DYNAMICAL SYSTEMS AND ERGODIC THEORY
BANACH CENTER PUBLICATIONS, VOLUME 23
PWN - POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1989

THE SIZE OF SCRAMBLED SETS: n-DIMENSIONAL CASE

I. MIZERA

Department of Mathematics, Komenský University, Bratislava, Czechoslovakia

This note extends the results from [4] and [5] for the continuous self-mappings of the n-dimensional cube I^n. The theory of Li–Yorke chaos in dimension $n > 1$ is not very much developed yet (cf. [1], [2], [3]).

Let f: $I^n \to I^n$ be continuous. Recall that S is called *scrambled set* (for f) if for every $x, y \in S$ and for every periodic point p of f

$$\limsup_{k\to\infty} d\big(f^k(x), f^k(y)\big) > 0, \tag{1}$$

$$\liminf_{k\to\infty} d\big(f^k(x), f^k(y)\big) = 0, \tag{2}$$

$$\limsup_{k\to\infty} d\big(f^k(x), f^k(p)\big) > 0. \tag{3}$$

Denote by μ the Lebesgue measure on I^n, by λ the linear Lebesgue measure on I.

THEOREM 1. *There exists a continuous mapping G: $I^n \to I^n$ for which there exists a scrambled set T with $\mu(T) = 1$.*

Proof. For simplicity of notation, suppose that $n = 2$; the extension to $n > 2$ is obvious. Define the product of maps h_1, h_2: $I \to I$ to be the map H: $I^2 \to I^2$ (write $H = h_1 \times h_2$) such that $H(x_1, x_2) = \big(h_1(x_1), h_2(x_2)\big)$ for all $(x_1, x_2) \in I^2$.

Let g be a Misiurewicz "chaos-almost-everywhere" transformation of the interval (cf. [4] – here and further in the proof we follow the notation from this paper). We show that $G = g \times g$ satisfies the statement of Theorem 1. The transformation g has a scrambled set $q(S)$ with $\lambda\big(q(S)\big) = 1$. Let $T = q(S) \times q(S)$. Clearly, $\mu(T) = 1$; we show that T is a scrambled set for G.

Since $q(S)$ is a linear scrambled set, (1) and (3) for T follow immediately. The homeomorphism q conjugates g with the "tent" map f, $f(x) = 1 - |2x - 1|$.

Hence, G is topologically conjugated by the product homeomorphism $Q = q \times q$, with the map $F = f \times f$. It suffices to establish (2) for F and for $Q^{-1}(T) = S \times S$ as its scrambled set.

Let $z \in S$. Let $\{z(k)\}_{k=0}^{\infty}$ be the itinerary of z:

$$z(k) = \begin{cases} 0 & \text{if } f^k(z) \in (0, 1/2), \\ 1 & \text{if } f^k(z) \in (1/2, 1) \end{cases}$$

(note that the orbit of z cannot contain 0, 1/2, 1). Let $A_n = \{n^2+1, n^2+2, \ldots, n^2+n\}$, $B_n = \bigcup_{i=1}^{\infty} A_{i2^{n-1}}$. We have $S = \bigcup_{n=1}^{\infty} S_n$; if $z \in S_n$, then $z(k) = 1$ for all $k \in B_n$ (cf. [4]).

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ be elements of $S \times S$. Let $x_1 \in S_a$, $x_2 \in S_b$, $y_1 \in S_c$, $y_2 \in S_d$. Let $e = \max\{a, b, c, d\}$. Since $B_{n+1} \subset B_n$ for all n, $x_1(k) = x_2(k) = y_1(k) = y_2(k) = 1$ for all $k \in B_e$. By the properties of f we obtain that

$$|f^{l(i)}(x_1) - f^{l(i)}(y_1)| < m(i),$$

$$|f^{l(i)}(x_2) - f^{l(i)}(y_2)| < m(i)$$

where $l(i) = i^2 4^{e-1} + 1$, $m(i) = 2^{-i2^{e-1}}$. This implies (2) and completes the proof.

Let C^0 be the space of all continuous self-mappings of I^n endowed with the uniform metric.

THEOREM 2. *There exists a first Baire category set $M \subseteq C^0$ such that any $f \in C^0 \setminus M$ has only (if any) nowhere dense scrambled sets of zero Lebesgue measure.*

Proof. Again, we restrict ourselves to the case $n = 2$ for simplicity. The proof is similar to that given in [5]. For $k = 1, 2, 3, \ldots$, let $0 < \delta_k < 2^{-2k-1}$. For $i = 1, 2, 3, \ldots, 2^k$, let $a(k, i) = (i-1)2^{-k}$, $a(k, 2^k+1) = 1$, $b(k, i) = i2^{-k} - \delta_k$. Let

$$I(k, i, j) = [a(k, i), b(k, i)] \times [a(k, j), b(k, j)],$$

$$I(k) = \bigcup_{i,j} I(k, i, j).$$

Let

$$A_k = \{f \in C^0 \colon f(I(k)) \subseteq \operatorname{int} I(k)\},$$

let $B_k = \bigcup_{m \geq k} A_m$ (note that here A_k and B_k have different meaning than in the proof of Theorem 1). Clearly, B_k is open. To show that B_k is dense in C^0, fix $f \in C^0$ and its ε-neighbourhood $U_\varepsilon(f)$. There exists an integer $m \geq n$ such that $2^{-m+1/2} < \varepsilon/4$ and

(4) $$d(f(x), f(y)) < \varepsilon/4 \quad \text{whenever } d(x, y) \leq 2^{-m+1/2}.$$

We construct a function $g \in A_m \cap U_\varepsilon(f)$. First we define an auxiliary function $h\colon I(m) \to I(m)$ letting $f(a(m, i), a(m, j))$ be the image of the whole $I(m, i, j)$ under h, provided $f(a(m, i), a(m, j)) \in \operatorname{int} I(m)$. If this is not the case, we take some perturbed value $f(a(m, i), a(m, j)) + \eta$ which already lies in $\operatorname{int} I(m)$; it suffices to take η such that

$$d(0, \eta) \leqslant \delta_m < \varepsilon/8. \tag{5}$$

LEMMA. *Let a rectangular lattice, containing the boundary, be given in I^2, let N be the set of the lattice points. Let $h\colon N \to I^2$ be given. Then h can be extended to a continuous function $g\colon I^2 \to I^2$ such that if R is a rectangle with vertices $a, b, c, d \in N$ containing no other points from N, then*

$$d(g(x), g(y)) \leqslant \operatorname{diam}\{g(a), g(b), g(c), g(d)\} \quad \textit{for all } x, y \in R. \tag{6}$$

Proof. Let a, b, c, d be the vertices of R taken in the anti-clockwise direction. Every $x \in R$ can be uniquely written in the form

$$x = (1-s)(1-t)a + (1-s)tb + s(1-t)d + stc,$$

where $s, t \in (0, 1)$. Put

$$g(x) = (1-s)(1-t)h(a) + (1-s)th(b) + s(1-t)h(d) + sth(c).$$

We can see that $h(x) = g(x)$ for $x \in N$; g is continuous on rectangles; the definition of g on the edges of rectangles depends only on the boundary points of the corresponding edge – hence it is consistent. The second part of the statement follows from the fact that $g(x)$ and $g(y)$ both lie in the convex hull of $\{g(a), g(b), g(c), g(d)\}$.

Now we use Lemma for the rectangular lattice formed by all points with coordinates $a(m, i)$ or $b(m, i)$. Let N be the set of lattice points. There are some $x \in N$ for which h is not defined: we then put $h(x) = f(x)$. Let g be a continuous extension of h as in Lemma. Note that in view of (6) g is constant on $I(m, i, j)$ for every i, j. Clearly $g \in A_m$. It remains to show that $d(f(x), g(x)) < \varepsilon$ for all $x \in I^2$. Let R be a rectangle with the vertices a, b, c, d, as in Lemma, let $x \in R$. We have

$$d(f(x), g(x)) \leqslant d(f(x), f(a)) + d(f(a), g(a)) + d(g(a), g(x)). \tag{7}$$

By (5), the middle term on the right of (7) is less than $\varepsilon/8$. By (4), the left term on the right of (7) is less than $\varepsilon/4$. By Lemma, (4) and (5),

$$\begin{aligned} d(g(a), g(x)) &\leqslant \operatorname{diam}\{g(a), g(b), g(c), g(d)\} \\ &\leqslant 2\varepsilon/8 + \operatorname{diam}\{f(a), f(b), f(c), f(d)\} \leqslant \varepsilon/2. \end{aligned}$$

Summing, we have $d(f(x), g(x)) < \varepsilon$. Note that $\mu(I^2 \setminus I(k)) < 2^{-k}$. The rest of the proof is the same as in [5] (relying of the fact that each component of $I(k)$ is mapped into some component of $I(k)$).

Remark. Li–Yorke chaos in C^0 is generic. To see this, note that the "horseshoe", i.e. the configuration consisting of two closed disjoint sets U, V with nonempty interiors such that $U \cup V \subseteq \operatorname{int}(f(U) \cap f(V))$, is stable under small perturbations of f and can be created near a fixed point of any C^0 function in any neighbourhood of that function.

Acknowledgement. The author is very grateful to Stefan Banach International Mathematical Centre and to the Organizing Committee of the Semester on Dynamical Systems and Ergodic Theory for their hospitality.

References

[1] C. Blanco García, *Chaos and topological entropy in dimension $n > 1$*, Ergod. Th. Dynam. Syst. 6 (1986), 163–165.

[2] P. E. Kloeden, *Chaotic difference equations in R^n*, J. Austral. Math. Soc. (Series A) 31 (1981), 217–225.

[3] F. R. Marotto, *Snap-back repellers imply chaos in R^n*, J. Math. Anal. Appl. 63 (1978), 199–223.

[4] M. Misiurewicz, *Chaos almost everywhere*, in: *Iteration theory and its functional equations*, ed. R. P. Liedl, L. P. Reich, G. Targonski, Lecture Notes in Math. 1163, Springer, Berlin 1986.

[5] I. Mizera, *Continuous chaotic functions of the interval have generically small scrambled sets*, Bull. Austral. Math. Soc. 37 (1988), 89–92.