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On a generalization of the functional equation for the harmonic ratio of four points on a projective line over an arbitrary commutative field

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INTRODUCTION

Aczél, Gołąb, Kuczma and Siwek gave in [1] the general solution of the functional equation

$$\varphi\left(\frac{a_{11}\overset{\lambda}{x}_1+a_{12}\overset{\lambda}{x}_2}{a_{21}\overset{\lambda}{x}_1+a_{22}\overset{\lambda}{x}_2}\right)=\varphi(\overset{\lambda}{x}_1,\overset{\lambda}{x}_2),\qquad \overset{\lambda}{x}_k,\ a_{ij}\in R,\qquad \det\|a_{ij}\|\neq 0 \qquad (\lambda=1,2,3,4), \tag{1}$$

where φ is an unknown real-valued function (R denotes the set of all real numbers).

A function φ satisfying (1) is an invariant of 4 points on the real projective line with respect to the corresponding projective group. One of the invariants is the harmonic ratio.

Benz determines in [2] all the invariants of 4 points on the projective line over an arbitrary, in general non-commutative field (Schiefkörper) with respect to the corresponding projective group.

In the present paper we give the general solution of functional equation (11), which is a generalization of (1).

The main result is contained in Theorem 4.

I. DESCRIPTION OF THE GENERALIZATION OF (1)

1. Suppose we are given an arbitrary commutative field D. Let us denote by L the linear group of matrices of order 2 over the field D. By L_0 we denote the group of matrices kE, where $k\in D$, $k\neq 0$ and E is the unit matrice of L. L_0 is a normal subgroup of L. By Λ we denote the factor group L/L_0. The elements of L and Λ we denote by $a=\|a_{ij}\|$, $b=\|b_{ij}\|,\ldots$ and $\alpha,\beta,\ldots,$ respectively.

Λ defines in L an equivalence relation the classes of which are the elements of Λ. We denote this relation by R_Λ.

2. Let us introduce the set $\mathscr{X} = D\times D\setminus_{\{0,0\}}$. We denote its elements by $x = (x_1, x_2)$, $y = (y_1, y_2), \ldots$

In the set $\mathscr{X}$ we consider the following representation group F of the group L

$$y_i = \sum_{j=1}^{2} a_{ij}x_j, \tag{2}$$

or shortly

$$y = ax, \quad x, y\in\mathscr{X}, \quad a\in L. \tag{2'}$$

In (2') $a\in L$ may be treated as an operator acting on the elements of $\mathscr{X}$. The relation R_1 defined by

$$xR_1y \Leftrightarrow \mathop{\exists}_{k\in D} (k \neq 0,\ y_i = kx_i), \quad x, y\in\mathscr{X}$$

is compatible with this operator. We denote by Ξ the factor space $\mathscr{X}/R_1$ and the factor group F/R_1 by $\mathscr{F}$. The elements of Ξ are denoted by $\xi, \eta, \ldots$

The group $\mathscr{F}$ may be written in the form $\eta = a\xi$, $\xi, \eta\in\Xi$, $a\in L$.

The relation R_Λ, defined in Section 1, is compatible with the group operation of the group $\mathscr{F}$. We denote by $\hat{\mathscr{F}}$ the factor group $\mathscr{F}/R_\Lambda$. The group $\hat{\mathscr{F}}$ may be written in the form

$$\eta = \alpha\cdot\xi, \quad \xi, \eta\in\Xi, \quad \alpha\in\Lambda. \tag{3}$$

3. We consider the following representation group of the group L

$$\overset{\lambda}{y}_i = a_{ij}\overset{\lambda}{x}_j, \quad \overset{\lambda}{x}, \overset{\lambda}{y}\in\mathscr{X}, \quad a\in L \quad (\lambda = 1, 2, 3, 4; i, j = 1, 2), \tag{4}$$

or shortly

$$\overset{\lambda}{y} = a\overset{\lambda}{x}, \quad \overset{\lambda}{x}, \overset{\lambda}{y}\in\mathscr{X}, \quad a\in L, \tag{4'}$$

which we denote by $\mathscr{F}^4$.

Using the relations $\overset{\lambda}{R} \overset{\mathrm{df}}{=} R_1$ $(\lambda = 1, 2, 3, 4)$ and R_Λ we may obtain from (4), in the way described in Section 2, the following group of transformations:

$$\overset{\lambda}{\eta} = \alpha\overset{\lambda}{\xi}, \quad \overset{\lambda}{\xi}, \overset{\lambda}{\eta}\in\Xi, \quad \alpha\in\Lambda \quad (\lambda = 1, 2, 3, 4), \tag{5}$$

which we denote by $\hat{\mathscr{F}}^4$.

4. Let us introduce the set

$$\Xi_*^4 = \{(\overset{\lambda}{\xi})\colon\ \overset{\lambda}{\xi}\in\Xi, \mathop{\forall}_{\lambda\neq\mu} (\overset{\lambda}{\xi} \neq \overset{\mu}{\xi})\}. \tag{6}$$

If we restrict the transformations belonging to the group $\hat{\mathscr{F}}^4$ to the set Ξ_*^4 then we obtain — as can easily to be proved — a new group of transformations, which we denote by $\hat{\mathscr{F}}_*^4$.

We give without proof the following

Remark 1. The restrictions of the group $\hat{\mathscr{F}}^4_*$ of transformations to its domains of transitivity are simple transitive groups (see [3]).

5. Let G be given representation group of the group L into an arbitrary space U

$$v = g(u, a), \quad u, v \in U; \quad a \in L. \tag{7}$$

Under the assumption

A. *For every* $u \in U$ *the function* $g(u, a)$ *in* (7) *is homogeneous of order* 0 *with respect to the whole group of variables* $a = (a_{ij})$, the relation R_Λ is compatible with the group operation of the group G. We denote the factor group G/R_Λ by $\hat{G}$. This group may be written in the form

$$v = \hat{g}(u, a), \quad u, v \in U, \quad a \in \Lambda. \tag{8}$$

The transformation group $\hat{G}$ is a representation group of the group Λ.

6. Now we consider the functional equation

$$\varphi(a\overset{\lambda}{x}) = g\big(\varphi(\overset{\lambda}{x}), a\big), \quad \overset{\lambda}{x} \in \mathscr{X}, \quad a \in L, \tag{9}$$

where $\varphi\colon \mathscr{X}^4 \to U$ is an unknown function.

Besides (9) let us consider the functional equation

$$\hat{\varphi}(a\overset{\lambda}{\xi}) = \hat{g}\big(\varphi(\overset{\lambda}{\xi}), a\big), \quad \overset{\lambda}{\xi} \in \Xi, \quad a \in \Lambda, \tag{10}$$

where $\hat{\varphi}\colon \Xi^4 \to U$ is unknown and $\hat{g}$ is a given function of the type (8) (e.g. $\hat{g} \in \hat{G}$).

We have the following connections between (9) and (10) (we omit the proofs):

THEOREM 1. *If assumption* A *for the function* g *is satisfied and if* $\hat{\varphi}(x)$ *is a solution of* (9) *and is homogeneous function of order* 0 *with respect to each variable* $\overset{\lambda}{x}$ $(\lambda = 1, 2, 3, 4)$, *then the factorization*(1) $\hat{\varphi}(\overset{\lambda}{\xi})$ *of* $\varphi(\overset{\lambda}{x})$ *with respect to the relation* R^4 *is a solution of* (10).

Conversely,

THEOREM 2. *If* $\hat{\varphi}(\overset{\lambda}{\xi})$ *is a solution of* (10) *and assumption* A *for the function* g *in* (9) *is satisfied, then there exists one and only one solution*

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(1) Suppose we are given a function $\varphi\colon \mathscr{X} \to U$ and let R be an equivalence relation in $\mathscr{X}$ such that for every $x \in \mathscr{X}$ and every $y \in [x]_R$ we have $\varphi(x) = \varphi(y)$. We may introduce a function $\hat{\varphi}(\xi)$, $\xi \in \mathscr{X}/R$, such that $\hat{\varphi}([x]_R) = \varphi(x)$ for every $x \in \mathscr{X}$. The function $\hat{\varphi}(\xi)$ is named the factorization of the function $\varphi(x)$ with respect to the relation R or shortly R-factorization of $\varphi(x)$.

$\varphi(x)$ of (9), homogeneous of order 0 with respect to each variable $\overset{\lambda}{x}$, $\lambda = 1, 2, 3, 4$, such that its R^4-factorization coincides with $\hat{\varphi}(\overset{\lambda}{\xi})$.

7. If we denote by (x_k) the projective coordinates of a point X on the projective line P over the field D in an arbitrary projective coordinates system, then (2), or more exactly (3), represents the projective group of transformations on P. In the case where $U = D$ and $g(u, a) \equiv u$ equation (10) is the equation for the invariants of 4-points on P.

In this paper we shall consider 4-points such that every two points are different. This means that we shall say about the functional equation

$$(11) \qquad \hat{\varphi}(a\overset{\lambda}{\xi}) = \hat{g}\big(\hat{\varphi}(\overset{\lambda}{\zeta}), a\big), \quad \overset{\lambda}{\xi} \in \Xi^4_*, \quad a \in \Lambda.$$

II. THE SOLUTION OF EQUATION (11)

The functional equation (11) will be solved here by using the method described in [4]. Now we are going to prepare some formulae which will be needed for the realization of this method.

1. We consider the family Σ of the domains of transitivity of the group $\hat{\mathscr{F}}^4_*$ [3].

DEFINITION 1 (see [4]). A set $\underset{0}{\Xi^4_*} \subset \Xi^4_*$ with the property of having one and only one point in common with every element of Σ will be called the *generator* of the set Ξ^4_* with respect to the family Σ (or with respect to the group $\hat{\mathscr{F}}^4_*$).

Without proof we give the following

THEOREM 3. *Two points of Ξ^4_* (as 4-points on P) have the same harmonic ratio*(²) *if and only if they belong to the same domain of transitivity of the group $\hat{\mathscr{F}}^4_*$.*

(²) If P is a projective line over D with an arbitrarily fixed corrdinate system and if there is given a point $(\overset{\lambda}{\xi}) \in \Xi^4_*$, then there exists one and only one 4 point $(\overset{\lambda}{x})$ in P with the coordinates $\overset{\lambda}{x} = (\overset{\lambda}{x}_i)$ such that $\overset{\lambda}{\xi} = [\overset{\lambda}{x}]_{R_1}$ $(\lambda = 1, 2, 3, 4)$.

We give the following well-known definition: The harmonic ratio s of a point $(\overset{\lambda}{\xi})$ is

$$(13') \qquad s(\overset{1}{\xi}, \overset{2}{\xi}, \overset{3}{\xi}, \overset{4}{\xi}) \overset{\text{df}}{=} \frac{W^{13}W^{24}}{W^{14}W^{23}},$$

where

$$W^{\lambda\mu} \overset{\text{df}}{=} \det \begin{Vmatrix} \overset{\lambda}{x}_1 & \overset{\lambda}{x}_2 \\ \overset{\mu}{x}_1 & \overset{\mu}{x}_2 \end{Vmatrix}.$$

From Theorem 3 it follows that every generator of the set Ξ^4_* can be parametrized by the harmonic ratio, which we denote by s.

Let us introduce the set

$$\underset{0}{\Xi^4_*} \overset{\mathrm{df}}{=} \{(\overset{\lambda}{\xi}(s)):\ \overset{\lambda}{\xi}\in\overset{\lambda}{x} = (\overset{\lambda}{k}\overset{\lambda}{\sigma}_i(s)),\ \overset{\lambda}{k}\in D,\ \overset{\lambda}{k}\neq 0,\ s\in S\}, \tag{12}$$

where

$$S = \{s:\ s\in D,\ s\neq 0,1\} \tag{13}$$

and

$$\begin{aligned} &\overset{1}{\sigma}_1 = 1, \quad \overset{1}{\sigma}_2 = 0,\\ &\overset{2}{\sigma}_1 = 0, \quad \overset{2}{\sigma}_2 = 1,\\ &\overset{3}{\sigma}_1 = 1, \quad \overset{3}{\sigma}_2 = 1,\\ &\overset{4}{\sigma}_1 = s, \quad \overset{4}{\sigma}_2 = 1. \end{aligned} \tag{14}$$

The parametrization of the set $\underset{0}{\Xi^4_*}$ given by (12), (13) and (14) is one-to-one. The harmonic ratio of $(\overset{\lambda}{\xi}(s))$ is equal to s. Indeed, we have

$$W^{\lambda\mu}(\overset{\lambda}{x}, \overset{\mu}{x}) = \overset{\lambda}{k}\overset{\mu}{k}\cdot\omega^{\lambda\mu}, \tag{15}$$

where

$$\omega^{\lambda\mu} = \det\begin{Vmatrix} \overset{\lambda}{\sigma}_1 & \overset{\lambda}{\sigma}_2\\ \overset{\mu}{\sigma}_1 & \overset{\mu}{\sigma}_2 \end{Vmatrix}, \qquad \omega^{\lambda\mu} = -\omega^{\mu\lambda}. \tag{16}$$

From (14) and (16) we obtain

$$\begin{aligned} &\omega^{12} = 1, \quad \omega^{23} = -1,\\ &\omega^{13} = 1, \quad \omega^{24} = -s,\\ &\omega^{14} = 1, \quad \omega^{34} = 1-s. \end{aligned} \tag{17}$$

Using (15) and (17) we obtain

$$(\overset{1}{\xi}, \overset{2}{\xi}, \overset{3}{\xi}, \overset{4}{\xi}) = \frac{W^{13}W^{24}}{W^{14}W^{23}} = \frac{\omega^{13}\omega^{24}}{\omega^{14}\omega^{23}} = \frac{1\cdot(-s)}{1\cdot(-1)} = s.$$

From this it follows that the set $\underset{0}{\Xi^4_*}$ defined by (12), (13) and (14) is a generator of the set Ξ^4_*. This means that the transformation

$$\overset{\lambda}{\eta} = a\overset{\lambda}{\xi}(s), \quad s\in S, \quad a\in\Lambda, \quad (\overset{\lambda}{\eta})\in\Xi^4_*, \tag{18}$$

obtained by putting $\overset{\lambda}{\xi}(s)$ in (5) instead of $\overset{\lambda}{\xi}$, is one-to-one.

Now it is necessary to find the inverse transformation to (18). Let us write (18) using the representatives of classes $\overset{\lambda}{\xi}(s), \overset{\lambda}{\eta}, a$. We have

$$\overset{\lambda}{y}_i = \overset{\lambda}{r} a_{ij} \overset{\lambda}{\sigma}_j(s), \quad i = 1, 2;\ \lambda = 1, 2, 3, 4. \tag{19}$$

Before calculations let us notice that

1° s does not depend on $\overset{\lambda}{r}$ but only on $\overset{\lambda}{y}_i$,

2° a_{ij} depend not only on $\overset{\lambda}{y}_i$ but also on $\overset{\lambda}{r}$ and the dependence is of the type

$$\bar{a}_{ij} = \varrho(\overset{\lambda}{r}) a_{ij}, \quad i, j = 1, 2,$$

where ϱ may be an arbitrary function. The quantities $\overset{\lambda}{r}$ are functions of representatives of classes $\overset{\lambda}{\xi}(s), \overset{\lambda}{\eta}, a$. If we change them, we do not change classes.

From Theorem 3 it follows that s must be equal to the harmonic ratio of the points $\overset{\lambda}{\eta}$, being connected with s by (18). Using the representatives of the classes $\overset{\lambda}{\eta}$ we may write

$$s = \frac{W^{13}(y)\, W^{24}(y)}{W^{14}(y)\, W^{23}(y)}, \tag{20}$$

where $W^{\lambda\mu}$ are defined in (13′).

The same result may be obtained by direct calculations, which are following:

By the theorem of Cauchy concerning the determinants of the products of matrices, related to (19), we have

$$W^{\lambda\mu}(y) = \overset{\lambda}{r}\overset{\mu}{r} \Delta \omega^{\lambda\mu}(s), \tag{21}$$

where

$$\Delta = \det\|a_{ij}\|; \quad \Delta \neq 0,$$

and $\omega^{\lambda\mu}(s)$ defined by (16) are given by (17).

From (21) we obtain

$$\overset{\lambda}{r}\overset{\mu}{r} = \frac{1}{\Delta} \cdot \frac{W^{\lambda\mu}(y)}{\omega^{\lambda\mu}(s)}. \tag{22}$$

If we put $\lambda = 1, 2$ in (19) and use (14), (17) and (22), then we obtain

$$a_{i1} = \frac{\overset{1}{y}_i}{\overset{1}{r}} = \overset{3}{r} \cdot \frac{\overset{1}{y}_i}{\overset{1}{r}\overset{3}{r}} = \overset{3}{r}\Delta \cdot \frac{\overset{1}{y}_i}{W^{13}(y)} = l \cdot \frac{\overset{1}{y}_i}{W^{13}(y)}, \tag{23}$$

$$a_{i2} = \frac{\overset{2}{y}_i}{\overset{2}{r}} = \overset{3}{r} \cdot \frac{\overset{2}{y}_i}{\overset{2}{r}\overset{3}{r}} = \overset{3}{r}\Delta \cdot \frac{\overset{2}{y}_i}{W^{32}(y)} = l \cdot \frac{\overset{2}{y}_i}{W^{32}(y)}, \quad l = \overset{3}{r}\Delta. \tag{24}$$

Now if we put $\lambda = 4$ in (19) and if we use (23), (24) and (14), then we have

$$\text{(25)}\qquad \begin{aligned} \overset{4}{y}_1 &= l\cdot\frac{\overset{4}{r}}{\overset{1}{r}}\left(s\cdot\frac{\overset{1}{y}_1}{W^{13}(y)}+\frac{\overset{2}{y}_1}{W^{32}(y)}\right),\\ \overset{4}{y}_2 &= l\cdot\frac{\overset{4}{r}}{\overset{1}{r}}\left(s\cdot\frac{\overset{1}{y}_2}{W^{13}(y)}+\frac{\overset{2}{y}_2}{W^{32}(y)}\right). \end{aligned}$$

Multiplying the first equation of (25) by $\overset{4}{y}_2$ and the second by $-\overset{4}{y}_1$ and adding, we obtain

$$0 = l\cdot\frac{\overset{4}{r}}{\overset{1}{r}}\left(s\cdot\frac{W^{14}(y)}{W^{13}(y)}-\frac{W^{24}(y)}{W^{32}(y)}\right),$$

whence follows immediately formula (20).

Formulae (20), (23) and (24) give us the inverse transformation to (18).

2. From Theorems 1 and 2 it follows that the solving of functional equation (11) is equivalent to the solving of equation (9) in the family of homogeneous functions of order 0 with respect to each variable $\overset{\lambda}{x}$ by assumption A about the function g and the assumption that the points $\overset{\lambda}{x}$ are different.

If we put arbitrary representatives of classes $\overset{\lambda}{\xi}(s)$ in formula (9) instead of $\overset{\lambda}{x}$'s, then we obtain (remember that φ is homogeneous of order 0)

$$\text{(26)}\qquad \varphi\big(\overset{\lambda}{r}a_{ij}\overset{\lambda}{\sigma}_j(s)\big) = g\big(\varphi\big(\overset{\lambda}{\sigma}_i(s)\big), a_{ij}\big).$$

Let us put, by definition,

$$\text{(27)}\qquad \Phi(s) \overset{\text{df}}{=} \varphi\big(\overset{\lambda}{\sigma}(s)\big) = \varphi(1,0;0,1;1,1;s,1), \qquad s\in S.$$

Using formulae (19) and (27) we may write (26) in the form

$$\text{(28)}\qquad \varphi(\overset{\lambda}{y}_i) = g\big(\Phi(s), a_{ij}\big).$$

Now we use formulae (20), (23), (24) and from (28) we get

$$\text{(29)}\quad \varphi(\overset{\lambda}{y}_i) = g\left(\Phi\left(\frac{W^{13}(y)\cdot W^{24}(y)}{W^{14}(y)\cdot W^{23}(y)}\right), \frac{\overset{1}{y}_1}{W^{13}(y)}, \frac{\overset{1}{y}_2}{W^{13}(y)}, \frac{\overset{2}{y}_1}{W^{32}(y)}, \frac{\overset{2}{y}_2}{W^{32}(y)}\right),$$

where

$$W^{\lambda\mu}(y) \overset{\mathrm{df}}{=} \det\begin{Vmatrix} \overset{\lambda}{y}_1 & \overset{\lambda}{y}_2 \\ \overset{\mu}{y}_1 & \overset{\mu}{y}_2 \end{Vmatrix}.$$

We obtain the following

THEOREM 4. *If in* (9) *there is given a function* $g(u, a_{ij})$, $u \in U$, $\|a_{ij}\| \in L$ *satisfying condition* A, *then the general solution* φ *of* (9) *in the set of all* 4-*points* $(\overset{\lambda}{x})$ *with different points and in the family of homogeneous functions of order* 0 *with respect to each variable* $\overset{\lambda}{x}$ *is of form* (29), *where* Φ *is an arbitrary function*

$$\Phi\colon S \to U,$$

where U *is the space of acting of the transformation group* $G \ni g$ *and* S *is defined by* (13). *The function* $\hat{\varphi}$ *obtained from function* (29) *by using the operation of factorization with respect to the relation* R^4 *introduced in Section* 3 *is the general solution of* (11), *where* $\hat{g}$ *denotes the factorization of* g *with respect to the relation* R_A.

Remark 2. In the case where $g(u, a_{ij}) \equiv u$, $u \in D = R$ we obtain from (29) the result presented in [1].

Remark 3. Moreover, if we assume for the unknown function φ the initial condition

$$\varphi(1, 0; 0, 1; 1, 1; s, 1) \equiv s,$$

then we obtain for the solution of (11)

$$\varphi(\overset{\lambda}{y}_i) = \frac{W^{13}(y)\cdot W^{24}(y)}{W^{14}(y)\cdot W^{23}(y)}.$$

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