-pa)/(p-1)}$ if $a+b < 0$, where a and b are arbitrary constants such that $a+b \neq 0$. From Theorems 2 and 3 we obtain the inequality

$$|a+b| \int_0^\infty t^{a+b-1}|h|^p dt \leqslant p\Big(\int_0^\infty t^{pa}|\dot{h}|^p dt\Big)^{1/p}\Big(\int_0^\infty t^{qb}|h|^p dt\Big)^{1/q} \tag{26}$$

which is valid for an arbitrary function $h \in \hat{W}$; and $\hat{W} = W$ if $a+b > 0$ and $(p-1)(1-a)+b \geqslant 0$ or $a+b < 0$ and $(p-1)(1-a)+b \leqslant 0$; $\hat{W} = W_0$ if $a+b < 0$ and $(p-1)(1-a)+b > 0$; and $\hat{W} = W^0$ if $a+b > 0$ and $(p-1)(1-a)+b < 0$. From the Corollary we infer that if $h \not\equiv 0$, then only in the cases $a+b > 0$ and $(p-1)(1-a)+b > 0$ or $a+b < 0$ and $(p-1)(1-a)+b < 0$ inequality (26) becomes an equality only for the function

$$h = c\exp\{-\lambda t^{[(p-1)(1-a)+b]/(p-1)}\},$$

where $c = \text{const} \neq 0$ and $\lambda = \text{const} > 0$. In the case $a+b > 0$, $(p-1)(1-a)+ +b \geqslant 0$, and $p = 2$ we obtain the inequalities considered in [6]. If $a = 0$, $b = 1$, and $p = 2$, we obtain the well-known Weyl inequality (cf. [8], p. 272, [4], Theorem 226, and [5], p. 128).

Now, we enlarge the class of considered functions r and w and we derive integral inequalities of the form (2).

Let $\alpha = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_n < x_{n+1} = \beta$ and let r and w be some given real functions which are defined and absolutely continuous in each of the open intervals (x_i, x_{i+1}), $i = 0, 1, \ldots, n$, and such that $w \not\equiv 0$ in one of those intervals, $r > 0$ and $v \equiv r|w|^{p-1}\operatorname{sgn} w$ is absolutely continuous in each of the intervals, and the limit conditions

$$\limsup_{t\to x_i-} v > -\infty, \qquad \liminf_{t\to x_i+} v < \infty, \qquad i = 1, \ldots, n, \tag{27}$$

are satisfied.

Let $v_j(x) \equiv \limsup_{t\to x-} v - \liminf_{t\to x+} v$ denote the jump of the function v at the point $x \in I$. It follows from (27) that $v_j(x_i) > -\infty$ for $i = 1, \ldots, n$. If $v_j(x_i) = \infty$, then we assume in the sequel that

$$v_j(x_i)|h(x_i)|^p = \begin{cases} 0 & \text{if } h(x_i) = 0, \\ \infty & \text{if } h(x_i) \neq 0. \end{cases}$$

Further, let W, $\tilde{W}$, W_0, and W^0 denote classes of the functions $h \in abs\, C$ defined as previously.

THEOREM 5. *For every function $h \in \tilde{W}$ the inequality*

$$(28) \quad \sum_{i=1}^{n} v_j(x_i)|h(x_i)|^p + \lim_{t\to\beta} v|h|^p - \lim_{t\to\alpha} v|h|^p \leqslant p\Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I s|h|^p dt\Big)^{1/q} + \int_I u|h|^p dt$$

holds. If $h \not\equiv 0$, then (28) *becomes an equality if and only if $h = c\varphi$, where $c = \text{const} \neq 0$ and $\varphi \not\equiv 0$ in I is an absolutely continuous function in I such that in each of the intervals (x_i, x_{i+1}), $i = 0, 1, \ldots, n$, we have*

$$\varphi \equiv c_i \exp\Big(\lambda \int_{t_i}^{t} w\, dt\Big),$$

where $t_i \in (x_i, x_{i+1})$ is an arbitrary fixed point, $c_i = \text{const}$, and $\lambda = \text{const} > 0$, provided the integral conditions

$$(29) \quad \begin{gathered} \int_{x_i}^{x_{i+1}} r|w|^p \exp\Big(p\lambda \int_{t_i}^{t} w\, dt\Big) dt < \infty, \\ \int_{x_i}^{x_{i+1}} \big|(r|w|^{p-1} \operatorname{sgn} w)^{\cdot}\big| \exp\Big(p\lambda \int_{t_i}^{t} w\, dt\Big) dt < \infty \end{gathered}$$

are satisfied or $\varphi \equiv 0$ otherwise.

Proof. Let $h \in \tilde{W}$. By (27) and Remark 2 (a) we have

$$(30) \quad \liminf_{t\to x_i+} v|h|^p < \infty, \qquad \limsup_{t\to x_{i+1}-} v|h|^p > -\infty$$

for $i = 0, 1, \ldots, n$. From Lemma 2 it follows that the limits in (30) exist as proper and finite ones. From the proof of Theorem 1 we get

$$(31) \quad \lim_{t\to x_{i+1}-} v|h|^p - \lim_{t\to x_i+} v|h|^p \leqslant \lambda^{1-p} \int_{x_i}^{x_{i+1}} r|\dot{h}|^p dt + (p-1)\lambda \int_{x_i}^{x_{i+1}} s|h|^p dt + \int_{x_i}^{x_{i+1}} u|h|^p dt$$

for $i = 0, 1, \ldots, n$, where λ is an arbitrary positive constant. Take an arbitrary $i = 1, \ldots, n$. By the continuity of h at the point x_i and from the existence of finite limits $\lim_{t\to x_i-} v|h|^p$ and $\lim_{t\to x_i+} v|h|^p$ it follows that if there exists $h\in\tilde{W}$ such that $h(x_i)\neq 0$, then there exist finite limits $\lim_{t\to x_i-} v$ and $\lim_{t\to x_i+} v$, and $v_j(x_i)<\infty$. Hence, in that case we have

$$\lim_{t\to x_i-} v|h|^p - \lim_{t\to x_i+} v|h|^p = v_j(x_i)|h(x_i)|^p. \tag{32}$$

If $h(x_i)=0$ for every $h\in\tilde{W}$, then in the case where $\lim_{t\to x_i-} v|h|^p\neq 0$ by (27) we have $\lim_{t\to x_i-} v=\infty$, and therefore

$$\lim_{t\to x_i-} v|h|^p > 0.$$

Similarly we deduce that

$$\lim_{t\to x_i+} v|h|^p \leqslant 0.$$

Thus in the considered case $v_j(x_i)=\infty$ and

$$\lim_{t\to x_i-} v|h|^p - \lim_{t\to x_i+} v|h|^p \geqslant 0 = v_j(x_i)|h(x_i)|^p. \tag{33}$$

Adding by sides inequalities (31) and using (32) and (33) we obtain

$$\sum_{i=1}^{n} v_j(x_i)|h(x_i)|^p + \lim_{t\to\beta} v|h|^p - \lim_{t\to\alpha} v|h|^p \leqslant \lambda^{1-p}\int_I r|\dot{h}|^p\,dt + (p-1)\lambda\int_I s|h|^p\,dt + \int_I u|h|^p\,dt \tag{34}$$

for $\lambda>0$. By (34), in an analogous way as in the proof of Theorem 1 we get inequality (28).

Now, let (28) be an equality for some function $h\in\tilde{W}$ and $h\not\equiv 0$. In that case also (34) becomes an equality with $\lambda=\lambda_h$, where $\lambda_h>0$ satisfies (11). Simultaneously, inequalities (31) hold with $\lambda=\lambda_h$. Thus all the inequalities (31) become equalities for that function h with $\lambda=\lambda_h$. From Theorem 1 it follows that

$$h = c_i\exp\left(\lambda_h\int_{t_i}^{t} w\,dt\right)$$

in the interval (x_i, x_{i+1}), where $t_i\in(x_i, x_{i+1})$ and $c_i=\text{const}\neq 0$, provided conditions (29) are satisfied for $\lambda=\lambda_h$ or $h=0$ in (x_i, x_{i+1}) if for $\lambda=\lambda_h$ at least one of the conditions (29) does not hold. The function h of the above-stated form satisfies (11) identically, and therefore $\lambda_h>0$ is an arbitrary constant. Now, it is easy to complete the proof.

We denote by $\check{W}$ the class of functions $h\in W$ satisfying the following limit condition:

$$\liminf_{t\to\alpha} v|h|^p \leqslant \limsup_{t\to\beta} v|h|^p. \tag{35}$$

From Theorem 5 we easily obtain (see Theorem 2)

THEOREM 6. *For every function $h \in \check{W}$ the inequality*

$$\sum_{i=1}^{n} v_j(x_i)|h(x_i)|^p \leqslant p\Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I s|h|^p dt\Big)^{1/q} + \int_I u|h|^p dt \tag{36}$$

holds. Inequality (36) *becomes an equality if and only if $h = c\varphi$, where c = const and φ is a function satisfying all the conditions of Theorem* 5 *and the additional condition*

$$\lim_{t\to\alpha} v|\varphi|^p = \lim_{t\to\beta} v|\varphi|^p.$$

In an interesting case where $u > 0$ almost everywhere in I the class $\check{W}$ can be described similarly as the class $\hat{W}$ (see Theorems 3 and 4) because by changing w into $-w$ the class $\hat{W}$ becomes $\check{W}$.

Example 4. Let $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$, and let $x \in I$ be an arbitrary fixed point. Let $r > 0$ be an arbitrary absolutely continuous function in I. We put $w = r^{-q/p}$ in (α, x) and $w = -r^{-q/p}$ in (x, β). In that case $u = 0$ in I and from Theorem 6 we obtain the inequality

$$|h(x)|^p \leqslant \frac{p}{2}\Big(\int_I r|\dot{h}|^p dt\Big)^{1/p}\Big(\int_I r^{-q/p}|h|^p dt\Big)^{1/q} \tag{37}$$

which is valid for $h \in \check{W}$. Using Theorem 3 we infer the following:

if $\int_\alpha^t r^{-q/p} dt = \infty$ and $\int_t^\beta r^{-q/p} dt = \infty$ for some $t \in I$, then $\check{W} = W$;

if $\int_\alpha^t r^{-q/p} dt < \infty$ and $\int_t^\beta r^{-q/p} dt = \infty$ for $t \in I$, then $\check{W} = W_0$;

if $\int_\alpha^t r^{-q/p} dt = \infty$ and $\int_t^\beta r^{-q/p} dt < \infty$ for $t \in I$, then $\check{W} = W^0$;

if $\int_I r^{-q/p} dt < \infty$, then $\check{W} = W_0 \cap W^0$.

If $h \not\equiv 0$, then only in the case

$$\int_\alpha^t r^{-q/p} dt = \infty \quad \text{and} \quad \int_t^\beta r^{-q/p} dt = \infty \quad \text{for } t \in I$$

inequality (37) becomes an equality if and only if

$$h = c\exp\Big(-\lambda\Big|\int_x^t r^{-q/p} dt\Big|\Big),$$

where $c = \text{const} \neq 0$ and $\lambda = \text{const} > 0$ provided

$$\int_I r^{1-q}\exp\Big(-p\lambda\Big|\int_x^t r^{-q/p} dt\Big|\Big) dt < \infty.$$

Taking $\alpha = -\infty$, $\beta = \infty$, and $r = 1$ we obtain the inequality

$$|h(x)|^p \leqslant \frac{p}{2}\Big(\int_{-\infty}^{\infty} |\dot{h}|^p dt\Big)^{1/p}\Big(\int_{-\infty}^{\infty} |h|^p dt\Big)^{1/q}, \qquad -\infty < x < \infty, \tag{38}$$

which is valid for an arbitrary function h absolutely continuous on $(-\infty, \infty)$ for which the right-hand side of the inequality is finite. Inequality (38) becomes an equality only for the function $h = ce^{-\lambda|t-x|}$, where $c = \text{const}$ and $\lambda = \text{const} > 0$.

Taking $\alpha = 0$, $\beta = \infty$, and $r = t^{p/q}$ we obtain for $0 < x < \infty$ the inequality

$$|h(x)|^p \leqslant \frac{p}{2}\Big(\int_{0}^{x} t^{p/q} |\dot{h}|^p dt\Big)^{1/p}\Big(\int_{0}^{\infty} t^{-1} |h|^p dt\Big)^{1/q} \tag{39}$$

which is valid for an arbitrary function h absolutely continuous on $(0, \infty)$ for which the right-hand side of (39) is finite. Inequality (39) becomes an equality only for the function $h = c(t/x)^{\lambda}$ for $t \in (0, x\rangle$ and $h = c(t/x)^{-\lambda}$ for $t \in (x, \infty)$, where $c = \text{const}$ and $\lambda = \text{const} > 0$.

Inequalities (38) and (39) were considered in [6] in the case $p = 2$.

REFERENCES

[1] D. C. Benson, *Inequalities involving integrals of functions and their derivatives*, Journal of Mathematical Analysis and Applications 17 (1967), p. 292-308.

[2] B. Florkiewicz, *Some integral inequalities of Hardy type*, Colloquium Mathematicum 43 (1980), p. 321-330.

[3] — and A. Rybarski, *Some integral inequalities of Sturm-Liouville type*, ibidem 36 (1976), p. 127-141.

[4] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, London 1951.

[5] D. S. Mitrinović, *Analytic inequalities*, Berlin 1970.

[6] R. Redheffer, *Inequalities with three functions*, Journal of Mathematical Analysis and Applications 16 (1966), p. 219-242.

[7] — *Integral inequalities with boundary terms*, p. 261-291 in: *Inequalities, II* (ed. O. Shisha), New York 1970.

[8] H. Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig 1928.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW

Reçu par la Rédaction le 2. 7. 1980;
en version modifiée le 1. 4. 1982