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CLIFFORD ALGEBRAS AND THE HIGHER DIMENSIONAL CAUCHY INTEGRAL

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1. Introduction

Clifford algebras were introduced over one hundred years ago in an attempt by Clifford to develop higher dimensional number systems analogous to the real and complex numbers. They have subsequently been used in various guises by mathematical physicists. In recent years however, they have been found to be useful in a number of different areas of mathematics. One particular application is in proving the L_2-boundedness of certain singular integral operators.

In this paper we shall first show how complex analysis can be extended to higher dimensions in the context of Clifford algebras. The material is taken from the book [1] by Brackx, Delanghe and Sommen. In particular, we shall discuss the higher-dimensional Cauchy integral. We shall then show that the Cauchy singular integral operator on Lipschitz surfaces is L_2-bounded. A corollary is that the double layer potential operator on Lipschitz surfaces is L_2-bounded. This result was proved previously using the Calderón rotation method in [2]. The idea of proving it directly using Clifford algebras is due to R. Coifman, and carried through by M. Murray for surfaces with small Lipschitz constants [6].

I would like to thank Professor Ciesielski for giving me the opportunity of presenting this material at the Banach Center. I would also like to thank J. Picton-Warlow for writing up an earlier version of parts of these notes.

2. Clifford algebras

The vector space $\mathbf{R}^{n+1}$ is embedded in a 2^n-dimensional algebra $\mathbf{R}_{(n)}$ as follows. Let $e_0, e_1, \ldots, e_n$ be the standard basis of $\mathbf{R}^{n+1}$ and denote the basis vectors of $\boldsymbol{R}_{(n)}$ by e_S, where S is any subset of $\{1, 2, \ldots, n\}$. Make the identifications $e_0 = e_\emptyset$ and $e_j = e_{\{j\}}$ for $1 \leqslant j \leqslant n$, and define the multiplication on $\boldsymbol{R}_{(n)}$ by

$$\begin{aligned}
&e_0 = 1,\\
&e_j^2 = -1 \text{ for } 1 \leqslant j \leqslant n,\\
&e_j e_k = -e_k e_j = e_{\{j,k\}} \text{ for } 1 \leqslant j < k \leqslant n, \text{ and}\\
&e_{j_1} e_{j_2} \ldots e_{j_s} = e_S \text{ if } 1 \leqslant j_1 < j_2 < \ldots < j_s \leqslant n \text{ and } S = \{j_1, \ldots, j_s\}.
\end{aligned}$$

The product of two elements $\lambda = \sum_S \lambda_S e_S$ and $\mu = \sum_S \mu_S e_S$, λ_S, $\mu_S \in \boldsymbol{R}$, is $\lambda\mu = \sum_{S,R} \lambda_S \mu_R e_S e_R$. Note that $e_S e_R$ is again a basis vector of $\boldsymbol{R}_{(n)}$. The term $\lambda_\emptyset$, also denoted λ_0, is called the *scalar part* of λ.

The Clifford algebras $\boldsymbol{R}_{(0)}$, $\boldsymbol{R}_{(1)}$ and $\boldsymbol{R}_{(2)}$ are the real numbers, complex numbers and quaternions respectively. An important property of these algebras is that every non-zero element has an inverse. This is not the case for the algebras $\boldsymbol{R}_{(n)}$ when $n \geqslant 3$. An important reason for considering Clifford algebras however is that every non-zero element in $\boldsymbol{R}^{n+1}$ does have an inverse. The inverse of the vector $x = x_0 e_0 + x_1 e_1 + \ldots + x_n e_n \in \boldsymbol{R}^{n+1}$ is defined in terms of the *conjugate vector* $\bar{x} = x_0 e_0 - x_1 e_1 - \ldots - x_n e_n$ as we shall now see.

PROPOSITION 1. *Let* $x, y \in \boldsymbol{R}^{n+1} \subset \boldsymbol{R}_{(n)}$. *Then*

(i) $\bar{x}y = \langle x, y\rangle e_0 + \sum_{0 \leqslant j < k \leqslant n} e_k e_j (x_j y_k - x_k y_j)$,

(ii) $x\bar{x} = \bar{x}x = |x|^2$,

(iii) *if* $x \neq 0$ *then* x *is invertible and* $x^{-1} = \bar{x}|x|^{-2}$.

3. Clifford analysis

The results of this section are taken from the book [1] by Brackx, Delanghe and Sommen, to which we refer the reader for details.

Let Ω denote an open subset of $\boldsymbol{R}^{n+1}$ and consider C^1-functions

$$f\colon \Omega \to \boldsymbol{R}_{(n)}.$$

Define

$$D = \sum_{j=0}^{n} \frac{\partial}{\partial x_j} e_j$$

acting on such f by

$$Df = \sum_{j=0}^{n} \sum_S \frac{\partial f_S}{\partial x_j} e_j e_S, \quad \text{where} \quad f = \sum_S f_S e_S.$$

By analogy with previous usage we define

$$\bar{D} = \frac{\partial}{\partial x_0} e_0 - \frac{\partial}{\partial x_1} e_1 - \ldots - \frac{\partial}{\partial x_n} e_n,$$

and note that

$$D\bar{D} = \bar{D}D = \left(\sum_{j=0}^{n} \frac{\partial^2}{\partial x_j^2}\right) e_0 = \Delta.$$

We define f to be *(left)-monogenic* if $Df = 0$. We remark that this is an elliptic system of 2^n equations in 2^n unknowns, so every monogenic function has (real)-analytic components. Indeed somewhat more follows from the identity $\bar{D}D = \Delta$.

PROPOSITION 2. *If f is monogenic, then each component f_S is harmonic.*

In the special case when $n = 1$, $\boldsymbol{R}^{1+1} = \boldsymbol{R}_{(1)} = \boldsymbol{C}$, and

$$\begin{aligned} Df &= \left(\frac{\partial}{\partial x_0} e_0 + \frac{\partial}{\partial x_1} e_1\right)(f_0 e_0 + f_1 e_1) \\ &= \left(\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1}\right) e_0 + \left(\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0}\right) e_1. \end{aligned}$$

So $Df = 0$ if and only if the Cauchy–Riemann equations are satisfied. That is, f is monogenic if and only if f is holomorphic.

Examples of monogenic functions are

$$g_y(x) = \frac{\overline{x-y}}{|x-y|^{n+1}}, \qquad x \neq y.$$

This can be verified directly, or deduced as follows. If $n \geqslant 2$, then

$$Dg_y(x) = (1-n)^{-1} D\bar{D}\left(\frac{1}{|x-y|^{n-1}}\right) = (1-n)^{-1} \Delta \left(\frac{1}{|x-y|^{n-1}}\right) = 0.$$

If $n = 1$, then $g_y(x) = (x-y)^{-1}$.

Further examples can be constructed as follows. Let Σ be a smooth n-dimensional oriented submanifold of $\boldsymbol{R}^{n+1}$, let $n(y)$ be a consistent unit normal at $y \in \Sigma$, and let u be an absolutely integrable function from Σ to $\boldsymbol{R}$ or $\boldsymbol{R}_{(n)}$. For $x \notin \Sigma$ we define

$$(\mathscr{T}_\Sigma u)(x) = \frac{1}{\sigma_n} \int_\Sigma \frac{\overline{y-x}}{|y-x|^{n+1}} n(y) u(y) \, dS_y,$$

where σ_n is the volume of the unit n-sphere in $\boldsymbol{R}^{n+1}$. Monogenicity follows by differentiating under the integral sign.

A generalization of Cauchy's theorem can now be stated. See [1] for its proof and numerous consequences.

THEOREM. *Let Ω be a bounded open subset of $\boldsymbol{R}^{n+1}$ with smooth boundary Σ and an exterior unit normal $n(y)$ defined at each point $y \in \Sigma$. If f is* *monogenic on a neighbourhood of* $\bar{\Omega} = \Omega \cup \Sigma$ *then*

$$(\mathscr{T}_\Sigma f)(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \boldsymbol{R}^{n+1} \sim \bar{\Omega}. \end{cases}$$

Because of this theorem we call $\mathscr{T}_\Sigma$ the *Cauchy operator*. Let us look at its components. By part (i) of Proposition 1 we see that

$$(\mathscr{T}_\Sigma u)(x) = \frac{1}{\sigma_n} \int_\Sigma \frac{\langle y-x, n(y) \rangle}{|y-x|^{n+1}} u(y)\, dS_y$$

$$+ \sum_{0 \leq j < k \leq n} e_k e_j \frac{1}{\sigma_n} \int_\Sigma \frac{(y-x)_j n_k - (y-x)_k n_j}{|y-x|^{n+1}} u(y)\, dS_y.$$

By Proposition 2, each component is a harmonic function of $x \in \boldsymbol{R}^{n+1} \sim \Sigma$. For the first term this is no surprise, for as we can see, the scalar part $(\mathscr{T}_\Sigma)_0$ of $\mathscr{T}_\Sigma$ is none other than the *double layer potential operator*.

Our interest in this paper is actually with the singular integral operators defined on Σ. We take the *Cauchy singular integral operator* to be T_Σ where

$$(T_\Sigma u)(x) = \frac{2}{\sigma_n} \text{ p.v. } \int_\Sigma \frac{\overline{y-x}}{|y-x|^{n+1}} n(y)\, u(y)\, dS_y$$

for $x \in \Sigma$, whenever the principal value integral exists. Its scalar part $(T_\Sigma)_0$ is the singular double layer potential operator,

$$((T_\Sigma)_0\, u)(x) = \frac{2}{\sigma_n} \text{p.v. } \int_\Sigma \frac{\langle y-x, n(y) \rangle}{|y-x|^{n+1}} u(y)\, dS_y.$$

4. Harmonic analysis on $\boldsymbol{R}^n$ and self-adjoint operators

We first introduce some notation.

If X is any Banach space, defined over either the real or complex field, we let $X_{(n)} = X \otimes \boldsymbol{R}_{(n)}$. This means that

$$X_{(n)} = \{x = \sum_S x_S e_S \mid x_S \in X\}$$

with norm $\|x\| = \{\sum_S \|x_S\|^2\}^{1/2}$. If X is a Hilbert space, then so is $X_{(n)}$, where the inner product is $\langle x, y \rangle = \sum_S \langle x_S, y_S \rangle$.

In order to study harmonic analysis on $\boldsymbol{R}^n$, we regard $\boldsymbol{R}^n$ as the subspace of $\boldsymbol{R}_{(n)}$ spanned by $e_1, e_2, \ldots, e_n$, and introduce the Hilbert space $H = L_2(\boldsymbol{R}^n)_{(n)}$ and the Dirac operator

$$\boldsymbol{D} = \sum_{j=1}^n D_j e_j = \sum_{j=1}^n \frac{\partial}{\partial x_j} e_j$$

defined by

$$Du = D\Big(\sum_S u_S e_S\Big) = \sum_{j=1}^{n} \sum_S \frac{\partial u_S}{\partial x_j} e_j e_S.$$

It is not difficult to show that D is a self-adjoint operator in H with domain $H^1(\boldsymbol{R}^n)_{(n)}$, where $H^1(\boldsymbol{R}^n)$ is the first order Sobolev space.

The important thing is that we have replaced the n-tuple of operators $\left(\frac{\partial}{\partial x_j}\right)$ by the self-adjoint operator D, and so can use the functional calculus of self-adjoint operators.

For example, the signum function gives rise to the self-adjoint operator $\operatorname{sgn} D \in \mathscr{L}(H)$. Proceeding formally, we have that

$$\operatorname{sgn} D = |D|^{-1} D = (D^2)^{-1/2} D.$$

Now

$$D^2 = \Big(\sum_{j=1}^{n} D_j e_j\Big)\Big(\sum_{k=1}^{n} D_k e_k\Big) = -\Delta$$

where

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

So

$$\operatorname{sgn} D = \sum_{j=1}^{n} (-\Delta)^{-1/2} D_j e_j = \sum_{j=1}^{n} R_j e_j$$

where R_j is the jth Riesz transform. That is,

$$(\operatorname{sgn} D)\, u(x) = -\sum_{j=1}^{n} e_j \frac{2}{\sigma_n} \text{p.v.} \int_{\boldsymbol{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} u(y)\, dy.$$

Compare this with the Cauchy operator defined at the end of Section 3 with $\Sigma = \boldsymbol{R}^n$ and $n(y) = -e_0$ for all $y \in \boldsymbol{R}^n$. We have the identity

$$\operatorname{sgn} D = T_{\boldsymbol{R}^n}.$$

This is a surprising generalization to higher dimensions of the well-known fact that when $n = 1$ (and $\boldsymbol{R}_{(1)} = \boldsymbol{C}$), $\operatorname{sgn} D$ is the Hilbert transform (suitably normalized).

It is also possible to represent $\mathscr{T}_{\boldsymbol{R}^n}$ using the functional calculus of D. Let E_+ and E_- be the spectral projections defined by $E_+ = \frac{1}{2}(I + \operatorname{sgn} D)$ and $E_- = \frac{1}{2}(I - \operatorname{sgn} D)$. Let $R_\tau = (I + i\tau D)^{-1}$ for $\tau \in \boldsymbol{R}$. Then, for $u \in L_2(\boldsymbol{R}^n)_{(n)}$

$$\frac{i}{2\pi} \int_{-\infty}^{\infty} R_\tau e^{-it/\tau} u \frac{d\tau}{\tau} = \begin{cases} e^{-tD} E_+ u =: u_+(t), & t > 0, \\ \frac{1}{2}(\operatorname{sgn} D) u = \frac{1}{2}(u_+(0) - u_-(0)), & t = 0, \\ -e^{-tD} E_- u =: -u_-(t), & t < 0, \end{cases}$$

where the integral in the case $t = 0$ is computed using the principal value. So $u_+(t)$ is defined for $t \geqslant 0$, $u_-(t)$ for $t \leqslant 0$, $u_+(0)+u_-(0) = u$, and

$$\frac{du_\pm}{dt}(t)+Du_\pm(t) = 0, \quad t \neq 0.$$

That is, $u_\pm(t)(x)$ are monogenic functions of $t+x = te_0+\sum x_j e_j$. It is not difficult to verify that

$$u_\pm(t)(x) = \pm(\mathscr{T}_{\mathbb{R}^n} u)(t+x), \quad t \neq 0.$$

5. Non self-adjoint operators

Let us develop the operator theory needed in studying Cauchy integrals on Lipschitz graphs.

Let T be a closed densely-defined linear operator in a complex Hilbert space H which is one-to-one, has spectrum $\sigma(T)$ in a double sector S_w where $0 \leqslant w < \frac{1}{2}\pi$ and

$$S_w = \{z \in \mathbb{C} \mid |\arg z| \leqslant w \text{ or } |\arg(-z)| \leqslant w\},$$

and satisfies $\|(I+itT)^{-1}\| \leqslant M < \infty$ for all real t.

Suppose moreover that T satisfies the quadratic estimates

$$\text{(Q)} \qquad \begin{aligned} \left\{\int_0^\infty \|\Psi(tT)u\|^2 \frac{dt}{t}\right\}^{1/2} &\leqslant \varkappa \|u\|, \\ \left\{\int_0^\infty \|\Psi(tT^*)u\|^2 \frac{dt}{t}\right\}^{1/2} &\leqslant \varkappa \|u\| \end{aligned}$$

where

$$\Psi(tT) = tT(I+t^2T^2)^{-1} = \tfrac{1}{2}i\{(I+itT)^{-1}-(I-itT)^{-1}\}.$$

We remark that $\|\Psi(tT)\| \leqslant M$.

Self-adjoint operators which are one-to-one satisfy all of the above conditions with $w = 0$, $M = 1$ and $\varkappa = 1/\sqrt{2}$. The results on self-adjoint operators used in the last section generalize to operators T as specified above, except that the projections E_+ and E_- are not orthogonal projections.

For example, consider the holomorphic function $\operatorname{sgn} z$ defined on $S_w \sim \{0\}$ by $\operatorname{sgn} z = +1$ if $\operatorname{Re} z > 0$ and $\operatorname{sgn} z = -1$ if $\operatorname{Re} z < 0$. This function satisfies

$$\operatorname{sgn} z = \frac{16}{\pi}\int_0^\infty \Psi^3(tz)\frac{dt}{t}.$$

So we can define $\operatorname{sgn} T$ by

$$\langle (\operatorname{sgn} T)u, v\rangle = \frac{16}{\pi}\int_0^\infty \langle \Psi^3(tT)u, v\rangle \frac{dt}{t}$$

for $u, v \in H$ provided the right-hand side makes sense. This is a consequence of (Q):

$$\begin{aligned}\int_0^\infty |\langle \Psi^3(tT)u, v\rangle| \frac{dt}{t} &\leqslant \sup_t \|\Psi(tT)\| \left\{\int_0^\infty \|\Psi(tT)u\|^2 \frac{dt}{t}\right\}^{1/2} \\ &\quad \times \left\{\int_0^\infty \|\Psi(tT^*)v\|^2 \frac{dt}{t}\right\}^{1/2} \\ &\leqslant M\varkappa^2 \|u\| \, \|v\|.\end{aligned}$$

As before, we define $E_+ = \frac{1}{2}(I + \operatorname{sgn} T)$ and $E_- = \frac{1}{2}(I - \operatorname{sgn} T)$. These can be thought of as spectral projections associated with each piece of S_w. They are bounded but not orthogonal projections.

As in Section 4 we can also define, for each $u \in H$, elements

$$u_+(t) = \exp(-tT)E_+ u \in H \quad \text{for } t \geqslant 0$$

and

$$u_-(t) = \exp(-tT)E_- u \in H \quad \text{for } t \leqslant 0$$

which satisfy

$$\frac{du_\pm}{dt}(t) + Tu_\pm(t) = 0, \quad t \neq 0.$$

The boundedness of the operators $\exp(-tT)E_\pm$ also follows from (Q). For a discussion of the relationship between quadratic estimates and the boundedness of $f(T)$ for H_∞-functions f, see [5].

We next give conditions on operators of the form $T = (I-B)^{-1}A$ which imply that they are of the above type.

THEOREM. *Suppose A is a one-to-one self-adjoint operator and B is a bounded skew-adjoint operator, and let $T = (I-B)^{-1}A$. Also let*

$$P_t = (I + t^2A^2)^{-1}, \quad Q_t = tA(I + t^2A^2)^{-1} \quad \text{and } B_\lambda = (1-\lambda)I + \lambda B$$

for $t \in \mathbf{R}$ and $0 < \lambda \leqslant 1$. Assume, for some constants $c < \infty$, $m < \infty$, that

$$\left\{\int_0^\infty \|Q_t(B_\lambda P_t)^k u\|^2 \frac{dt}{t}\right\}^{1/2} \leqslant c(1+k^m)\|B_\lambda\|^k \|u\|$$

and

$$\left\{\int_0^\infty \|Q_t(B_\lambda^* P_t)^k u\|^2 \frac{dt}{t}\right\}^{1/2} \leqslant c(1+k^m)\|B_\lambda\|^k \|u\|$$

for all $u \in H$, $\lambda \in (0, 1]$ *and* $k = 0, 1, 2, \ldots$ *Then there exist* $w \in [0, \frac{1}{2}\pi)$, $M \in \mathbf{R}$ *and* $\varkappa \in \mathbf{R}$ *(depending only on* c, m *and* $\|B\|$*) such that* $\sigma(T) \subset S_w$, $\|(I+itT)^{-1}\| \leqslant M$ *for all real* t, *and condition* (Q) *holds.*

Proof. First suppose $\|B\| < 1$, in which case we take $\lambda = 1$. (Here there is no need to invoke the assumption that $B = -B^*$.) Then

$$\begin{aligned}(I+itT)^{-1} &= \big(I+it(I-B)^{-1}A\big)^{-1} = (I-B+itA)^{-1}(I-B)\\ &= (I+itA)^{-1}\big(I-B(I+itA)^{-1}\big)^{-1}(I-B)\\ &= R_t \sum_{k=0}^{\infty} (BR_t)^k (I-B)\end{aligned}$$

where

$$R_t = (I+itA)^{-1} = P_t - iQ_t$$

and the series converges because $\|R_t\| \leqslant 1$ and $\|B\| < 1$. Indeed

$$\|(I+itT)^{-1}\| \leqslant (1+\|B\|)/(1-\|B\|).$$

Therefore

$$\begin{aligned}\Psi(tT) &= \frac{i}{2}\{(I+itT)^{-1} - (I-itT)^{-1}\}\\ &= \frac{i}{2}\Big\{R_t \sum_{k=0}^{\infty}(BR_t)^k - R_{-t}\sum_{k=0}^{\infty}(BR_{-t})^k\Big\}(I-B)\\ &= \tfrac{1}{2}\sum_{k=0}^{\infty}\sum_{s=0}^{k}\{(R_tB)^{k-s}Q_t(BP_t)^s + (R_{-t}B)^{k-s}Q_t(BP_t)^s\}(I-B).\end{aligned}$$

Hence

$$\begin{aligned}\Big\{\int_0^\infty \|\Psi(tT)u\|^2\frac{dt}{t}\Big\}^{1/2} &\leqslant \sum_{k=0}^{\infty}\sum_{s=0}^{k}\|B\|^{k-s}\Big\{\int_0^\infty \|Q_t(BP_t)^s(I-B)u\|^2\frac{dt}{t}\Big\}^{1/2}\\ &\leqslant \sum_{k=0}^{\infty}\sum_{s=0}^{k}\|B\|^k c(1+s^m)\,\|I-B\|\,\|u\| = \varkappa\|u\|\end{aligned}$$

as required. The dual estimate is proved similarly. We leave it to the reader to show that the spectrum belongs to a double sector.

It remains for us to remove the restriction that $\|B\| < 1$. Recall that $B_\lambda = (1-\lambda)I + \lambda B$, so $I - B_\lambda = \lambda(I-B)$, and

$$(I+itT)^{-1} = \big(I+it\lambda(I-B_\lambda)^{-1}A\big)^{-1}.$$

We can now proceed as before (with t replaced by λt and B replaced by B_λ) provided that we can find $\lambda \in (0, 1]$ for which $\|B\| < 1$. Indeed it suffices to take $\lambda = (1+\|B\|^2)^{-1}$, for then (using the skew-adjointness of B)

$$\|B_\lambda\| = \{(1-\lambda)^2 + \lambda^2\|B\|^2\}^{1/2} = \sqrt{1-\lambda} < 1.$$

6. The Cauchy integral on a Lipschitz graph

Let Σ be the graph of a Lipschitz function g: $\boldsymbol{R}^n \to \boldsymbol{R}$. It is our aim to show that the Cauchy singular integral operator T_Σ defined at the end of Section 3 is L_2-bounded. An immediate corollary is that the singular double-layer potential operator is also L_2-bounded, for it is the scalar part of T_Σ.

When $n = 1$, this result was first proved in [2] and [3]. The result for $n > 1$ can be reduced to one-dimensional estimates using the Calderón rotation method as indicated in [2]. Our aim here is to show that an analogue of the one-dimensional proof actually holds in higher dimensions, provided Clifford algebras are used. The idea of using Clifford algebras is R. Coifman's, though it was M. Murray who first proved this result in the case of graphs with small Lipschitz constants [6].

Coifman noted the following remarkable identity:

$$T_\Sigma = \operatorname{sgn} T$$

where $T = (I-B)^{-1}\boldsymbol{D}$, and B denotes multiplication by $b = \boldsymbol{D}g = \sum_{j=1}^n b_j e_j$ with $b_j = \partial g/\partial x_j \in L_\infty(\boldsymbol{R}^n)$. Observe that B is a bounded skew-adjoint operator and $\boldsymbol{D}$ is a one-to-one self-adjoint operator in $H = L_2(\boldsymbol{R}^n)_{(n)}$, so $\sigma(T) \subset S_w$ for some $w < \frac{1}{2}\pi$ and $\|(I+itT)^{-1}\| \leqslant M < \infty$. So $\operatorname{sgn} T$ has at least a formal meaning. Its boundedness is a consequence of the quadratic estimates (Q) which are in turn a consequence of the estimates stated in the Theorem in Section 5, with $A = \boldsymbol{D}$.

These estimates will be proved in Section 7.

We conclude this section with a brief discussion of the identity $T_\Sigma = \operatorname{sgn} T$ with $T = (I-B)^{-1}\boldsymbol{D}$. Recall from Section 4 that it is true in the special case when $g = 0$.

In the middle of Section 5 we showed, given $u \in H = L_2(\boldsymbol{R}^n)_{(n)}$, how to construct functions $u_+(t)$ for $t \geqslant 0$ and $u_-(t)$ for $t \leqslant 0$ which satisfy

$$\begin{cases} u_+(0) + u_-(0) = u, \\ u_+(0) - u_-(0) = (\operatorname{sgn} T)u, \\ \dfrac{du_\pm}{dt}(t) + Tu_\pm(t) = 0, \qquad t \neq 0. \end{cases}$$

This final equation can be rewritten as

$$(1-b)\frac{du_\pm}{dt}(t) + \boldsymbol{D}u_\pm(t) = 0$$

or

$$\frac{\partial u_\pm}{\partial t}(t, x) + \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} e_j - b_j e_j \frac{\partial}{\partial t}\right) u_\pm(t, x) = 0.$$

Using the change of variable $x_0 = t+g(\mathbf{x})$, and writing $x = x_0 e_0 + \mathbf{x}$, this becomes

$$\frac{\partial u_\pm}{\partial x_0}(x) + \sum_{j=1}^{n} \frac{\partial u_\pm}{\partial x_j}(x)\, e_j = 0$$

or

$$Du_\pm(x) = 0, \qquad x_0 \neq g(\mathbf{x}).$$

In other words, $u_\pm$ are monogenic functions on $\mathbf{R}^{n+1} \sim \Sigma$.

By analogy with Section 4, this leads us to guess that

$$\frac{i}{2\pi} \int_{-\infty}^{\infty} (I + i\tau T)^{-1} e^{-it/\tau} u \frac{d\tau}{\tau}(\mathbf{x})$$

$$= \begin{cases} u_+(t, \mathbf{x}) = (\mathscr{T}_\Sigma u)(g(\mathbf{x})+t+\mathbf{x}), & t > 0, \\ \frac{1}{2}(\operatorname{sgn} T) u(\mathbf{x}) = \frac{1}{2}(T_\Sigma u)(g(\mathbf{x})+\mathbf{x}), & t = 0, \\ -u_-(t, \mathbf{x}) = (\mathscr{T}_\Sigma u)(g(\mathbf{x})+t+\mathbf{x}), & t < 0 \end{cases}$$

which provides an explanation, if not a proof, of the formula $T_\Sigma = \operatorname{sgn} T$.

7. Square-function estimates

We now present a proof of the estimates stated in the Theorem in Section 5, in the case when $H = L_2(\mathbf{R}^n)_{(n)}$, $A = \mathbf{D}$ and B denotes multiplication by a function $b = \sum_{j=1}^{n} b_j e_j$ with $b_j \in L_\infty(\mathbf{R}^n, \mathbf{R})$. Actually we note that B_λ and B_λ^* are also of this form with the summation going from 0 to n, so in the sequel we use the symbol B for such an operator. Elements of $\mathbf{R}^n$ are denoted by x rather than $\mathbf{x}$.

The proof uses a combination of ideas from [2] and [3], and from [4].

The following spaces and norms will be used. Each space consists of equivalence classes of measurable functions for which the corresponding norm is finite.

(i) $L_2 = L_2(\mathbf{R}^n)$; $\|u\|_2$,

(ii) $L_{\infty,2} = L_{\infty,2}(\mathbf{R}_+^{n+1})$; $\|u\|_{\infty,2} = \big\|\sup_{t>0} |u(x,t)|\big\|_{L_2(dx)}$,

(iii) $L_{2,2} = L_2\left(\mathbf{R}_+^{n+1}, \dfrac{dx\,dt}{t}\right)$; $\|u\|_{2,2}$,

(iv) $T_{\infty,2} = T_{\infty,2}(\mathbf{R}_+^{n+1})$; $\|u\|_{\infty,2} = \big\|\sup_{|y-x|\leqslant t} |u(y,t)|\big\|_{L_2(dx)}$.

Each space can be embedded in its corresponding "Cliffordized" space. For example, $L_2 \subset L_{2(n)}$, $L_{\infty,2} \subset L_{\infty,2(n)}$, etc.

We note the estimates for $P_t = (I + t^2 \mathbf{D}^2)^{-1}$, $Q_t = t\mathbf{D}(I + t^2 \mathbf{D}^2)^{-1}$ and $R_t = P_t - iQ_t = (I + it\mathbf{D})^{-1}$:

$$\|P_t u\|_2 \leqslant \|u\|_2, \qquad \|Q_t u\|_2 \leqslant \tfrac{1}{2}\|u\|_2, \qquad \|R_t u\|_2 \leqslant \|u\|_2,$$

$$\|Q_t u\|_{2,2} = \left\{\int_0^\infty \|Q_t u\|_2^2 \frac{dt}{t}\right\}^{1/2} \leqslant \left\|\int_0^\infty Q_t^2 \frac{dt}{t}\right\|^{1/2} \|u\|_2 = \frac{1}{\sqrt{2}}\|u\|_2.$$

Let B denote multiplication by $b = \sum_{j=0}^n b_j e_j$ with $b_j \in L_\infty(\boldsymbol{R}^n, \boldsymbol{R})$.

THEOREM.

$$Q_t(BP_t)^k \in \mathscr{L}(L_{2(n)}, L_{2,2(n)})$$

and

$$\left\{\int_0^\infty \|Q_t(BP_t)^k u\|_2^2 \frac{dt}{t}\right\}^{1/2} \leqslant c_n(1+k)\|b\|_\infty^k \|u\|_2 \quad \textit{for } k = 0, 1, 2, \ldots$$

Proof. Let $M = \|B\| = \|b\|_\infty$. If

$$Q_t = \sum_{j=1}^n Q_{t,j} e_j,$$

then

$$Q_{t,j}(BP_t)^k = (P_t B)^k Q_{t,j} + \sum_{r=0}^{k-1} (P_t B)^{k-r-1}(Q_{t,j} BP_t - P_t BQ_{t,j})(BP_t)^r.$$

The result follows from the facts that

$$\|P_t\|_{L_{2,2}\to L_{2,2}} = 1, \quad \|Q_{j,t}\|_{L_2\to L_{2,2}} = 1/\sqrt{2},$$
$$\|P_t B\|_{L_{2,2(n)}\to L_{2,2(n)}} \leqslant M,$$

together with the estimates (which still need to be proved):

(i) $$\|(BP_t)^r u\|_{\infty,2} \leqslant c_n M^r \|u\|_2, \quad u \in L_{2(n)},$$

(ii) $$\|(Q_{j,t} BP_t - P_t BQ_{j,t})u\|_{2,2} \leqslant c_n M \|u\|_{\infty,2}, \quad u \in L_{\infty,2}.$$

To prove (i), let $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$, where $\hat\varphi(\xi) = (1+|\xi|^2)^{-1}$, and note that $P_t u = \varphi_t * u$ and that φ_t is a radially decreasing function with $\|\varphi_t\|_1 = 1$ (cf. p. 132 of [7]). Then

$$|(BP_t)^r u(x)| \leqslant M^r \varphi_t * \varphi_t * \ldots * \varphi_t * |u|(x) \leqslant M^r u^*(x),$$

where u^* denotes the Hardy–Littlewood maximal function of u. The last inequality follows from an n-dimensional version of Lemma 3.6 of [3]. Therefore

$$\|(BP_t)^r u\|_{\infty,2} \leqslant M^r \|u^*\|_2 \leqslant c_n M^r \|u\|_2.$$

This completes the proof of (i).

To prove (ii) we introduce a Littlewood–Paley decomposition. Fix a function $\hat\theta \in C^\infty(\boldsymbol{R}^n)$ such that $\hat\theta(\xi) = 1$ if $|\xi| \leqslant 1/2$ and $\hat\theta(\xi) = 0$ if $|\xi| \geqslant 1$, and define $\hat{\theta}_{(k)}$ by

$$\hat{\theta}_{(k)}(\xi) = \hat{\theta}(2^{-k-1}\xi) - \hat{\theta}(2^{-k}\xi), \qquad k = 0, 1, 2, \ldots,$$
$$\hat{\theta}_{(-1)}(\xi) = \hat{\theta}(\xi).$$

Then

$$\sum_{k=-1}^{\infty} \hat{\theta}_{(k)}(\xi) = 1, \qquad \xi \in \mathbf{R}^n,$$
$$\operatorname{sppt} \hat{\theta}_{(k)} \subset \{\xi \mid 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}, \qquad k \geqslant 0,$$
$$\operatorname{sppt} \hat{\theta}_{(-1)} \subset \{\xi \mid |\xi| \leqslant 1\},$$

and

$$\left\| \frac{\partial^\alpha}{\partial \xi^\alpha} \hat{\theta}_{(k)} \right\|_\infty \leqslant c_\alpha 2^{-k|\alpha|}.$$

We also write $\hat{\varphi}_{(k)} = \hat{\theta}_{(k)} \hat{\varphi}$ where $\hat{\varphi}$ is defined above, and note that for all multi-indices α,

$$\left\| \frac{\partial^\alpha}{\partial \xi^\alpha} \hat{\varphi}_{(k)} \right\|_\infty \leqslant c_\alpha 2^{-k(|\alpha|+2)},$$
$$\left\| \frac{\partial^\alpha}{\partial \xi^\alpha} \hat{\varphi}_{(k)} \right\|_1 \leqslant c_\alpha 2^{-k(|\alpha|+2-n)},$$
$$|\varphi_{(k)}(x)| \leqslant c_n 4^{-k} \underline{\varphi}_{2^{-k}}(x), \qquad x \in \mathbf{R}^n,$$

where $\underline{\varphi}_t(x) = t^{-n} \underline{\varphi}(x/t)$ and $\underline{\varphi}(x) = (1+|x|^{n+1})^{-1}$.

Define the operators $P_{(k)t}$ by $P_{(k)t} = \hat{\varphi}_{(k)}(tD)$ and $\mathscr{E}_{(k)}$ by

$$\mathscr{E}_{(k)} u(x, t) = u(x, 2^k t),$$

and note that $\mathscr{E}_{(k)}$ is an isometry on both $L_{2,2}$ and $L_{\infty,2}$.

LEMMA.

$$\|\mathscr{E}_{(k)} P_{(k)t}\|_{L_{\infty,2} \to T_{\infty,2}} \leqslant c_n 4^{-k}.$$

Proof.

$$|P_{k2^k t} u(x, t)| \leqslant c_n 4^{-k} \varphi_t * |u_t|(x)$$
$$\sup_{|x-y| \leqslant t} |P_{(k)2^k t} u(y, t)| \leqslant c_n 4^{-k} \underline{\varphi}_t * |u_t|(x), \qquad t > 0,$$

where

$$\underline{\varphi}(x) = \sup_{|y| \geqslant |x|-1} \underline{\varphi}(y) = \begin{cases} 1, & |x| \leqslant 1, \\ \dfrac{1}{1+\{|x|-1\}^{n+1}}, & |x| \geqslant 1. \end{cases}$$

$$\begin{aligned}|||P_{(k)2^k t}u|||_{\infty,2} &\leqslant c_n 4^{-k}\|\sup_t \underline{\varphi}_t * |u_t|\|_2 \\ &\leqslant c_n 4^{-k}\|\sup_t \underline{\varphi}_t * v\|_2 \quad (\text{where } v = \sup_t |u_t|) \\ &\leqslant c_n 4^{-k}\|\underline{\varphi}\|_1 \|v^*\|_2 \quad (\text{as in proof of (i)}) \\ &\leqslant c_n 4^{-k}\|\underline{\varphi}\|_1 \|u\|_{\infty,2}.\end{aligned}$$

It is readily checked that $\|\underline{\varphi}\|_1 < \infty$. Therefore

$$|||\mathscr{E}_{(k)} P_{(k)t} u|||_{\infty,2} = |||P_{(k)2^k t} \mathscr{E}_{(k)} u|||_{\infty,2} \leqslant c_n 4^{-k} \|\mathscr{E}_{(k)} u\|_{\infty,2} = c_n 4^{-k}\|u\|_{\infty,2}.$$

The lemma is now proved.

We now record some facts concerning Carleson measures, where we define a measure μ on $\boldsymbol{R}^{n+1}_+$ to be a Carleson measure if, for each $x_0 \in \boldsymbol{R}^n$ and $t > 0$,

$$\mu\{(x, t) \mid 0 < t + |x - x_0| < d\} \leqslant c_\mu d^n \omega_n$$

where ω_n is the volume of the unit ball in $\boldsymbol{R}^n$. The first two lemmas below are n-dimensional analogues of results in [3].

LEMMA. *Suppose $u \in T_{\infty,2}$ and μ is a Carleson measure. Then*

$$\Big\{ \int_{\boldsymbol{R}^{n+1}_+} |u(x, t)|^2 d\mu \Big\}^{1/2} \leqslant c_n \sqrt{c_\mu}\, |||u|||_{\infty,2}.$$

LEMMA. *Suppose $\hat{\psi} \in C_0^\infty(\boldsymbol{R}^n)$, $|\hat{\psi}(\xi)| \leqslant |\xi|$ and* $\operatorname{sppt}\hat{\psi} \subset \{\xi \mid |\xi| \leqslant 2\}$. *If $b \in L_\infty(\boldsymbol{R}^n)$, then*

$$d\mu = |\hat{\psi}_{\tau t}(D) b|^2 \frac{dx\, dt}{t}$$

defines a Carleson measure with

$$c_\mu \leqslant \begin{cases} c_n \|b\|^2_\infty, & 0 \leqslant \tau \leqslant 1, \\ c_n \tau^2 \|b\|^2_\infty, & \tau \geqslant 1.\end{cases}$$

LEMMA. *Suppose $1 \leqslant i \leqslant n$. If $j, k \geqslant -1$, then*

$$d\mu_{jk} = |\mathscr{E}_{(k)} \hat{\theta}_{(j)t}(D)\, tD_i\, b(x)|^2 \frac{dx\, dt}{t}$$

defines a Carleson measure with

$$c_{\mu_{jk}} \leqslant \begin{cases} c_n 4^k \|b\|^2_\infty, & k \geqslant j, \\ c_n 4^j \|b\|^2_\infty, & j \geqslant k.\end{cases}$$

Proof. We consider the case when $j \geqslant 0$. (The case $j = -1$ is similar.) Let $\hat{\psi}(\xi) = \xi_i \hat{\theta}_{(0)}(\xi)$. Then $|\hat{\psi}(\xi)| \leqslant |\xi|$ and $\operatorname{sppt}\hat{\psi} \subset \{\xi \mid |\xi| \leqslant 2\}$. So

$$d\mu_{jk} = |\mathscr{E}_{(k)}\,\hat{\theta}_{(j)t}(D)\,tD_i\,b(x)|^2\frac{dx\,dt}{t} = 4^j|\hat{\psi}_{2^k - j_t}(D)\,b(x)|^2\frac{dx\,dt}{t}$$

and hence μ_{jk} is a Carleson measure with the required bound.

We are now in a position to prove the estimate (ii). In doing so we drop the subscripts on B and b. So $b \in L_\infty(\boldsymbol{R}^n)$. We also define α_{jk} as follows

$$\alpha_{jk} = \begin{cases} 4^{-j+2}, & k \leqslant j-3, \\ 4^{-k+2}, & j \leqslant k-3, \\ 1, & k-2 \leqslant j \leqslant k+2. \end{cases}$$

The first inequality below follows on considering the spectral supports of the various functions. Let $v \in L_{\infty,2}$. Then

$$\begin{aligned}
\|(Q_{i,t}BP_t - P_tBQ_{i,t})v\|_{2,2} &= \Big\|\sum_{j,k\geqslant -1} P_t\,\{\hat{\theta}_{(j)t}(D)\,tD_i\,b\}\,P_{(k)t}\,v\Big\|_{2,2} \\
&\leqslant \sum_{j,k\geqslant -1} \alpha_{jk}\,\|\{\hat{\theta}_{(j)t}(D)\,tD_i\,b\}\,P_{(k)t}\,v\|_{2,2} \\
&= \sum_{j,k\geqslant -1} \alpha_{jk}\,\|\mathscr{E}_{(k)}\,\{\hat{\theta}_{(j)t}(D)\,tD_i\,b\}\,P_{(k)t}\,v\|_{2,2} \\
&= \sum_{j,k\geqslant -1} \alpha_{jk}\,\Big\{\int_{\boldsymbol{R}^{n+1}_+} |\mathscr{E}_{(k)}\,P_{(k)t}\,v(x,t)|^2\,d\mu_{jk}\Big\}^{1/2} \\
&\leqslant c_n \sum_{j,k\geqslant -1} \alpha_{jk}\sqrt{c_{\mu_{jk}}}\,|||\mathscr{E}_{(k)}\,P_{(k)t}\,v|||_{\infty,2} \\
&\leqslant c_n \sum_{-1\leqslant k\leqslant j-3} 4^{-j+2}\,2^j M 4^{-k}\,\|v\|_{\infty,2} \\
&\quad + c_n \sum_{-1\leqslant j\leqslant k-3} 4^{-k+2}\,2^k M 4^{-k}\,\|v\|_{\infty,2} \\
&\quad + c_n \sum_{k-2\leqslant j\leqslant k+2} 2^{k+2} M 4^{-k}\,\|v\|_{\infty,2} \\
&\leqslant c_n M\,\|v\|_{\infty,2}.
\end{aligned}$$

This completes the proof of (ii), and hence of the theorem.

8. Conclusion

In this paper we have given an introduction to the theory of Clifford algebras and Clifford analysis, with particular reference to the Cauchy integral. We have then shown that the Cauchy singular integral operator T_Σ is L_2-bounded in the case when Σ is the graph of a Lipschitz function g: $\boldsymbol{R}^n \to \boldsymbol{R}$. We did this by proving the quadratic estimates (Q) for $T = (I-B)^{-1}\boldsymbol{D}$ where B denotes multiplication by $\boldsymbol{D}g$, and using the identity $T_\Sigma = \operatorname{sgn} T$.

A localization argument gives the corresponding result for strongly Lipschitz compact surfaces:

THEOREM. *Let Σ denote an n-dimensional strongly Lipschitz compact surface in $\mathbf{R}^{n+1}$. Then the Cauchy singular integral operator T_Σ is a bounded operator in $L_2(\Sigma)_{(n)}$.*

A corollary is that the singular double-layer potential operator $(T_\Sigma)_0$ is bounded in $L_2(\Sigma)$. This result, which was already proved in [2] by reduction to one-dimensional estimates, was used by G. Verchota in solving the Dirichlet and Neumann problems for Laplace's equation in domains with Lipschitz boundaries.

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