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HAMILTONIAN SHORTAGE, PATH PARTITIONS OF VERTICES, AND MATCHINGS IN A GRAPH

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1. Introduction. The main purpose of this paper is to give some new results concerning problems indicated in the title (some related results were published in [16]). We shall formulate conditions concerning an ordinary graph G_n (on n vertices) and depending upon a parameter s, study interrelations between them, and prove that each ensures the existence in G_n of a *path s-covering* of vertices, that is, a covering of vertices by s disjoint paths if $s \geqslant 1$ or by a Hamiltonian circuit if $s = 0$, where

$$n > s \geqslant 0, \quad n+s \geqslant 3, \quad n, s \in \boldsymbol{Z}. \tag{1.1}$$

In what follows we shall show how to prove that each of those conditions is in a sense best possible. Moreover, we shall formulate three new conditions and find all implications between all conditions. The weakest condition is of Las Vergnas type. It coincides with that of Las Vergnas for $s \leqslant 1$ and is essentially weaker than the condition of Chvátal type, the second weakest condition. Finally, we shall prove that all those conditions ensure the existence of a matching in G_n with a given deficiency and that, for $s \geqslant 1$ and odd $n-s$, they are sharp.

2. Measures of non-hamiltonity. Results of [16] will be put here in a new setting. For that purpose consider the notion of a *vertex-path partition number* of a graph G. This number, denote it by $\pi_0^{(P)}(G)$, is the minimal number of disjoint paths (possibly trivial) in G which contain all vertices of G. In fact, a path s-covering with $s \geqslant 1$ is a spanning subgraph with s components each of which is a path.

So, for $s \geqslant 1$, it is clear that each of the conditions mentioned in Introduction implies that $\pi_0^{(P)}(G) \leqslant s$. Each of those conditions with $s = 0$, however, ensures the existence of a Hamiltonian circuit in G_n.

A *Hamiltonian circuit* was called in [16] a path 0-covering of vertices. To avoid this, one can consider the parameter s in the above-mentioned conditions as an upper bound for the number of new edges whose addition to G_n in the case $n \geqslant 3$ results in a Hamiltonian supergraph. This suggests introducing a new invariant which we shall denote by $s_{\mathrm{H}}(G)$ and call the ***Hamiltonian shortage*** in a graph G. The Hamiltonian shortage $s_{\mathrm{H}}(G)$ in G is the minimal number of new edges in a Hamiltonian spanning supergraph of G if the order $|V(G)|$ of G exceeds 2. Naturally, the Hamiltonian shortage in a Hamiltonian graph is equal to 0. There are three exceptional graphs, namely K_1, K_2, and $\bar{K}_2$, in which the Hamiltonian shortage is not defined yet. Therefore, we assume additionally that

$$s_{\mathrm{H}}(K_2) = 1 \quad \text{and} \quad s_{\mathrm{H}}(K_1) = s_{\mathrm{H}}(\bar{K}_2) = 2.$$

Thus we have the following relations between the vertex-path partition number $\pi_0^{(P)}$ and the Hamiltonian shortage s_{H} in a non-empty graph G:

$$s_{\mathrm{H}}(G) = \begin{cases} \pi_0^{(P)}(G) - 1 = 0 & \text{if } G \text{ is Hamiltonian}, \\ \pi_0^{(P)}(G) + 1 = 2 & \text{if } G = K_1, \\ \pi_0^{(P)}(G) & \text{otherwise}. \end{cases}$$

One may doubt whether the definition $s_{\mathrm{H}}(K_1) = 2$ is reasonable. There is, however, another natural definition of $s_{\mathrm{H}}(G)$, compatible with that given above. Assume that, for two given graphs $G^{(1)}$ and $G^{(2)}$, we can write $G^{(1)} * G^{(2)}$ only if $G^{(1)}$ and $G^{(2)}$ are disjoint, and that it stands for the *join* of $G^{(1)}$ and $G^{(2)}$ (defined by A. A. Zykov in 1949, cf. Harary [9]), i.e., for the graph consisting of both $G^{(1)}$ and $G^{(2)}$ and of all possible edges with one end-vertex in $G^{(1)}$ and the other in $G^{(2)}$. Now the above--defined Hamiltonian shortage in a graph G satisfies the condition

$$s_{\mathrm{H}}(G) = \min\{p\colon\ G * K_p \text{ is a Hamiltonian graph}\}$$

and this equality can be taken as a simple, uniform, and natural definition of s_{H} (cf. [17]).

Note that the Hamiltonian shortage is a parameter analogous to the *Hamiltonian index* introduced by Chartrand in 1965 (cf. [6] and [7]). Definition of the latter parameter is implied by the observation that, for a connected graph $G = G_n$ different from a path (so with $n \geqslant 3$), the k-th iterated line graph $L^k(G)$ of G is Hamiltonian for all integers $k \geqslant n-3$, with L^0 being the identity operator. The minimal exponent k for which $L^k(G)$ is a Hamiltonian graph is the value of the parameter. We shall denote it by $\log_L(\mathfrak{H}/G)$ and call it the (minimal) ***Hamiltonian L-exponent*** of G.

Now, replacing L by another unary operator on graphs (e.g., raising to a power) or considering a property different from that of being Hamiltonian which above is denoted by $\mathfrak{H}$, one can introduce new analogous invariants. We do not follow this idea here but remark that, for the ***total graph operator*** T: $G \mapsto T(G)$ ($T(G)$ denotes the ***total graph*** of a graph G, introduced by M. Behzad in 1965, cf. Harary [9], Chapter 8), the ***Hamiltonian T-exponent*** of a connected and non-trivial graph G does not exceed 2, i.e.,

$$\log_T(\mathfrak{H}/G) \leqslant 2 \quad \text{if } G \text{ is connected and } G \neq K_1.$$

This is a new formulation of a result of Behzad and Chartrand [1].

Each of the three quasi-Hamiltonian invariants (s_H and the two exponents) attains 0 precisely for Hamiltonian graphs. However, only s_H has a definite value for any graph, while the two exponents may have not. So, from among the three parameters only the Hamiltonian shortage s_H (and for a large part also the vertex-path partition number $\pi_0^{(P)}$) yields a satisfactory classification of all ordinary graphs (connected and disconnected) with respect to Hamiltonian (or rather quasi-Hamiltonian) properties. Also, only the Hamiltonian shortage in an arbitrary graph can be viewed as a measure of non-hamiltonity of that graph.

However, one more (at least) measure of the non-hamiltonity of a graph G can be introduced in another natural way. This measure can be determined by means of an invariant defined as the minimal number of hanging vertices (with degree 1 or 0) among all spanning forests of G. Upper bounds for the last invariant (in the case where the graph G is connected) follow from results of Las Vergnas [12] and Bermond [4].

3. Terminology and notation. By a *graph* we mean a finite ordinary graph. So a graph G is an ordered pair $\langle V, E\rangle$ consisting of two finite sets: V (the *vertex set* $V(G)$ of G) and E (the *edge set* $E(G)$ of G), where

$$V \cap \mathscr{P}_2(V) = \emptyset \quad \text{and} \quad E \subseteq \mathscr{P}_2(V),$$

$\mathscr{P}_2(V)$ being the set of all two-member subsets of V. The cardinality $n = |V|$ of the vertex set $V(G)$ and the cardinality $|E|$ of the edge set $E(G)$ are said to be the *order* of G and the *size* of G, respectively. If n is the order of G, we write G_n instead of G. The graph $\langle V, \mathscr{P}_2(V)\rangle$ is called *complete* and is denoted by K, $\langle V\rangle$, or $\langle G\rangle$ in the case where $V(K) = V(G)$. It is also denoted either by K_n or by $\langle n\rangle$ if n is its order.

The terms subgraph, supergraph, factor (i.e., spanning subgraph), k-factor, Hamiltonian graph, complementary graph of a graph G (denoted by $\bar{G}$), disjoint graphs, union, join, isomorphism of graphs etc. will be used in the usual sense. However, the terms *path* and *circuit* are names of graphs consisting of vertices and edges in a simple open chain and in a simple closed chain, respectively. In connection with this convention, note that a Hamiltonian path of a graph is not any Hamiltonian graph.

If $x, y \in V$, we write xy (or yx) to denote the edge $\{x, y\}$. The *degree* of a vertex x in G is denoted by $d(x, G)$. The *minimal degree* among all vertices of G is denoted by $\delta(G)$.

The symbols $\boldsymbol{Z}$ and $\boldsymbol{R}$ denote the set of integers and the set of real numbers, respectively. Given a number $x \in \boldsymbol{R}$, the symbols $[x]$ and $[^*x]$ stand for the integer nearest to x in the sets $\{y\colon\ y \leqslant x\}$ and $\{y\colon\ y \geqslant x\}$, respectively. Thus, $[^*x] = -[-x]$.

4. Conditions on graphs. At present there are known many non--trivial sufficient conditions for an ordinary graph G_n to contain a Hamiltonian circuit (for $n \geqslant 3$) or a Hamiltonian path (for $n \geqslant 2$). Eleven types of such conditions (including two new ones, both being modifications of the known conditions due to Erdös and Ore, respectively) are considered in [16]. Each of those conditions appears, however, as a special case (namely, with $s = 0$ or $s = 1$) of a certain new condition that depends upon an additional parameter s. Recall that (1.1) is assumed throughout this paper.

In some conditions one more additional numerical parameter k appears, and so all conditions from [16] are uniformly denoted by $A_{ns}^{(r)}(k)$ (or by $A_{ns}^{(r)}(k; G_n)$ if G_n is not specified), where r is a distinguishing parameter, $r = 1, 2, \ldots, 11$. The parameter k appears only if $r = 1, 2, 4$. In what follows we shall quote some of those conditions together with formally new conditions $A_{ns}^{(\varrho)}$, $\varrho = 1.0, 2.0, 4.0$, each of which is a generalized or-connection of conditions $A_{ns}^{(r)}(k)$ with different values of k ($r = 1, 2, 4$, respectively). The conditions we shall deal with will be denoted by $A_{ns}^{(\varrho)}$ with $\varrho = 01, 02, 1.0, 2.0, 3, 4.0, 5, 6, \ldots, 11$. In conditions $A_{ns}^{(\varrho)}$ the variable k will not be free: for $\varrho = 1.0, 2.0, 4.0$ it will be bounded by the existential quantifier $\exists$. We shall not quote the conditions with $r = \varrho = 5, 6, \ldots, 9$; we shall only give some information about them.

To quote conditions $A_{ns}^{(r)}(k)$ with $r = 1, 2$ we need to recall the following definitions:

$$\varphi_{ns}(t) := 1 + \binom{n-t-s}{2} + (t+s)t,$$

$$\varkappa \equiv \varkappa_{ns} := [(n-s-1)/2] \quad (\text{hence } \max\{1-s, 0\} \leqslant \varkappa),$$

$$\bar{k}_s := \max\{1-s, 0, [^*k]\} \quad \text{with } k \in \boldsymbol{R},$$

$$\Phi_{ns}(k) := \begin{cases} \max\{\varphi_{ns}(t) \mid \bar{k}_s \leqslant t \leqslant \varkappa\} & \text{if } \bar{k}_s \leqslant \varkappa, \\ 0 & \text{otherwise}, \end{cases}$$

$$\Phi_{ns}^*(k) := \begin{cases} \max\{\varphi_{ns}(t) + k - t \mid k \leqslant t \leqslant \varkappa\} & \text{if } k \leqslant \varkappa, \\ 0 & \text{otherwise}, \end{cases}$$

$$a \equiv a_{ns} := \begin{cases} (n-5s+1)/6 & \text{for odd } n+s, \\ (n-5s+4)/6 & \text{otherwise}. \end{cases}$$

Now one can prove (see [16]) that

$$a \neq \varkappa, \quad [a] \leqslant \varkappa, \quad [{}^{*}a] = [a+4/6],$$

$$a+4/6 \neq \varkappa \quad \text{and} \quad [a+4/6] \leqslant \varkappa \quad \text{iff } n+s \neq 4;$$

$$\Phi_{ns}(k) = \Phi_{ns}(\bar{k}_s).$$

Moreover,

$$\Phi_{ns}(k) = \varphi_{ns}(\tau) \text{ and } \tau \geqslant k \Leftrightarrow \tau \in \begin{cases} \{\bar{k}_s\} & \text{if } \bar{k}_s < a, \\ \{a, \varkappa\} & \text{if } \bar{k}_s = a, \\ \{\varkappa\} & \text{if } a < \bar{k}_s \leqslant \varkappa \end{cases}$$

and, analogously,

$$\Phi^{*}_{ns}(k) = \varphi_{ns}(\tau) + k - \tau \text{ and } \tau \geqslant k \Leftrightarrow$$

$$\Leftrightarrow \tau \in \begin{cases} \{k\} & \text{if } k < \min\{a+4/6, \varkappa\}, \\ \{a+4/6, \varkappa\} & \text{if } k = a+4/6 \leqslant \varkappa, \\ \{\varkappa\} & \text{if } a+4/6 < k \leqslant \varkappa. \end{cases}$$

Hence, for $k \in \boldsymbol{Z}$ and $k \geqslant \max\{2-s, 0\}$, one can easily deduce that either

(4.1) $$\Phi^{*}_{ns}(k) < \Phi_{ns}(k) \equiv \varphi_{ns}(\varkappa)$$

if

(4.2) $$\max\{1-s, -1, a\} < k < \varkappa$$

or

$$\Phi^{*}_{ns}(k) = \Phi_{ns}(k) \quad \text{otherwise}.$$

It is easy to prove the following equivalences:

An integer k satisfying (4.2) *does exist* $\Leftrightarrow$

$$\Leftrightarrow \varkappa \geqslant \max\{3-s, 1\} \textit{ and } a+1 < \varkappa$$

$$\Leftrightarrow n-s \geqslant 3 \textit{ and } 7 \leqslant n+s \neq 8.$$

Now we are able to formulate the conditions

$$A^{(1)}_{ns}(k;\, G_n)\colon \big(\forall x \in V(G_n)\colon d(x, G_n) \geqslant k \wedge |E(G_n)| \geqslant \Phi_{ns}(k)\big) \quad (\boldsymbol{R} \ni k \leqslant n-1),$$

$$A^{(2)}_{ns}(k;\, G_n)\colon \delta(G_n) = k \wedge |E(G_n)| \geqslant \Phi^{*}_{ns}(k) \quad (k \in \boldsymbol{Z} \cap [\max\{2-s, 0\}, n-1]).$$

Remark. For a given integer $k \geqslant \max\{2-s, 0\}$, if k satisfies (4.2) (cf. (4.1)), then the increase of information about G_n in the first logical factor of $A_{ns}^{(2)}(k;\ G_n)$, compared with that of $A_{ns}^{(1)}(k;\ G_n)$, is followed by the decrease of information in the second factor.

For certain values of k, one can bring the condition $A_{ns}^{(r)}(k)$ $(r = 1, 2)$ to a *simplified form*, in [16] denoted by $\tilde{A}_{ns}^{(r)}(k)$, by omitting an unessential logical factor. For instance, we can put

$$A_{ns}^{(01)}(G_n) := \tilde{A}_{ns}^{(1)}\big((n-s)/2; G_n\big):\ \big(\forall x \in V(G_n):\ d(x, G_n) \geqslant (n-s)/2\big),$$

$$A_{ns}^{(02)}(G_n) := \tilde{A}_{ns}^{(1)}(0; G_n):\ |E(G_n)| \geqslant \Phi_{ns}(0).$$

In connection with the latter condition, notice that in [16] it is proved that

$$\Phi_{ns}(0) = \Phi_{ns}(k) = \varphi_{ns}(1-s) = \binom{n-1}{2} + 2 - s \quad \text{if } 0 \leqslant s \leqslant 1 \text{ and } k \leqslant 1-s. \tag{4.3}$$

We have denoted the above conditions by 01 and 02 to obtain a uniform numbering of all conditions $A_{ns}^{(\varrho)}$ and to keep (if possible) the notation the same as that in [16]; it is precisely the same for $\varrho = 3, 5, 6, \ldots, 11$.

Now define

$$A_{ns}^{(1.0)}(G_n):\ \big(\exists k \leqslant n-1:\ A_{ns}^{(1)}(k; G_n)\big);$$

$$A_{ns}^{(2.0)}(G_n):\ \big(\exists k \in \boldsymbol{Z} \cap [\max\{2-s, 0\}, n-1]:\ A_{ns}^{(2)}(k; G_n)\big);$$

$$A_{ns}^{(3)}(G_n):\ \big(\forall x, y \in V(G_n):\ xy \notin E(G_n) \Rightarrow d(x, G_n) + d(y, G_n) \geqslant a_{ns}^{(3)}\big)$$

with

$$a_{ns}^{(3)} = \begin{cases} n-s-1 & \text{if } s \geqslant 2 \text{ and } n-s \text{ is even,} \\ n-s & \text{otherwise;} \end{cases}$$

$$A_{ns}^{(4.0)}(G_n):\ \big(\exists k \leqslant n:\ A_{ns}^{(4)}(k; G_n)\big),$$

where

$$A_{ns}^{(4)}(k; G_n):\ |\{x \in V(G_n):\ d(x, G_n) = n-1\}|$$
$$\geqslant k^+ \wedge \big(k \leqslant \varkappa \Rightarrow \forall x, y \in V(G_n):\ xy \notin E(G_n) \Rightarrow d(x, G_n) + d(y, G_n) \geqslant a_{ns}^{(4)}(k)\big)$$

with

$$k^+ := \max\{0, k\}$$

and

$$a_{ns}^{(4)}(k) = \begin{cases} n-s-1 & \text{if } s + k^+ > 1 \text{ and } n-s \text{ is even,} \\ n-s & \text{otherwise.} \end{cases}$$

In all conditions with $r \geqslant 6$, an *arrangement* ψ of vertices of G_n is involved. Therein, ψ is a bijection

$$\{1, 2, \ldots, n\} \to V(G_n),$$

$\psi(i)$ being denoted by v_i $(i = 1, 2, \ldots, n)$, and ψ itself by (v_i). All conditions to be considered can be uniformly formulated by the use of the following *scheme of conditions* Ω'_n dependent upon a further condition, say $\hat{\omega}_{ns}$, on G_n and (v_i):

$A_{ns}^{(r)}(G_n)$ is defined to be the condition

$$\Omega'_n(G_n, \hat{\omega}_{ns}^{(r)}): \big(\exists(v_i): \hat{\omega}_{ns}^{(r)}(G_n, (v_i))\big),$$

where

$$\hat{\omega}_{ns}^{(r)} := \begin{cases} d(v_1, G_n) \leqslant d(v_2, G_n) \leqslant \ldots \leqslant d(v_n, G_n) \wedge \omega_{ns}^{(r)} & \text{for } 6 \leqslant r \leqslant 10, \\ \omega_{ns}^{(r)} & \text{for } r = 11 \end{cases}$$

and, in particular,

$$\omega_{ns}^{(10)}(G_n, (v_i)): \big(\forall i \in \boldsymbol{Z}: \max\{1, s\} \leqslant i < (n+s)/2 \Rightarrow$$
$$\Rightarrow d(v_i, G_n) > i-s \text{ or } d(v_{n+s-i}, G_n) \geqslant n-i\big).$$

The above definitions of $A_{ns}^{(r)}$ with $6 \leqslant r \leqslant 10$ can be restated. In fact, the scheme Ω'_n can be replaced by another one, say Ω_n, which is obtained from Ω'_n by substituting the phrase $\forall(v_i)$ for the phrase $\exists(v_i)$. However, the last condition (with $r = 11$) does not enjoy this property. Namely, we have

$$A_{ns}^{(11)}(G_n): \big(\exists(v_i)\forall i, j \in \boldsymbol{Z}: \max\{1, s\} \leqslant i < j \leqslant n,\ j \geqslant n+s-i,$$
$$d(v_i, G_n) \leqslant i-s,\ d(v_j, G_n) \leqslant j-s-1,\ v_i v_j \notin E(G_n) \Rightarrow$$
$$\Rightarrow d(v_i, G_n) + d(v_j, G_n) \geqslant n-s\big).$$

As stated at the beginning of this section, most of the conditions in question, for $s = 0$ and $s = 1$ or for $s = 0$ only, coincide (see references in Sachs [15] and Berge [2], [3]) with known conditions due to Dirac (1952, for $\varrho = 01$ and $s = 0$), Ore (1961, for $\varrho = 02$ and $0 \leqslant s \leqslant 1$ and for $r = 3$ and $s = 1$; 1960, for $r = 3$ and $s = 0$), Erdös (1962, for $\varrho = 1.0$ and $s = 0$), Pósa (1962, for $r = 5$ and $s = 0$), Bondy (1969, for $r = 7$ and $s = 0$, cf. Corollary 1.1 (with a misprint, however) in Bondy [5]), Bondy (and Nash-Williams) (1969-1970, for $r = 8$ and $s = 0$), Chvátal (1972, for $r = 10$ and $0 \leqslant s \leqslant 1$), and Las Vergnas (1971, for $r = 11$ and $s \leqslant 1$). Moreover, the condition with $r = 8$ and $s = 0$, that was found by Bondy and Nash-Williams (cf. Berge [2], p. 199), is a slight refinement of the original condition of Bondy [5]. Similarly, the condition with $r = 9$ and $s = 0$ is an analogous refinement of the original condition formulated (with errata, however) in a paper by Wojda and the present author [19]. Both conditions $A_{ns}^{(6)}$ and $A_{ns}^{(7)}$ are reformulations of the condition of Pósa type $A_{ns}^{(5)}$.

5. Interrelations between conditions. Some interrelations (implications) between the conditions with $0 \leqslant s \leqslant 1$ are obvious, some others are already proved or can be proved, and most of those implications can be extended for all $s \geqslant 0$. This is done in [16] in Lemmas 1-9. Implications between 13 different conditions $A_{ns}^{(\varrho)}(G_n)$ with any admissible integers n and s satisfying (1.1) ($\varrho = 01, 02, 1.0, \ldots, 11$) can be visualized by means of the digraph D shown in Fig. 1.

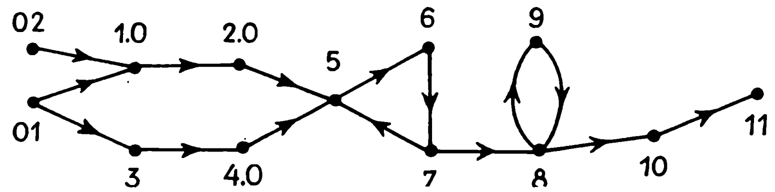

Fig. 1. The digraph D

Vertices and arcs of D represent conditions and implications between them, respectively. For the sake of simplicity, many arcs are omitted. However, the digraph which represents all implications between the conditions can easily be obtained. It is the digraph, call it $\hat{D}$, obtainable from the transitive closure of the digraph D by adding loops so that each vertex is the end-vertex of exactly one loop. The following conclusion is of great importance:

(5.1) *Each condition* $A_{ns}^{(\varrho)}(G_n)$ ($\varrho = 01, 02, 1.0, \ldots, 10$) *implies the condition of Las Vergnas type* $A_{ns}^{(11)}(G_n)$.

One can give examples of graphs which show that the implications corresponding to non-loop arcs in the complementary 1-digraph of $\hat{D}$ do not hold true for some admissible values of n and s. All of those implications, excepting two ones which correspond to arcs $02 \to 3$ and $02 \to 4.0$, are not true also in the most interesting case, namely, for $s = 0$. In fact, a graph which may be taken as a counterexample to any one of such implications with $s = 0$ can have at most 7 vertices (see Table 1, where C_n and P_n denote a circuit with n vertices and a path with n vertices, respectively, and $G_1 * G_2 * G_3 := (G_1 * G_2) \cup (G_2 * G_3)$).

Using formula (4.3) one can easily obtain Ore's result which says (see [14]) that, for $0 \leqslant s \leqslant 1$, the condition with $\varrho = 02$ implies that with $\varrho = 3$. However, the implication is not true for $s \geqslant 2$. On the other hand, the implication

$$A_{ns}^{(4.0)}(G_n) \Rightarrow A_{ns}^{(3)}(G_n)$$

is clearly true for $s \geqslant 2$ (and any admissible n) but is not true for $0 \leqslant s \leqslant 1$ and even $n+s \geqslant 6$.

Graphs which satisfy only the condition of Las Vergnas type (i.e., the condition with $\varrho = 11$) for any $n \geqslant s+5$ ($s = 0, 1, \ldots$) are given in [17]. So, in the case $n \geqslant s+5 \geqslant 5$ the condition of Las Vergnas type is essentially weaker than any other condition in question.

In [16] there were studied interrelations between conditions $A_{ns}^{(r)}(k; G_n)$ and the operation of joining the complete graph K_p to G_n, that is, between the conditions $A_{ns}^{(r)}(k; G_n)$ and $A_{n+p,s-p}^{(r)}(k+p; G_n * K_p)$ with $n > s \geqslant p \geqslant 0$, $p \in \boldsymbol{Z}$.

Table 1

Implications (arcs)	Counterexamples (for $s = 0$)
$01 \to 02$	C_4
$02 \to 01$	$K_1 * K_2 * K_2$
$1.0 \to 4.0$	$\bar{K}_2 * K_3 * K_2$; $K_1 * K_2 * (K_2 * \bar{K}_2)$
$2.0 \to 1.0$	$K_1 * K_2 * C_4$ (9 graphs with $n = 7$)
$3 \to 2.0$	$K_1 * \bar{K}_2 * K_2$
$4.0 \to 3$	$K_1 * K_2 * P_3$
$8 \to 5$	$K_1 * K_2 * K_2 * K_1$
$10 \to 8$	C_6 with an inscribed triangle C_3
$11 \to 10$	C_5 with 1 diagonal

Some of the results formulated in Lemmas 10_r of [16] can be summarized as follows.

If $r = 2, 4, 5, \ldots, 11$ (i.e., $r \neq 1$ and $r \neq 3$) and $k \leqslant n-1$ except for $r = 4$ when $k \leqslant n$, then

$$A_{ns}^{(r)}(k; G_n) \Leftrightarrow A_{n+p,s-p}^{(r)}(k+p; G_n * K_p)$$

and, furthermore,

$$A_{ns}^{(1)}(k; G_n) \Leftrightarrow A_{ns}^{(1)}(\bar{k}_s; G_n) \Leftrightarrow A_{n+p,s-p}^{(1)}(\bar{k}_s + p; G_n * K_p) \qquad \forall k \leqslant n-1$$

with $\bar{k}_s = \max\{1-s, 0, [^*k]\}$.

Hence one can easily obtain the equivalences

$$A_{ns}^{(\varrho)}(G_n) \Leftrightarrow A_{n+p,s-p}^{(\varrho)}(G_n * K_p) \qquad \text{for } \varrho \neq 02, 3.$$

The situation in cases $\varrho = 02$ and $\varrho = 3$ is a little involved. However, only the case where $\varrho = 11$ is of importance for our purposes. Indeed, owing to (5.1) and the above equivalence with $\varrho = 11$ and $p = s$, we have the following proposition:

$$\forall \varrho: \ A_{ns}^{(\varrho)}(G_n) \Rightarrow A_{n+s,0}^{(11)}(G_n * K_s). \tag{5.2}$$

6. Upper bounds for Hamiltonian shortage. It is well known that the condition $A_{n,0}^{(11)}(G_n)$ (with $s = 0$) is sufficient for a graph G_n to be Hamiltonian (see Las Vergnas [12], cf. also [20]). Therefore, by (5.2), each of the conditions $A_{ns}^{(\varrho)}(G_n)$ implies that the graph $G_n * K_s$ is Hamiltonian, which is clearly equivalent to the following implication:

(6.1) *Each condition* $A_{ns}^{(\varrho)}(G_n)$ *implies the inequality*

$$s_H(G_n) \leqslant s. \tag{6.2}$$

Hence we obtain the following upper bounds for the Hamiltonian shortage $s_H(G_n)$ in a graph G_n with n ($n \geqslant 2$) vertices:

$$s_H(G_n) \leqslant \min\big(\{n\} \cup \{s \in \{0, 1, \ldots, n-1\} \mid A_{ns}^{(\varrho)}(G_n)\}\big), \tag{6.3}$$
where $\varrho = 01, 02, 1.0, \ldots, 11$.

By (5.1), the smallest upper bound is that for $\varrho = 11$. For certain values of ϱ, the upper bounds can be expressed more explicitly, e.g.,

$$s_H(G_n) \leqslant \begin{cases} \max\{0, n-2\,\delta(G_n)\} & (\varrho = 01),\ G_n \neq K_1, K_2, \\ \max\{0, w(G_n)\} & (\varrho = 3), \end{cases}$$

where, for $u := \min\{d(x, G_n) + d(y, G_n) \mid xy \notin E(G_n)\}$ if $G \neq K_n$, and $u := 2n-3$ if $G = K_n$,

$$w(G_n) = \begin{cases} n-u-1 & \text{for odd } u \leqslant n-3, \\ n-u & \text{otherwise}. \end{cases} \tag{6.4}$$

There is one more upper bound for $s_H(G)$. To quote it, let $a(G)$ and $\varkappa(G)$ be the *independence number* of G and the *connectivity* of G, respectively. In [17] it is proved that the inequality

$$s_H(G) \leqslant \max\{0, a(G) - \varkappa(G)\} \quad \text{for } G \neq K_1, K_2$$

is equivalent to the following *theorem of Chvátal and Erdös* [8] (see also Berge [3], p. 213):

$$a(G) \leqslant \varkappa(G) \text{ and } G \neq K_2 \text{ implies that } s_H(G) = 0,$$

i.e., that the graph G is Hamiltonian.

7. Sharpness of conditions. One can show, by means of examples of graphs, that each condition $A_{ns}^{(\varrho)}(G_n)$ is in a sense best possible as the sufficient condition for inequality (6.2). For instance, the condition $A_{ns}^{(10)}$ of Chvátal type is best possible in the following sense.

The *degree sequence* (being the non-decreasing sequence of degrees of vertices) of any graph G_n which does not satisfy the condition $A_{ns}^{(10)}$ (with n and s satisfying (1.1)) is majorized by a non-decreasing sequence

$$\underbrace{t, \ldots, t}_{t+s \text{ times}}, \underbrace{n-s-t-1, \ldots, n-s-t-1}_{n-s-2t \text{ times}}, \underbrace{n-1, \ldots, n-1}_{t \text{ times}}, \tag{7.1}$$

where $\max\{1-s, 0\} \leqslant t < (n-s)/2$.

Moreover, (7.1) is the degree sequence of the unique graph, say $G(t)$,

(7.2) $$G(t) \equiv G(t; n, s) := \bar{K}_{t+s} * K_t * K_{n-s-2t}$$

with $s_H(G(t)) \geqslant s+1$ (factually, $s_H(G(t)) = s+1$).

Hence, putting $t = i - s$ in (7.1) and considering the definition of $A_{ns}^{(10)}$, we obtain the following conclusion with $\varrho = 10$:

(7.3) *Any essentially weakened condition obtained by changing only numerical bounds which appear in the condition $A_{ns}^{(\varrho)}$ is not sufficient for inequality* (6.2).

Proposition (7.3) can be proved for all ϱ ($\varrho = 01, 02, 1.0, \ldots, 11$).

(7.4) The examples of graphs needed for the proof of (7.3) can be found among graphs $G(t; n, s)$ (see (7.2)) and, only for $\varrho = 2.0$, also among their spanning subgraphs.

This follows from properties of $G(t)$, e.g., we have

$$\delta(G(t)) = t < (n-s)/2, \qquad |E(G(t))| = \varphi_{ns}(t) - 1,$$

the minimal sum of degrees of two non-adjacent vertices of $G(t)$ is equal to

$$\begin{cases} n-s-1 & \text{if } t+s = 1, \\ 2t < n-s & \text{otherwise.} \end{cases}$$

Note that the statement "the condition, say A, which is sufficient for the property, say P, of graphs is in a sense best possible" or "sharp" is often used in the literature on graph theory. The following general definition related to Proposition (7.3) seems to be satisfactory in general:

(7.5) A condition A which is sufficient for a property P is best possible if any condition A', which has the same logical structure as that of A and is essentially weaker than A, is not sufficient for the property P. The statement that the logical structure of A' is the same as that of A means usually that A' is obtained by changing some numerical bounds which appear in A.

8. Sufficient conditions for a given deficiency. It is worthy of noting that graphs $G(t)$ appeared already in paper [4] by Bermond. He proved that each $G(t; n, s)$ with $t > 0$ is a maximal (connected) graph which has no Hamiltonian cycle and no spanning tree with less than $s+2$ hanging vertices.

For odd $n-s$, graphs $G(t; n, s)$ have also another interesting property. Namely, each of them is a maximal graph with the deficiency $s+1$. The *deficiency* of a graph G, denoted by $a(G)$, is the minimal number of vertices whose deletion results in a subgraph with a perfect matching (i.e., with a 1-factor).

It is obvious that

(8.1) $$a(G_n) \equiv n \pmod 2.$$

One can prove the following proposition:

(8.2) *Assume that* $s \equiv n \pmod 2$. *Then*

(8.3) $$A^{(\varrho)}_{n,s+1}(G_n) \Rightarrow a(G_n) \leqslant s \quad \textit{for } \varrho = 01, 02, 1.0, \ldots, 11$$

and, moreover, each $A^{(\varrho)}_{n,s+1}(G_n)$ (*with* $s+1$ *instead of* s) *is the best possible sufficient condition* (*in the sense of* (7.5)) *for the inequality* $a(G_n) \leqslant s$.

To sketch a proof, assume that $A^{(\varrho)}_{n,s+1}(G_n)$ holds true for a graph G_n with n and s satisfying (1.1) and such that $s \equiv n \pmod 2$ (so $0 \leqslant s \leqslant n-2$ and $n \geqslant 3$). Hence, by (6.1), $s_{\mathrm{H}}(G_n) \leqslant s+1$.

Therefore, there is a partition of vertices of the graph G_n into $s+1$ paths. However, if $s \equiv n \pmod 2$, then any such partition contains a path L_0 of odd length (i.e., with even $|V(L_0)|$). Hence $a(G_n) \leqslant s$ since, for any path L, the deficiency $a(L)$ is 0 or 1 according to whether $|V(L)|$ is even or odd. Thus (8.3) is proved.

Further, from Mader's description of maximal graphs with a given deficiency (see [13], cf. also [18]) we infer that each graph $G(t; n, s+1)$ is a maximal graph with n vertices and the deficiency $s+2$. Hence (cf. (7.4)) we obtain the second part of Proposition (8.2).

Moreover, we have proved the inequality

$$a(G_n) \leqslant \begin{cases} s_{\mathrm{H}}(G_n) & \text{if } s_{\mathrm{H}}(G_n) \equiv n \pmod 2, \\ \max\{\varepsilon_n, s_{\mathrm{H}}(G_n)-1\} & \text{otherwise,} \end{cases}$$

where

(8.4) $$\varepsilon_n := \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

By (6.1), we have also the following generalization of implication (8.3):

$$A^{(\varrho)}_{ns}(G_n) \Rightarrow a(G_n) \leqslant \begin{cases} s & \text{if } s \equiv n \pmod 2, \\ 1 & \text{if } s = 0 \text{ and } n \text{ is odd,} \\ s-1 & \text{otherwise.} \end{cases}$$

Now, by (8.1) and (8.2), we have the following analogue to inequality (6.3):

$$a(G_n) \leqslant \min\big(\{n\} \cup \{s \mid 0 \leqslant s < n,\ s \equiv n \pmod 2, \text{ and } A^{(\varrho)}_{n,s+1}(G_n)\}\big).$$

Hence, for ε_n defined by (8.4),

$$a(G_n) \leqslant \begin{cases} \max\{\varepsilon_n, n-2\,\delta(G_n)\} & (\varrho = 01), \\ \max\{\varepsilon_n, w(G_n)\} & (\varrho = 3), \end{cases}$$

where $w(G_n)$ is defined by (6.4). Owing to (8.1), $w(G_n)$ in the last inequality can be replaced by 1 if $w(G_n) = 2$ and n is odd.

Remark. After completing this paper, I obtained manuscripts of new papers by Jolivet. One of them (see [10]) contains some closely related results on partitions of vertices into paths. Furthermore, in [10] and [11] Jolivet has introduced and investigated a new interesting measure of that a connected graph G of order not less than 3 is not Hamiltonian. This invariant is defined as the difference between the length of a Hamiltonian pseudo-cycle of the graph G and the order of G.

Added in proof. Las Vergnas studied partitions of vertices into paths and proved (see [12a], Chapitre IV, §4, Proposition 3 and Corollaire 2 to Proposition 3) that $A^{(r)}_{n,n-2q+1}(G_n)$ for $r = 10, 11$ are sufficient conditions for G_n to contain a matching of size q.

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