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A MIXED GAME OF TIMING: PROBLEM OF OPTIMALITY

The present paper is a continuation of paper [2]. Therefore, all notation, Definitions 1-8, Lemmas 1-10, and relations (1)-(41) given therein are binding in what follows.

1. Proof of Theorem 1. Before proving the theorem we formulate some lemmas.

LEMMA 11. *If* $S \in \{S(\varepsilon)\}_{\varepsilon>0} \in N_\lambda$ $(\|\lambda\| \geqslant 2)$, *then the strategy* S *is admissible in the game* Γ_λ.

This lemma follows easily from Lemmas 5-8 and from the relation $S^*_{\lambda_1}(\varepsilon/2) \in B^0_{|\lambda_1|}(\lambda_1)$.

LEMMA 12. *For a strategy* $F \in M_\pi$ *and a family* $\{S(\varepsilon)\}_{\varepsilon>0} \in N_\pi$ $(\|\pi\| \geqslant 2)$ *described by* (38) *and* (41),

(i) $K(F;\ y\,|\,\pi)$ *is a continuous function of the variable* y *in the interval* $\langle a_2, a_{\pi_1})$;

(ii) $K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S(\varepsilon)\,|\,\pi\big)$ *is a continuous function of the variable* x *in the interval* $\langle b_2, a_{\pi_1})$ *for* $\varepsilon > 0$.

Proof. Let $x \in \langle b_2, a_{\pi_1})$. Then, with the aid of (5), (13), (9), (6), (3) and (4), we get

$$\begin{aligned} K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S(\varepsilon)\,|\,\pi\big) &= (1-a)\int\Big\{\int\limits_{y>x_1}\{P(x)+[1-P(x)][1-2Q(y)]\}\,dT_{b_2}(y)+ \\ &\quad + \int\limits_{y<x}[1-2Q(y)]\,dT_{b_2}(y)\Big\}\,dF^*_{\pi_1}(\bar{x}_{n,1})+ \\ &\quad + a\int\Big\{P(x)+[1-P(x)]K\Big(\bar{x}_{n,1};\ S^*_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi_1\Big)\Big\}\,dF^*_{\pi_1}(\bar{x}_{n,1}), \end{aligned}$$

whence it follows that statement (ii) of the lemma is true. Statement (i) can be concluded similarly.

LEMMA 13. *Under the assumptions of Theorem* 1, *if* $w = c$ *and the family* $\{S(\varepsilon)\}_{\varepsilon>0}$ *is of the form* (41), *then* $a < 1$.

Proof. Let us assume that $a = 1$. Then for any $\varepsilon > 0$ we have $S(\varepsilon) = S^*_{\pi_1}(\varepsilon/2)$.

Hence we infer the validity of the inequalities

$$K\left(F;\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right) \geqslant v_\pi \tag{42}$$

and

$$K\left(F';\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right) \leqslant v_\pi + \varepsilon \tag{43}$$

for any positive ε and for all $F' \in A_{\|\pi\|}$, where v_π is the value of the game Γ_π.

Let us put

$$F(\bar{x}_n) = U_{a_2}(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1}) \qquad (n = \|\pi\|),$$

$$S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right) = [G^m, \{s_m(y)\}_y] \qquad (m = |\pi|).$$

By the definition of the strategy $S^*_{\pi_1}(\varepsilon/2)$ it can be concluded, with the help of (7) and (11), that $\operatorname{supp} G^m \subset \langle a_{\pi_1}, 1\rangle$ and that G^m is a continuous measure.

Further, we put

$$F^1(\bar{x}_n) = D_{a_{\pi_1}}(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1}).$$

On account of the definitions of F, $S^*_{\pi_1}(\varepsilon/2)$ and F^1 the following sequence of equalities is valid:

$$\begin{aligned}
K\left(F^1(\bar{x}_n);\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right) &\overset{(5)}{=} \int K\left((a_{\pi_1}, \bar{x}_{n,1});\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right) dF^*_{\pi_1}(\bar{x}_{n,1}) \\
&\overset{(3)}{=} \int\int K\big((a_{\pi_1}, \bar{x}_{n,1});\ s_m(y)|\pi\big) dG^m(y)\, dF^*_{\pi_1}(\bar{x}_{n,1}) \\
&\overset{(4)}{=} \int\int \big\{P(a_{\pi_1}) + [1-P(a_{\pi_1})]K(\bar{x}_{n,1};\ s_m(y)|\pi_1)\big\} dG^m(y)\, dF^*_{\pi_1}(\bar{x}_{n,1}) \\
&\overset{(3)(5)}{=} P(a_{\pi_1}) + [1-P(a_{\pi_1})]K\left(F^*_{\pi_1};\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi_1\right) \\
&\overset{(35)}{=} P(a_{\pi_1}) + [1-P(a_{\pi_1})][1-2Q(a_{\pi_1})] \\
&= \int\limits_{a_2}^{a_{\pi_1}} \{P(x_1) + [1-P(x_1)][1-2Q(a_{\pi_1})]\}\, dU_{a_2}(x_1) + h,
\end{aligned}$$

where h is a positive number independent of ε.

Analogously it can be shown that

$$K\left(F;\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right) = \int\limits_{a_2}^{a_{\pi_1}} \{P(x_1) + [1-P(x_1)][1-2Q(a_{\pi_1})]\}\, dU_{a_2}(x_1).$$

Therefore

$$K\left(F^1;\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right) = K\left(F;\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right)+h \quad \text{for any } \varepsilon > 0,$$

which contradicts relations (42) and (43).

Thus $a < 1$.

LEMMA 14. *Under the assumptions of Theorem* 1, *if* F *and* $\{S(\varepsilon)\}_{\varepsilon>0}$ *take the forms determined by* (37)-(41), *then*

(i) $a_1 = b_1$ *in the case* $w = g$,

(ii) $a_2 = b_2$ *in the case* $w = c$.

The proof is immediate because of the monotonicity of the functions $P(t)$ and $Q(t)$.

LEMMA 15. *If the assumptions of Theorem* 1 *are satisfied in the case* $w = c$ *and if the strategy* F *and the family* $\{S(\varepsilon)\}_{\varepsilon>0}$ *are of the forms given by* (38) *and* (41) *with the condition* $a_2 = b_2 = a$, *respectively, then*

$$K(F;\ y|\pi) = v_\pi \tag{44}$$

for any $y \in \langle a, a_{\pi_1})$,

$$K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S(\varepsilon)|\pi\big) = v_\pi \tag{45}$$

for any $\varepsilon > 0$ *and for all* $x \in \langle a, a_{\pi_1})$, *where* v_π *is the value of the game* Γ_π.

Proof. The assumptions of the lemma imply

$$K(F;\ y|\pi) \geqslant v_\pi, \quad y \in \langle a, a_{\pi_1}), \tag{46}$$

$$K\big(F;\ S(\varepsilon)|\pi\big) \leqslant v_\pi+\varepsilon \quad (\varepsilon > 0). \tag{47}$$

Assume that for a certain $y_0 \in \langle a, a_{\pi_1})$ we have $K(F;\ y_0|\pi) > v_\pi$. Then, by Lemma 12, there exist a number $\varepsilon_0 > 0$ and a neighbourhood Z_0 of the point y_0 such that

$$T_a(Z_0) > 0, \quad K(F;\ y|\pi) > v_\pi+\varepsilon_0,\ y \in Z_0. \tag{48}$$

Hence, in view of the inequality $a < 1$ (the result of Lemma 13), we can evaluate the following:

$$\begin{aligned} v_\pi+\varepsilon &\overset{(47)}{\geqslant} K\big(F;\ S(\varepsilon)|\pi\big) \\ &\overset{(5)(13)}{=} (1-a)\int K(\bar{x}_n;\ S^{T_a}_{|\pi|}|\pi)\,dF(\bar{x}_n)+a\int K\left(\bar{x}_n;\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right)dF(\bar{x}_n) \\ &\overset{(9)(5)}{=} (1-a)\iint K(\bar{x}_n;\ y|\pi)\,dT_a(y)\,dF(\bar{x}_n)+aK\left(F;\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right) \\ &\overset{(5)}{=} (1-a)\int K(F;\ y|\pi)\,dT_a(y)+aK\left(F;\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\pi\right) \end{aligned}$$

$$= (1-a)\int_{y\in Z_0} K(F;\ y|\pi)dT_a(y)+(1-a)\int_{y\notin Z_0} K(F;\ y|\pi)dT_a(y)+$$

$$+aK\Big(F;\ S^*_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)|\pi\Big)$$

$$\overset{(46)(48)}{>} (1-a)\int_{y\in Z_0}(v_\pi+\varepsilon_0)dT_a(y)+(1-a)\int_{y\notin Z_0} v_\pi dT_a(y)+aK\Big(F;\ S^*_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)|\pi\Big)$$

$$\overset{(46)}{\geqslant} (1-a)v_\pi+av_\pi+(1-a)\varepsilon_0 T_a(Z_0) = v_\pi+d,$$

where d is a positive number independent of ε.

Therefore $v_\pi+\varepsilon > v_\pi+d$ for any $\varepsilon>0$, which is impossible. Thus condition (44) is valid.

Now, let $x\in\langle a, a_{\pi_1})$. From the ε-optimality of the strategy $S(\varepsilon)$ it follows that for any $\varepsilon>0$

$$K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S(\varepsilon)|\pi\big)\leqslant v_\pi+\varepsilon.$$

We show that the left-hand side of this inequality does not depend on ε.

Indeed, we have

$$K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S(\varepsilon)|\pi\big)$$

$$\overset{(5)(13)}{=}(1-a)\int K(\bar{x}_n;\ S^{T_a}_{|\pi|}|\pi)d\{D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1})\}+$$

$$+a\int\Big\{P(x)+[1-P(x)]K\Big(\bar{x}_{n,1};\ S^*_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)|\pi_1\Big)\Big\}dF^*_{\pi_1}(\bar{x}_{n,1})$$

$$=(1-a)K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S^{T_a}_{|\pi|}|\pi\big)+$$

$$+a\Big\{P(x)+[1-P(x)]K\Big(F^*_{\pi_1};\ S^*_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)|\pi_1\Big)\Big\}$$

$$\overset{(35)}{=}(1-a)K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S^{T_a}_{|\pi|}|\pi\big)+$$

$$+a\{P(x)+[1-P(x)][1-2Q(a_{\pi_1})]\},$$

which is of course independent of ε.

Therefore, for any $\varepsilon>0$ and for all $x\in\langle a, a_{\pi_1})$ we get

$$K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S(\varepsilon)|\pi\big)\leqslant v_\pi.$$

Suppose that for a certain $x_0\in\langle a, a_{\pi_1})$ we have

$$K\big(D_{x_0}(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S(\varepsilon)|\pi\big)< v_\pi.$$

Then, by Lemma 12, there exist a number $\varepsilon_1>0$ and a neighbourhood Z_1 of the point x_0 such that $U_a(Z_1)>0$ and

$$K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S(\varepsilon)|\pi\big)< v_\pi-\varepsilon_1,\qquad x\in Z_1.$$

Hence

$$K(F;\ S(\varepsilon)|\pi) = \int_a^{a_{\pi_1}} K\big(D_x(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1});\ S(\varepsilon)|\pi\big)dU_a(x)$$

$$< \int_{x\notin Z_1} v_\pi dU_a(x) + \int_{x\in Z_1} (v_\pi - \varepsilon_1)dU_a(x) = v_\pi - \varepsilon_1 U_a(Z_1),$$

which contradicts the assumption of the optimality of the strategy F. Therefore, equality (45) holds.

LEMMA 16. *If the assumptions of Theorem* 1 *are satisfied in the case* $w = g$ *and if the strategy* F *and the family* $\{S(\varepsilon)\}_{\varepsilon>0}$ *are of the forms given by* (37) *and* (39), (40), *respectively, then under the condition* $a_1 = b_1 = b$ *the equalities*

$$P(b) + [1-P(b)][1-2Q(a_{\tau_1})] = v_\tau, \tag{49}$$

$$1-2Q(b) = v_\tau \tag{50}$$

are valid.

Proof. Using the optimality of the strategies F and $S(\varepsilon)$ we conclude that

$$v_\tau \leqslant \lim_{y\to a_{\tau_1}^-} K(F;\ y|\tau) \overset{(5)(6)}{=} P(b) + [1-P(b)][1-2Q(a_{\tau_1})]$$

$$\overset{(35)}{=} P(b) + [1-P(b)]K\left(F^*_{\tau_1};\ S^*_{\tau_1}\left(\frac{\varepsilon}{2}\right)|\tau\right)$$

$$\overset{(5)(4)(3)}{=} K\big(F;\ S(\varepsilon)|\tau\big) \leqslant v_\tau + \varepsilon.$$

Considering that ε is an arbitrary positive number we get equality (49).

Now we prove (50). Under the assumption $b = 0$, equalities (49) and (50) would imply the relation $Q(a_{\tau_1}) = 0$, which contradicts inequality (29). Therefore we have $b > 0$.

Hence, reasoning analogously as above, we get

$$v_\tau \leqslant \lim_{y\to b^-} K(F;\ y|\tau) \overset{(5)(6)}{=} 1-2Q(b)$$

$$= \lim_{\varepsilon\to 0^+} \int [1-2Q(y)]dH_{\langle b,b+\delta(\varepsilon)\rangle}(y) \overset{(5)(4)(3)}{=} \lim_{\varepsilon\to 0^+} K\big(D_{a_{\tau_1}}(x_1)\cdot F^*_{\tau_1}(\bar{x}_{n,1});\ S(\varepsilon)|\tau\big)$$

$$\leqslant \lim_{\varepsilon\to 0^+} (v_\tau + \varepsilon) = v_\tau,$$

which implies (50).

Proof of Theorem 1. First we consider the case $w = c$.

By Lemmas 13 and 14 it can be concluded that the strategy F and the family $\{S(\varepsilon)\}_{\varepsilon>0}$ are of the forms

$$F(\bar{x}_n) = U_a(x_1)\cdot F^*_{\pi_1}(\bar{x}_{n,1}) \qquad (\|\pi\| = n),$$

$$S(\varepsilon) = \left[(1-\alpha)S_{|\pi|}^{T_a} + \alpha S_{\pi_1}^{*}\left(\frac{\varepsilon}{2}\right)\right] \quad (\varepsilon > 0),$$

where $\operatorname{supp} U_a = \operatorname{supp} T_a = \langle a, a_{\pi_1}\rangle$, and a and α are certain numbers such that $0 \leqslant a < a_{\pi_1}$ and $0 \leqslant \alpha < 1$.

Let $y \in (a, a_{\pi_1})$. Then

$$\begin{aligned}
v_\pi &\overset{(44)(5)}{=} \int_a^{a_{\pi_1}} K\big(D_x(x_1)\cdot F_{\pi_1}^{*}(\bar{x}_{n,1});\ y\,|\,\pi\big)\, dU_a(x) \\
&\overset{(5)(6)}{=} \int_a^{y} \big\{P(x) + [1-P(x)]K\big(F_{\pi_1}^{*}(\bar{x}_{n,1});\ y\,|\,\pi_1\big)\big\}\, dU_a(x) + \\
&\quad + \int_y^{a_{\pi_1}} [1-2Q(y)]\, dU_a(x) \\
&\overset{(5)(6)}{=} \int_a^{y} \{P(x) + [1-P(x)][1-2Q(y)]\}\, dU_a(x) + \int_y^{a_{\pi_1}} [1-2Q(y)]\, dU_a(x) \\
&= 1-2Q(y) + \int_a^{y} 2Q(y)P(x)\, dU_a(x).
\end{aligned}$$

The above transformation and (5) lead to the identity

$$K(F_\pi^{*};\ y\,|\,\pi) = 1-2Q(y) + \int_{a_\pi}^{y} 2Q(y)P(x)\, dU_\pi(x), \qquad a_\pi \leqslant y < a_{\pi_1} \tag{51}$$

(it is sufficient to repeat the transformation for $U_a = U_\pi$ defined by (24) beginning with the second component of the sequence of equalities) and to the equation

$$\int_a^{y} P(x_1)\, dU_a(x_1) = \frac{v_\pi - 1}{2Q(y)} + 1. \tag{52}$$

Integrating by parts we see that $U_a(x_1)$ must be absolutely continuous in the interval (a, a_{π_1}), and then, differentiating with respect to y, we get

$$\frac{dU_a(x_1)}{dx_1} = \frac{(v_\pi - 1)Q'(x_1)}{2P(x_1)Q^2(x_1)}, \qquad a < x_1 < a_{\pi_1}.$$

Further, relation (52) is valid for $y = a$, which yields

$$v_\pi = 1-2Q(a), \tag{53}$$

whence

$$\frac{dU_a(x_1)}{dx_1} = \frac{Q(a)Q'(x_1)}{P(x_1)Q^2(x_1)}, \qquad a < x_1 < a_{\pi_1}.$$

Since U_a is a probability measure, we have

$$\int_a^{a_{\pi_1}} \frac{Q(a)Q'(x_1)}{P(x_1)Q^2(x_1)}\,dx_1 = 1.$$

Thus, in view of (31), $a = a_\pi$. Hence, using (24) and (53), we conclude that $F = F_\pi^*$ and

(54) $$v_\pi = 1-2Q(a_\pi).$$

Now we show that $S(\varepsilon) = S_\pi^*(\varepsilon)$ for any $\varepsilon > 0$.

Assuming $x \in \langle a_\pi, a_{\pi_1})$ we have

$$\begin{aligned}
v_\pi &\overset{(45)}{=} K\big(D_x(x_1)\cdot F_{\pi_1}^*(\bar{x}_{n,1});\ S(\varepsilon)\,|\,\pi\big)\\
&\overset{(9)(13)}{=} (1-a)\int_{a_\pi}^{a_{\pi_1}} K\big(D_x(x_1)\cdot F_{\pi_1}^*(\bar{x}_{n,1});\ y\,|\,\pi\big)dT_a(y)+\\
&\quad + aK\left(D_x(x_1)\cdot F_{\pi_1}^*(\bar{x}_{n,1});\ S_{\pi_1}^*\left(\frac{\varepsilon}{2}\right)|\,\pi\right)\\
&\overset{(3)\text{-}(6)}{=} (1-a)\int_{a_\pi}^{x} [1-2Q(y)]dT_a(y)+\\
&\quad +(1-a)\int_x^{a_{\pi_1}} \{P(x)+[1-P(x)][1-2Q(y)]\}dT_a(y)+\\
&\quad + a\left\{P(x)+[1-P(x)]K\left(F_{\pi_1}^*;\ S_{\pi_1}^*\left(\frac{\varepsilon}{2}\right)|\,\pi_1\right)\right\}\\
&\overset{(35)}{=} (1-a)\int_{a_\pi}^{a_{\pi_1}} [1-2Q(y)]dT_a(y)+(1-a)\int_x^{a_{\pi_1}} 2Q(y)P(x)dT_a(y)+\\
&\quad + a\{1-2Q(a_{\pi_1})[1-P(x)]\}.
\end{aligned}$$

The above transformation leads to the identities

(55) $$K\big(D_x(x_1)\cdot F_{\pi_1}^*(\bar{x}_{n,1});\ S_\pi^*(\varepsilon)\,|\,\pi\big) = (1-a_\pi)\int_{a_\pi}^{a_{\pi_1}} [1-2Q(y)]dT_\pi(y)+$$
$$+(1-a_\pi)\int_x^{a_{\pi_1}} 2Q(y)P(x)dT_\pi(y)+a_\pi\{1-2Q(a_{\pi_1})[1-P(x)]\},\qquad a_\pi \leqslant x < a_\pi$$

(it is sufficient to repeat that transformation for $T_a = T_\pi$, $a = a_\pi$ and $S(\varepsilon) = S_\pi^*(\varepsilon)$, beginning with the second component of the sequence of equalities) and

$$(56)\qquad v_\pi = (1-a)\int_{a_\pi}^{a_{\pi_1}} [1-2Q(y)]dT_a(y) + + (1-a)\int_{x}^{a_{\pi_1}} 2Q(y)P(x)dT_a(y) + a\{1-2Q(a_{\pi_1})[1-P(x)]\},$$

$$a \leqslant x < a_{\pi_1}.$$

Putting $x = a_{\pi_1}$ in the last identity we get

$$v_\pi = (1-a)\int_{a_\pi}^{a_{\pi_1}} [1-2Q(y)]dT_a(y) + a\{1-2Q(a_{\pi_1})[1-P(a_{\pi_1})]\}.$$

Hence identity (56) can be transformed to the following form:

$$\int_{x}^{a_{\pi_1}} Q(y)dT_a(y) = \frac{a}{1-a}\,\frac{Q(a_{\pi_1})[P(a_{\pi_1})-P(x)]}{P(x)}, \qquad a_\pi \leqslant x < a_{\pi_1}.$$

Integrating by parts we see that $T_a(y)$ must be absolutely continuous in the interval (a_π, a_{π_1}), and then, differentiating with respect to x, we have

$$\frac{dT_a(y)}{dy} = \frac{lP'(y)}{Q(y)P^2(y)}, \qquad a_\pi < y < a_{\pi_1},$$

where

$$l = P(a_{\pi_1})Q(a_{\pi_1})\frac{a}{1-a}.$$

Since T_a is a probability measure, we obtain

$$\int_{a_\pi}^{a_{\pi_1}} dT_a(y) = 1.$$

Summarizing, equations (25), (26) and (34) imply $a = a_\pi$ and $T_a = T_\pi$, which means that $S(\varepsilon) = S_\pi^*(\varepsilon)$ for any $\varepsilon > 0$.

Thus Theorem 1 has been proved in the case $w = c$.

Now we consider the case $w = g$. By Lemma 14 the strategy F and the family $\{S(\varepsilon)\}_{\varepsilon>0}$ are of the forms

$$F(\bar{x}_n) = D_b(x_1)\cdot F_{\tau_1}^*(\bar{x}_{n,1}) \qquad (\|\tau\| = n),$$

$$S(\varepsilon) = \left[H_{\langle b,b+\delta(\varepsilon)\rangle}, \left\{S_{\tau_1}^*\left(\frac{\varepsilon}{2}\right)\right\}_{\nu_1}\right] \qquad (\varepsilon > 0).$$

Comparing the left-hand sides of equations (49) and (50) we get

$$Q(a_{\tau_1}) = \frac{Q(b)}{1-P(b)},$$

which, in view of (21), gives $b = a_\tau$ and

$$v_\tau = 1-2Q(a_\tau). \tag{57}$$

Therefore, $F = F_\tau^*$, and if $\delta(\varepsilon) = \delta_\tau(\varepsilon)$, $H_{\langle b,b+\delta(\varepsilon)\rangle} = H_\tau^*(\varepsilon)$, then, finally, $S(\varepsilon) = S_\tau^*(\varepsilon)$.

Thus Theorem 1 is valid also in the case $w = g$.

At the end of this section we give a lemma which will be used in the proof of optimality.

LEMMA 17. *The following equalities are valid:*

$$K(F_\pi^*;\ y|\pi) = v_\pi, \qquad a_\pi \leqslant y < a_{\pi_1}, \tag{58}$$

$$K\big(D_x(x_1)\cdot F_{\pi_1}^*(\bar{x}_{n,1});\ S_\pi^*(\varepsilon)|\pi\big) = v_\pi, \qquad \varepsilon > 0,\ a_\pi \leqslant x < a_{\pi_1}. \tag{59}$$

Proof. Equality (58) follows immediately from (51) and (24), and equality (59) is a simple consequence of (55) and (25).

2. Proof of optimality of strategies F_λ^* and $S_\lambda^*(\varepsilon)$. In this section we show that the strategies F_λ^* and $S_\lambda^*(\varepsilon)$ are optimal for player A and ε-optimal for player B, respectively, and the number v_λ given by (54) and (57) is the value of the game Γ_λ.

LEMMA 18. *The following inequalities are valid:*

(i) $K\big(F_{1|g}^*;\ S\big|1|g\big) \geqslant v_{1|g}$ *for all* $S \in B_1^0(1|g)$,

(ii) $K\big(F;\ S_{1|g}^*(\varepsilon)\big|1|g\big) \leqslant v_{1|g}$ *for all* $F \in A_1$ $(\varepsilon > 0)$,

(iii) $K\big(F_{n|c}^*;\ S\big|n|c\big) \geqslant v_{n|c}$ *for all* $S \in B_0^0(n|c)$ $(n \geqslant 1)$,

(iv) $K\big(F;\ S_{n|c}^*(\varepsilon)\big|n|c\big) \leqslant v_{n|c}$ *for all* $F \in A_n$ $(\varepsilon > 0, n \geqslant 1)$.

Proof. The first two inequalities are a particular case of the result of paper [1], the next two were proved in [3].

THEOREM 2. *For every vector λ the strategies F_λ^* and $S_\lambda^*(\varepsilon)$ satisfy the inequalities*

$$K(F_\lambda^*;\ S|\lambda) \geqslant v_\lambda \tag{60}$$

for any $S \in B_{|\lambda|}^0(\lambda)$ *and*

$$K\big(F;\ S_\lambda^*(\varepsilon)|\lambda\big) \leqslant v_\lambda + \varepsilon \tag{61}$$

for any $F \in A_{\|\lambda\|}$ *and for all* $\varepsilon > 0$.

Proof. We prove by induction with respect to the number $n = \|\lambda\|$.

Step 1. If $n = 1$, the theorem follows immediately from Lemma 18.

Step 2. Assume that for some $n \geqslant 1$ and for every $(\bar{k}_r|w) = \lambda'$ such that $\|\lambda'\| = n$ inequalities (60) and (61) are satisfied for $\lambda = \lambda'$.

Step 3. Let us fix in an arbitrary way a vector $(\bar{k}_r|w) = \lambda$ such that $\|\lambda\| = n+1$. We show, under the inductive hypothesis, that inequalities (60) and (61) hold for the vector λ fixed above.

At first we consider the case $w = g$.

Let $S = [G^m, \{s_m(y)\}_y] = [G^m, \{S_{m-1}(\nu_1)\}_{\nu_1}] \in B^0_m(\tau)$ $(m = |\tau|)$ according to the notation given in the Remark in Section 2 of paper [2]. We investigate, using the statement of Step 2, the expression $K(F^*_\tau;\ s_m(y)|\tau)$.

(i) If $y < a_\tau$, we have

$$K(F^*_\tau;\ s_m(y)|\tau) \overset{(5)(4)}{=} 1-2Q(y) > 1-2Q(a_\tau) \overset{(57)}{=} v_\tau.$$

(ii) If $y = a_\tau$, we have

$$K(F^*_\tau;\ s_m(y)|\tau) \overset{(5)(4)}{=} 1-Q(a_\tau)-Q(a_\tau)[1-P(a_\tau)] > 1-2Q(a_\tau) \overset{(57)}{=} v_\tau.$$

(iii) If $y > a_\tau$, we have

$$K(F^*_\tau;\ s_m(y)|\tau) \overset{(5)(4)}{=} P(a_\tau)+[1-P(a_\tau)]K(F^*_{\tau_1};\ S_{m-1}(a_\tau)|\tau_1)$$
$$\geqslant P(a_\tau)+[1-P(a_\tau)]v_{\tau_1} \overset{(21)(57)}{=} v_\tau.$$

Summarizing cases (i)-(iii), we get

$$K(F^*_\tau;\ s_m(y)|\tau) \geqslant v_\tau, \qquad 0 \leqslant y \leqslant 1,$$

whence, by (5) and (19), for every $S \in B^0_m(\tau)$ we obtain

$$K(F^*_\tau;\ S|\tau) \geqslant v_\tau. \tag{62}$$

Now we prove inequality (61) in the case $w = g$ for the vector τ fixed in Step 3. For simplification we introduce the following notation: $b = a_\tau + \delta_\tau(\varepsilon)$ and $H = H^*_\tau(\varepsilon)$.

Analogously as before we estimate the expression $K(\bar{x}_{n+1};\ S^*_\tau(\varepsilon)|\tau)$ for $\bar{x}_{n+1} \in \bar{X}_{n+1}$ and $\varepsilon > 0$.

(i) If $x_1 \leqslant a_\tau$, we have

$$K(\bar{x}_{n+1};\ S^*_\tau(\varepsilon)|\tau) \overset{(3)(4)}{=} P(x_1)+[1-P(x_1)]K\left(\bar{x}_{n+1,1};\ S^*_{\tau_1}\left(\frac{\varepsilon}{2}\right)|\tau_1\right)$$
$$\leqslant P(x_1)+[1-P(x_1)]\left(v_{\tau_1}+\frac{\varepsilon}{2}\right)$$
$$\overset{(57)}{\leqslant} P(a_\tau)+[1-P(a_\tau)][1-2Q(a_{\tau_1})]+\varepsilon \overset{(21)(57)}{=} v_\tau+\varepsilon.$$

(ii) If $x_1 \in (a_\tau, b)$, we have

$$K(\bar{x}_{n+1};\ S_\tau^*(\varepsilon)\,|\,\tau)$$

$$\overset{(3)(4)}{=} \int\limits_{a_\tau}^{x_1} [1-2Q(y)]\,dH(y) +$$

$$+ \int\limits_{x_1}^{b} \left\{P(x_1) + [1-P(x_1)]K\left(\bar{x}_{n+1,1};\ S_{\tau_1}^*\left(\frac{\varepsilon}{2}\right)|\,\tau_1\right)\right\} dH(y)$$

$$\leqslant \int\limits_{a_\tau}^{x_1} [1-2Q(a_\tau)]\,dH(y) + \int\limits_{x_1}^{b} \left\{P(b) + [1-P(b)]K\left(\bar{x}_{n+1,1};\ S_{\tau_1}^*\left(\frac{\varepsilon}{2}\right)|\,\tau_1\right)\right\} dH(y)$$

$$\overset{(57)}{\leqslant} \int\limits_{a_\tau}^{x_1} [1-2Q(a_\tau)]\,dH(y) + \int\limits_{x_1}^{b} \left\{P(b) + [1-P(b)]\left[1-2Q(a_{\tau_1}) + \frac{\varepsilon}{2}\right]\right\} dH(y)$$

$$\overset{(22)}{\leqslant} \int\limits_{a_\tau}^{x_1} [1-2Q(a_\tau)]\,dH(y) + \int\limits_{x_1}^{b} \left\{1-2Q(a_{\tau_1})\left[1-P(a_\tau) - \frac{\varepsilon}{4}\right] + \frac{\varepsilon}{2}\right\} dH(y)$$

$$\leqslant \int\limits_{a_\tau}^{x_1} [1-2Q(a_\tau)]\,dH(y) + \int\limits_{x_1}^{b} \{1-2Q(a_{\tau_1})[1-P(a_\tau)] + \varepsilon\}\,dH(y)$$

$$\overset{(21)}{=} \int\limits_{a_\tau}^{x_1} [1-2Q(a_\tau)]\,dH(y) + \int\limits_{x_1}^{b} \{1-2Q(a_\tau) + \varepsilon\}\,dH(y)$$

$$\leqslant \int\limits_{a_\tau}^{b} [1-2Q(a_\tau)]\,dH(y) + \varepsilon \overset{(57)}{=} v_\tau + \varepsilon.$$

(iii) If $x_1 \geqslant b$, we have

$$K(\bar{x}_{n+1};\ S_\tau^*(\varepsilon)\,|\,\tau) \overset{(3)(4)}{=} \int\limits_{a_\tau}^{b} [1-2Q(y)]\,dH(y) \leqslant \int\limits_{a_\tau}^{b} [1-2Q(a_\tau)]\,dH(y) \overset{(57)}{=} v_\tau.$$

Summarizing, for any $x_{n+1} \in \bar{X}_{n+1}$ we have

$$K(\bar{x}_{n+1};\ S_\tau^*(\varepsilon)\,|\,\tau) \leqslant v_\tau + \varepsilon \qquad (\varepsilon > 0),$$

which, by (5), implies finally

$$K(F;\ S_\tau^*(\varepsilon)\,|\,\tau) \leqslant v_\tau + \varepsilon \tag{63}$$

for any $F \in A_{n+1}$ and for all $\varepsilon > 0$.

Thus, under the inductive hypothesis, inequalities (60) and (61) have been proved in the case $w = g$.

Now we consider the case $w = c$.

Let $S = [G^m, \{s_m(y)\}_y] \in B_m^0(\pi)$ $(m = |\pi|)$. We investigate, using the statement of Step 2, the expression $K(F_\pi^*;\ s_m(y)\,|\,\pi)$.

(i) If $y < a_\pi$, we have

$$K(F_\pi^*;\ s_m(y)|\pi) \overset{(5)(4)}{=} 1-2Q(y) > 1-2Q(a_\pi) \overset{(54)}{=} v_\pi.$$

(ii) If $y \in \langle a_\pi, a_{\pi_1})$, we have

$$K(F_\pi^*;\ s_m(y)|\pi) \overset{(5)(4)(6)}{=} K(F_\pi^*;\ y|\pi) \overset{(58)}{=} v_\pi.$$

(iii) If $y \geqslant a_{\pi_1}$, we have

$$K(F_\pi^*;\ s_m(y)|\pi) \overset{(5)(4)}{=} \int\limits_{a_\pi}^{a_{\pi_1}} \{P(x_1)+[1-P(x_1)]K(F_{\pi_1}^*;\ s_m(y)|\pi_1)\}\,dU_\pi(x_1)$$

$$\overset{(54)}{\geqslant} \int\limits_{a_\pi}^{a_{\pi_1}} \{P(x_1)+[1-P(x_1)][1-2Q(a_{\pi_1})]\}\,dU_\pi(x_1) \overset{(23)(24)}{=} 1-2Q(a_\pi) \overset{(54)}{=} v_\pi.$$

Therefore for all $y \in \langle 0, 1\rangle$ we get

$$K(F_\pi^*;\ s_m(y)|\pi) \geqslant v_\pi,$$

which, by (5) and (19), implies

$$K(F_\pi^*;\ S|\pi) \geqslant v_\pi \tag{64}$$

for all strategies $S \in B_m^0(\pi)$.

Now, taking into account relations (62)-(64), one can see that in order to complete the inductive proof of Theorem 2 it suffices to show that the inequality

$$K(F;\ S_\pi^*(\varepsilon)|\pi) \leqslant v_\pi+\varepsilon \tag{65}$$

is valid for any $F \in A_{n+1}$ and for all $\varepsilon > 0$.

In view of Lemma 18 (iii) we can restrict our investigation only to the case $r \geqslant 2$ $(\pi = (\bar{k}_r|c))$.

Inequality (65) is equivalent, by (5), to

$$K(\bar{x}_{n+1};\ S_\pi^*(\varepsilon)|\pi) \leqslant v_\pi+\varepsilon \tag{66}$$

for any $\varepsilon > 0$ and for all $\bar{x}_{n+1} \in \bar{X}_{n+1}$.

To prove this inequality we use the forthcoming Lemmas 19-23.

At first we introduce the following new notation, taking $\pi_i = (\bar{k}_{r,i}|c)$:

$$T(V) = \sum_{i=1}^{k_r} \prod_{j=1}^{i-1} a_{\pi_{j-1}}(1-a_{\pi_{i-1}})T_{\pi_{i-1}}(V) \quad \text{for } V \in \mathscr{B}(\langle 0,1\rangle)\ \Big(\prod_{j=1}^{0}(\cdot) = 1\Big),$$

$$p = a_\pi a_{\pi_1} \dots a_{\pi_{k_r-1}},$$

$$t_s = a_\pi a_{\pi_1} \dots a_{\pi_{s-1}}(1-a_{\pi_s})l_{\pi_s} \quad (1 \leqslant s < k_r), \tag{67}$$

where T_{π_i}, a_{π_i} and l_{π_i} $(i = 0, 1, \dots, k_r-1)$ are determined by (25)-(27).

We define the auxiliary strategies $S^j_\pi(\varepsilon)$ and $S^j_{\pi_1}(\varepsilon)$ $(\varepsilon > 0)$ for every $j = 0, 1, \ldots, \|\pi\|$ as follows:

$$S^j_{\pi_j}(\varepsilon) = \begin{cases} D_1 & \text{if } j = \|\pi\|, \\ S^*_{\pi_j}(\varepsilon) & \text{if } j < \|\pi\|, \end{cases}$$

$$S^j_{\pi_i}(\varepsilon) = \begin{cases} [(1-a_{\pi_i})S^{T_{\pi_i}}_{|\pi_i|} + a_{\pi_i} S^j_{\pi_{i+1}}(\varepsilon/2)] \\ \quad \text{if } \sum_{n=0}^{2k-1} k_{r-n} \leqslant i < \sum_{n=0}^{2k} k_{r-n} \text{ for some } k \geqslant 0 \ (\sum_{n=0}^{-1}(\cdot) = 0), \\ S^j_{\pi_{i+1}}(\varepsilon/2) \quad \text{otherwise}. \end{cases}$$

By Lemmas 5-7 it is easy to conclude that the strategies $S^j_\pi(\varepsilon)$ and $S^j_{\pi_1}$ are admissible in the game Γ_{π_j}.

Let us associate with every noisy action of player A in the game Γ_π a point in which this action is taken under the condition that player A uses the strategy F^*_π. Now, the strategy $S^j_\pi(\varepsilon)$ can be interpreted in the following manner. Player B, applying the strategy $S^j_\pi(\varepsilon)$ $(0 \leqslant j \leqslant \|\pi\|)$, behaves according to the strategy $S^*_\pi(\varepsilon)$ if all noisy actions of player A belonging to the group of j initial actions were taken not later than in the associated points.

For strategies $S^j_\pi(\varepsilon)$ and $S^j_{\pi_1}(\varepsilon)$ we obtain, with the aid of (3), (4), (9) and (13), the following equalities:

(68) $$K(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\,|\,\pi) = 1 - \prod_{i=1}^{j}[1-P(x_i)] + \prod_{i=1}^{j}[1-P(x_i)]K(\bar{x}_{n+1,j};\ S^j_\pi(\varepsilon)\,|\,\pi_j)$$

$$\text{if } 0 \leqslant x_1 \leqslant \ldots \leqslant x_j \leqslant a_\pi \quad (0 \leqslant j \leqslant \|\pi\|),$$

(69) $$K(\bar{x}_{n+1};\ S^*_{\pi_1}(\varepsilon)\,|\,\pi) = 1 - \prod_{i=1}^{j}[1-P(x_i)] + \prod_{i=1}^{j}[1-P(x_i)]K(\bar{x}_{n+1,j};\ S^j_{\pi_1}(\varepsilon)\,|\,\pi_j)$$

$$\text{if } 0 \leqslant x_1 \leqslant \ldots \leqslant x_j \leqslant a_{\pi_1} \quad (0 \leqslant j \leqslant \|\pi\|),$$

70) $$K(\bar{x}_{n+1,1};\ S^*_{\pi_1}(\varepsilon)\,|\,\pi_1) = 1 - \prod_{i=2}^{j}[1-P(x_i)] + \prod_{i=2}^{j}[1-P(x_i)]K(\bar{x}_{n+1,j};\ S^j_{\pi_1}(\varepsilon)\,|\,\pi_j)$$

$$\text{if } 0 \leqslant x_2 \leqslant \ldots \leqslant x_j \leqslant a_{\pi_1} \quad (0 \leqslant j \leqslant \|\pi\|).$$

In the forthcoming lemmas we use the additional notation:

$$\pi_{kr} = (\bar{k}_{r,k_r}|c).$$

LEMMA 19. *For any $\varepsilon > 0$ and for all $\bar{x}_{n+1} \in \bar{X}_{n+1}$ satisfying $x_1 > a_{\pi_{kr}}$ the inequality*

$$K(\bar{x}_{n+1};\ S_\pi^*(\varepsilon)|\pi) \leqslant K(\bar{x}_{n+1}^{(1)};\ S_\pi^*(\varepsilon)|\pi) + \frac{\varepsilon}{2}$$

is valid, where $\bar{x}_{n+1}^{(1)} = (a_{\pi_{kr}}, \bar{x}_{n+1,1})$.

Proof. By (3), (4), (9) and (13) we have

$$(71)\qquad K(\bar{x}_{n+1};\ S_\pi^*(\varepsilon)|\pi) = \int_{a_\pi}^{a_{\pi_{kr}}} [1-2Q(y)]\,dT(y) + pK\left(\bar{x}_{n+1};\ S_{\pi_{kr}}^*\left(\frac{\varepsilon}{2^{k_r}}\right)|\pi\right),$$

where T and p are given by (67).

For simplification we write

$$a = a_{\pi_{kr}}, \qquad b = a_{\pi_{kr}} + \delta_{\pi_{kr}}\left(\frac{\varepsilon}{2^{k_r}}\right),$$

$$H = H^*_{\langle a,b\rangle}, \qquad \varepsilon_r = \frac{\varepsilon}{2^{k_r+1}}, \qquad c = \min(x_1, b),$$

$$s^*(y) = [D_y, \{S^*_{\bar{k}_r,k_r+1|c}(\varepsilon_r)\}_{r_1}].$$

One can easily check, by (3) and (4), that

$$(72)\qquad K(\bar{x}_{n+1,1};\ s^*(y)|\pi_1) = 1-2Q(y), \qquad a < y < x_1.$$

Since $x_1 > a_{\pi_{kr}}$, we have

$$K\left(\bar{x}_{n+1};\ S_{\pi_{kr}}^*\left(\frac{\varepsilon}{2^{k_r}}\right)|\pi\right)$$

$$\overset{(3)(4)}{=} \int_a^c [1-2Q(y)]\,dH(y) + \int_c^b \{P(x_1) + [1-P(x_1)]K(\bar{x}_{n+1,1};\ s^*(y)|\pi_1)\}\,dH(y)$$

$$\leqslant \int_a^c \{P(a) + [1-P(a)][1-2Q(y)]\}\,dH(y) +$$

$$+ \int_c^b \{P(b) + [1-P(b)]K(\bar{x}_{n+1,1};\ s^*(y)|\pi)\}\,dH(y)$$

$$\overset{(22)}{\leqslant} \int_a^c \{P(a)+[1-P(a)][1-2Q(y)]\}dH(y)+$$
$$+\int_c^b \{P(a)+[1-P(a)]K(\bar{x}_{n+1,1};\ s^*(y)\,|\,\pi_1)+\varepsilon_r\}dH(y)$$
$$\overset{(72)}{\leqslant} \int_a^b \{P(a)+[1-P(a)]K(\bar{x}_{n+1,1};\ s^*(y)\,|\,\pi_1)\}dH(y)+\frac{\varepsilon}{2}$$
$$\overset{(3)(4)}{=} K\left(\bar{x}^{(1)}_{n+1};\ S^*_{\pi_{kr}}\left(\frac{\varepsilon}{2^{k_r}}\right)\Big|\,\pi\right)+\frac{\varepsilon}{2}.$$

Hence we conclude that the assertion of the lemma is true because of the validity of (71) for $\bar{x}_{n+1} = \bar{x}^{(1)}_{n+1}$.

LEMMA 20. *For every $\bar{x}_{n+1} \in \bar{X}_{n+1}$ such that $x_1 < a_\pi$ the inequality*

$$K(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\,|\,\pi) \leqslant K(\bar{x}^{(2)}_{n+1};\ S^*_\pi(\varepsilon)\,|\,\pi) \qquad (\varepsilon > 0)$$

is valid, where $\bar{x}^{(2)}_{n+1}$ is the vector obtained from $\bar{x}_{n+1}$ by putting the number a_π in place of all its components smaller than a_π.

Proof. It is easy to check using (68) that $K(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\,|\,\pi)$ considered as a function of variables $x_1, x_2, \ldots, x_j$ ($j = \max\{i\colon x_i < a_\pi\}$) is an increasing function on the set $0 \leqslant x_1 \leqslant \ldots \leqslant x_j < a_\pi$ with respect to each variable. This implies the assertion of the lemma.

LEMMA 21. *Under the inductive hypothesis the inequality*

$$K(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\,|\,\pi) \leqslant v_\pi + \frac{\varepsilon}{2} \qquad (\varepsilon > 0)$$

is valid if the vector $\bar{x}_{n+1}$ satisfies $a_{\pi_1} < x_1 \leqslant a_{\pi_{kr}}$.

Proof. Let us assume that $a_{\pi_s} < x_1 \leqslant a_{\pi_{s+1}}$ for some s ($1 \leqslant s < k_r-1$). Then, using (67), we evaluate

$$K(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\,|\,\pi)$$
$$\overset{(3)(4)(13)(9)}{=} \int_{a_\pi}^{a_{\pi_1}} [1-2Q(y)]dT(y)+\int_{a_{\pi_1}}^{x_1} [1-2Q(y)]dT(y)+$$
$$+\int_{x_1}^{a_{\pi_{kr}}} \{P(x_1)+[1-P(x_1)]K(\bar{x}_{n+1,1};\ y\,|\,\pi_1)\}dT(y)+$$
$$+p\left\{P(x_1)+[1-P(x_1)]K\left(\bar{x}_{n+1,1};\ S^*_{\pi_{kr}}\left(\frac{\varepsilon}{2^{k_r}}\right)\Big|\,\pi_1\right)\right\}$$
$$= \int_{a_\pi}^{a_{\pi_1}} [1-2Q(y)]dT(y)-\int_{a_{\pi_1}}^{x_1} 2P(x_1)Q(y)dT(y)+$$

$$+\int_{a_{\pi_1}}^{x_1}\{P(x_1)+[1-P(x_1)][1-2Q(y)]\}dT(y)+$$

$$+\int_{x_1}^{a_{\pi_{kr}}}\{P(x_1)+[1-P(x_1)]K(\bar{x}_{n+1,1};\ y\,|\,\pi_1)\}dT(y)+$$

$$+p\left\{P(x_1)+[1-P(x_1)]K\left(\bar{x}_{n+1,1};\ S^*_{\pi_{kr}}\left(\frac{\varepsilon}{2^{k_r}}\right)|\,\pi_1\right)\right\}.$$

From (34) and (67) it follows that

$$\int_{a_{\pi_1}}^{a_{\pi_{kr}}} dT(y)+p = a_\pi.$$

Hence, taking into account the inductive hypothesis, we obtain

$$K\left(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\,|\,\pi\right)$$

$$=\int_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT(y)-\int_{a_{\pi_1}}^{x}2P(x_1)Q(y)dT(y)+a_\pi P(x_1)+$$

$$+[1-P(x_1)]\left\{\int_{a_{\pi_1}}^{x_1}[1-2Q(y)]dT(y)+\int_{x_1}^{a_{\pi_{kr}}}K(\bar{x}_{n+1,1};\ y\,|\,\pi_1)dT(y)+\right.$$

$$\left.+pK\left(\bar{x}_{n+1,1};\ S^*_{\pi_{kr}}\left(\frac{\varepsilon}{2^{k_r}}\right)|\,\pi_1\right)\right\}$$

$$\overset{(3)(4)(9)(13)}{=}\int_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT(y)-\int_{a_{\pi_1}}^{x_1}2P(x_1)Q(y)dT(y)+$$

$$+a_\pi\left\{P(x_1)+[1-P(x_1)]K\left(\bar{x}_{n+1,1};\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)|\,\pi_1\right)\right\}$$

$$\leqslant\int_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT(y)-\sum_{i=1}^{s-1}\int_{a_{\pi_i}}^{a_{\pi_{i+1}}}2P(x_1)Q(y)dT(y)-$$

$$-\int_{a_{\pi_s}}^{x_1}2P(x_1)Q(y)dT(y)+a_\pi\{P(x_1)+[1-P(x_1)][1-2Q(a_{\pi_1})]\}+\frac{\varepsilon}{2}$$

$$=\int_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT(y)+2P(x_1)\left\{-\frac{t_1}{P(a_{\pi_1})}+\frac{t_1}{P(a_{\pi_2})}-\right.$$

$$\left.-\frac{t_2}{P(a_{\pi_2})}+\frac{t_2}{P(a_{\pi_3})}-\ldots-\frac{t_s}{P(a_{\pi_s})}+\frac{t_s}{P(x_1)}\right\}+$$

$$+a_\pi\{1-2Q(a_{\pi_1})[1-P(x_1)]\}+\frac{\varepsilon}{2}.$$

Let us consider the function

$$E(x,j) = 2P(x)\left\{-\frac{t_1}{P(a_{\pi_1})}+\frac{t_1}{P(a_{\pi_2})}-\ldots-\frac{t_j}{P(a_{\pi_j})}+\frac{t_j}{P(x)}\right\}+ \\ +a_\pi\{1-2Q(a_{\pi_1})[1-P(x)]\}$$

defined on the set $\langle a_{\pi_j}, a_{\pi_{j+1}}\rangle \times \{1, 2, \ldots, s\}$.

Since

$$-\frac{t_1}{P(a_{\pi_1})}+a_\pi Q(a_{\pi_1}) \overset{(26)(27)(67)}{=} -\frac{a_\pi P(a_{\pi_1})Q(a_{\pi_1})}{1-P(a_{\pi_1})}<0$$

and

$$\frac{t_i}{P(a_{\pi_i})}-\frac{t_{i+1}}{P(a_{\pi_{i+1}})} \overset{(26)(27)(67)}{=} -a_\pi a_{\pi_1}\ldots a_{\pi_i}\frac{P(a_{\pi_{i+1}})Q(a_{\pi_{i+1}})}{1-P(a_{\pi_{i+1}})}<0,$$

$E(x,j)$ is a decreasing function of the variable x in the interval $\langle a_{\pi_j}, a_{\pi_{j+1}}\rangle$ for every fixed $j = 1, 2, \ldots, s$ and, consequently,

$$E(x_1, s) \leqslant E(a_{\pi_s}, s) = E(a_{\pi_s}, s-1) < E(a_{\pi_{s-1}}, s-1) \\ = \ldots = E(a_{\pi_2}, 1) < E(a_{\pi_1}, 1).$$

Now, returning again to the previous evaluations, we get

$$K(\bar{x}_{n+1};\ S_\pi^*(\varepsilon)|\pi) \\ \leqslant \int_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT(y)+E(x_1, s)+\frac{\varepsilon}{2} \\ < \int_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT(y)+E(a_{\pi_1}, 1)+\frac{\varepsilon}{2} \\ \overset{(47)(18)(19)}{=} (1-a_\pi)\int_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT_\pi(y)+a_\pi\{1-2Q(a_{\pi_1})[1-P(a_{\pi_1})]\}+\frac{\varepsilon}{2} \\ \overset{(55)(59)}{=} v_\pi+\frac{\varepsilon}{2}.$$

The lemma has been proved.

LEMMA 22. *If the vector* $\bar{x}_{n+1} \in \bar{X}_{n+1}$ *satisfies* $x_1 \geqslant a_\pi$, $x_j < a_{\pi_1}$, $x_{j+1} \geqslant a_{\pi_1}$ *for some* j, $2 \leqslant j \leqslant n+1$ $(x_{n+2} = 1)$, *then, under the inductive hypothesis, the inequality*

$$K(\bar{x}_{n+1};\ S_\pi^*(\varepsilon)|\pi) \leqslant K(\bar{x}_{n+1}^{(3)};\ S_\pi^*(\varepsilon)|\pi)+\frac{\varepsilon}{2} \qquad (\varepsilon > 0)$$

holds, where $\bar{x}^{(3)}_{n+1}$ *is the vector obtained from* $\bar{x}_{n+1}$ *by putting the value* a_{π_1} *in place of the components* $x_2, x_3, \ldots, x_j$.

Proof. Let $\bar{x}_{n+1}$ be the vector satisfying the assumption of the lemma. Then for $y \in (x_j, a_{\pi_1})$ we have

$$K(\bar{x}_{n+1,j};\ y\,|\,\pi_j) = \begin{cases} 1-2Q(y) & \text{if } 2 \leqslant j < n+1, \\ -1 & \text{if } j = n+1,\ w_{n+1} = g, \\ -Q(y) & \text{if } j = n+1,\ w_{n+1} = c, \end{cases}$$

which implies the following inequality:

$$(73)\qquad K(\bar{x}_{n+1,j};\ y\,|\,\pi_j) \leqslant 1-2Q(y), \qquad x_j < y < a_{\pi_1}\ (j = 2, 3, \ldots, n+1).$$

Now we introduce the notation

(74)
$$E_j(x_1, \ldots, x_{j-1}) = \int_{a_\pi}^{x_{j-1}} K(\bar{x}_{n+1};\ y\,|\,\pi)\,dT(y) + \int_{x_{j-1}}^{a_{\pi_1}} \Big\{1 - \prod_{i=1}^{j-1}[1-P(x_i)] + \prod_{i=1}^{j-1}[1-P(x_i)][1-2Q(y)]\Big\}\,dT(y) + a_\pi\Big\{1 - \prod_{i=1}^{j-1}[1-P(x_i)]\Big\},$$

where T is given in (67) and $j = 2, 3, \ldots, n+1$.

As is easy to see, the expression $E_j(x_1, \ldots, x_{j-1})$ depends only on the variables $x_1, x_2, \ldots, x_{j-1}$.

Now, using the definition of the strategy $S^*_\pi(\varepsilon)$, we evaluate the following:

$$K\big(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\,|\,\pi\big)$$
$$\overset{(3)(4)(6)(9)(13)}{=} \int_{a_\pi}^{x_{j-1}} K(\bar{x}_{n+1};\ y\,|\,\pi)\,dT(y) + \int_{x_{j-1}}^{a_{\pi_1}} \Big\{1 - \prod_{i=1}^{j-1}[1-P(x_i)] + \prod_{i=1}^{j-1}[1-P(x_i)]K(\bar{x}_{n+1,j-1};\ y\,|\,\pi_{j-1})\Big\}\,dT(y) + a_\pi K\Big(\bar{x}_{n+1};\ S^*_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi\Big)$$
$$\overset{(4)(69)}{=} \int_{a_\pi}^{x_{j-1}} K(\bar{x}_{n+1};\ y\,|\,\pi)\,dT(y) + \int_{x_{j-1}}^{a_{\pi_1}} \Big\{1 - \prod_{i=1}^{j-1}[1-P(x_i)]\Big\}\,dT(y) + \prod_{i=1}^{j-1}[1-P(x_i)]\Big\{\int_{x_{j-1}}^{x_j}[1-2Q(y)]\,dT(y) + \int_{x_j}^{a_{\pi_1}}\{P(x_j) + [1-P(x_j)]K(\bar{x}_{n+1,j};\ y\,|\,\pi_j)\}\,dT(y)\Big\} + a_\pi\Big\{1 - \prod_{i=1}^{j}[1-P(x_i)] + \prod_{i=1}^{j}[1-P(x_i)]K\Big(\bar{x}_{n+1,j};\ S^j_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi_j\Big)\Big\}$$

$$\leqslant \int_{a_\pi}^{x_{j-1}} K(\bar{x}_{n+1};\; y\,|\,\pi)\,dT(y) + \int_{x_{j-1}}^{a_{\pi_1}} \Big\{1 - \prod_{i=1}^{j-1} [1-P(x_i)]\Big\}\,dT(y) +$$

$$+ \prod_{i=1}^{j-1} [1-P(x_i)] \Big\{ \int_{x_{j-1}}^{x_j} [1-2Q(y)]\,dT(y) +$$

$$+ \int_{x_j}^{a_{\pi_1}} \{P(x_j) + [1-P(x_j)][1-2Q(y)]\}\,dT(y)\Big\} +$$

$$+ a_\pi \Big\{1 - \prod_{i=1}^{j} [1-P(x_i)] + \prod_{i=1}^{j} [1-P(x_i)]\,K\Big(\bar{x}_{n+1,j};\; S^j_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi_j\Big)\Big\}$$

$$\overset{(74)}{=} E_j(x_1, \ldots, x_{j-1}) + \prod_{i=1}^{j-1} [1-P(x_i)] \Big\{ \int_{x_j}^{a_{\pi_1}} 2P(x_j)Q(y)\,dT(y) + a_\pi P(x_j) +$$

$$+ a_\pi [1-P(x_j)]\,K\Big(\bar{x}_{n+1,j};\; S^j_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi_j\Big)\Big\}$$

$$\overset{(67)}{=} E_j(x_1, \ldots, x_{j-1}) + \prod_{i=1}^{j-1} [1-P(x_i)] \Big\{ 2(1-a_\pi) l_\pi P(x_j) \Big\{\frac{1}{P(x_j)} - \frac{1}{P(a_{\pi_1})}\Big\} +$$

$$+ a_\pi P(x_j) + a_\pi [1-P(x_j)]\,K\Big(\bar{x}_{n+1,j};\; S^j_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi_j\Big)\Big\}$$

$$\overset{(26)}{=} E_j(x_1, \ldots, x_{j-1}) +$$

$$+ \prod_{i=1}^{j-1} [1-P(x_i)] \Big\{ 2(1-a_\pi) l_\pi + a_\pi K\Big(\bar{x}_{n+1,j};\; S^j_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi_j\Big) +$$

$$+ P(x_j)\Big[-2a_\pi Q(a_{\pi_1}) + a_\pi - a_\pi K\Big(\bar{x}_{n+1,j};\; S^j_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi_j\Big)\Big]\Big\}$$

$$\leqslant E_j(x_1, \ldots, x_{j-1}) + \prod_{i=1}^{j-1} [1-P(x_i)] \Big\{ 2(1-a_\pi) l_\pi +$$

$$+ a_\pi K\Big(\bar{x}_{n+1,j};\; S^j_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi_j\Big) +$$

$$+ P(x_j)\Big[-2a_\pi Q(a_{\pi_1}) + a_\pi + a_\pi \frac{\varepsilon}{2} - a_\pi K\Big(\bar{x}_{n+1,j};\; S^j_{\pi_1}\Big(\frac{\varepsilon}{2}\Big)\,|\,\pi_j\Big)\Big]\Big\}.$$

Further, we investigate, using relation (70) and the inductive assumption, the coefficient which stands at $P(x_j)$ in the last expression:

$$-2a_\pi Q(a_{\pi_1})+a_\pi+a_\pi\frac{\varepsilon}{2}-a_\pi K\left(\bar{x}_{n+1,j};\ S^j_{\pi_1}\left(\frac{\varepsilon}{2}\right)\mid\pi_j\right)$$

$$\geqslant -2a_\pi Q(a_{\pi_1})+a_\pi+a_\pi\frac{\varepsilon}{2}-a_\pi\Big\{1-[1-P(a_{\pi_1})]^{j-1}+$$

$$+[1-P(a_{\pi_1})]^{j-1}K\left(\bar{x}_{n+1,j};\ S^j_{\pi_1}\left(\frac{\varepsilon}{2}\right)\mid\pi_j\right)\Big\}$$

$$\overset{(70)}{=} -2a_\pi Q(a_{\pi_1})+a_\pi+a_\pi\frac{\varepsilon}{2}-a_\pi K\left((\underbrace{a_{\pi_1},\ldots,a_{\pi_1}}_{j-1\text{ times}},\bar{x}_{n+1,j});\ S^*_{\pi_1}\left(\frac{\varepsilon}{2}\right)\mid\pi_1\right)$$

$$\geqslant -2a_\pi Q(a_{\pi_1})+a_\pi+a_\pi\frac{\varepsilon}{2}-a_\pi\left(v_{\pi_1}+\frac{\varepsilon}{2}\right)\overset{(37)}{=}0.$$

Therefore, the above-investigated coefficient is non-negative. Consequently, returning to the previous evaluations, we have

$$K(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\mid\pi)$$

$$\leqslant E_j(x_1,\ldots,x_{j-1})+\prod_{i=1}^{j-1}[1-P(x_i)]\Big\{2(1-a_\pi)l_\pi+a_\pi K\left(\bar{x}_{n+1,j};\ S^j_{\pi_1}\left(\frac{\varepsilon}{2}\right)\mid\pi_j\right)+$$

$$+P(a_{\pi_1})\left[-2a_\pi Q(a_{\pi_1})+a_\pi+a_\pi\frac{\varepsilon}{2}-a_\pi K\left(\bar{x}_{n+1,j};\ S^j_{\pi_1}\left(\frac{\varepsilon}{2}\right)\mid\pi_j\right)\right]\Big\}$$

$$= K\big((x_1,\ldots,x_{j-1},a_{\pi_1},x_{j+1},\ldots,x_{n+1});\ S^*_\pi(\varepsilon)\mid\pi\big)+$$

$$+a_\pi P(a_{\pi_1})\prod_{i=1}^{j-1}[1-P(x_i)]\frac{\varepsilon}{2},$$

where the last equality is obtained analogously as in evaluating the expression $K(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\mid\pi)$ in the initial part of the proof.

Now, repeating step by step the procedure outlined above for the components $x_{j-1}, x_{j-2}, \ldots, x_2$, we get finally

$$K(\bar{x}_{n+1};\ S^*_\pi(\varepsilon)\mid\pi)\leqslant K(\bar{x}^{(3)}_{n+1};\ S^*_\pi(\varepsilon)\mid\pi)+a_\pi P(a_{\pi_1})\sum_{s=2}^{j}\prod_{i=1}^{s-1}[1-P(x_i)]\frac{\varepsilon}{2}.$$

However, the second term of the right-hand side of this inequality can be estimated as follows:

$$a_\pi P(a_{\pi_1}) \sum_{s=2}^{j} \prod_{i=1}^{s-1} [1-P(x_i)] \frac{\varepsilon}{2}$$

$$\leqslant a_\pi P(a_{\pi_1}) \sum_{s=2}^{\infty} [1-P(a_\pi)]^{s-1} \frac{\varepsilon}{2} = a_\pi P(a_{\pi_1}) \frac{1-P(a_\pi)}{P(a_\pi)} \frac{\varepsilon}{2}$$

$$\overset{(27)}{=} \frac{Q(a_\pi)}{Q(a_{\pi_1})} \frac{\varepsilon}{2} \overset{(30)}{<} \frac{\varepsilon}{2}.$$

Summarizing, we have

$$K(\bar{x}_{n+1};\ S_\pi^*(\varepsilon)\,|\,\pi) \leqslant K(\bar{x}_{n+1}^{(3)};\ S_\pi^*(\varepsilon)\,|\,\pi) + \frac{\varepsilon}{2} \qquad (\varepsilon > 0),$$

which completes the proof of Lemma 22.

LEMMA 23. *If the vector $\bar{x}_{n+1} \in \bar{X}_{n+1}$ satisfies $a_\pi \leqslant x_1 \leqslant a_{\pi_1}$ and $x_2 \geqslant a_{\pi_1}$, then under the inductive hypothesis the inequality*

$$K(\bar{x}_{n+1};\ S_\pi^*(\varepsilon)\,|\,\pi) \leqslant v_\pi + \frac{\varepsilon}{2}$$

is valid for any $\varepsilon > 0$.

Proof. Using the definition of the strategy $S_\pi^*(\varepsilon)$ and the assumption of the lemma we obtain

$$K(\bar{x}_{n+1};\ S_\pi^*(\varepsilon)\,|\,\pi)$$

$$\overset{(3)(4)(6)(9)(13)}{=} (1-a_\pi) \int_{a_\pi}^{a_{\pi_1}} K(\bar{x}_{n+1};\ y\,|\,\pi)\, dT_\pi(y) +$$

$$+ a_\pi \left\{ P(x_1) + [1-P(x_1)] K\left(\bar{x}_{n+1,1};\ S_{\pi_1}^*\left(\frac{\varepsilon}{2}\right) | \pi_1\right) \right\}$$

$$\overset{(54)}{\leqslant} (1-a_\pi) \int_{a_\pi}^{a_{\pi_1}} K(\bar{x}_{n+1}; y\,|\,\pi)\, dT_\pi(y) + a_\pi\{P(x_1) + [1-P(x_1)][1-2Q(a_{\pi_1})]\} + \frac{\varepsilon}{2}$$

$$\overset{(6)}{=} (1-a_\pi) \left\{ \int_{a_\pi}^{x_1} [1-2Q(y)]\, dT_\pi(y) + \int_{x_1}^{a_{\pi_1}} \{P(x_1) + \right.$$

$$\left. + [1-P(x_1)][1-2Q(y)]\} dT_\pi(y) \right\} + a_\pi\{1-2Q(a_{\pi_1})[1-P(x_1)]\} + \frac{\varepsilon}{2}$$

$$= (1-a_\pi)\int\limits_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT_\pi(y)+$$

$$+(1-a_\pi)\int\limits_{x_1}^{a_{\pi_1}}2Q(y)P(x_1)dT_\pi(y)+a_\pi\{1-2Q(a_{\pi_1})[1-P(x_1)]\}+\frac{\varepsilon}{2}$$

$$\overset{55)}{=} K\big(D_{x_1}\cdot F^*_{\pi_1}(\bar{x}_{n+1,1});\ S^*_\pi(\varepsilon)\,|\,\pi\big)+\frac{\varepsilon}{2}\overset{(59)}{=}v_\pi+\frac{\varepsilon}{2}.$$

This completes the proof of the lemma.

Now, summarizing the results of Lemmas 19-23 we infer that the inductive hypothesis (Step 2) implies inequality (66).

Hence, by (5), we get finally

$$K\big(F;\ S^*_\pi(\varepsilon)\,|\,\pi\big)\leqslant v_\pi+\varepsilon$$

for any $\varepsilon > 0$ and for all $F\in A_{n+1}$. This inequality, together with (62)-(64), completes the inductive proof of Theorem 2.

We end our reasonings with a conclusion being equivalent to Theorem 2:

For an arbitrary vector λ the game Γ_λ has the value equal to $v_\lambda = 1-2Q(a_\lambda)$, F^*_λ is the optimal strategy for player A and $S^*_\lambda(\varepsilon)$ is the ε-optimal strategy for player B for any $\varepsilon > 0$.

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MIESZANA GRA CZASOWA: PROBLEM OPTYMALNOŚCI

STRESZCZENIE

Niniejsza praca jest kontynuacją pracy [2]. Jest ona dalszym ciągiem rozważań dotyczących modelu gry czasowej na zbiorze $\langle 0, 1\rangle$, w której gracz A dysponuje dowolną skończoną liczbą akcji cichych i głośnych, a gracz B — jedną akcją głośną. W szczególności dowodzi się twierdzenia o jednoznaczności strategii optymalnych, sformułowanego w [2], oraz wykazuje się, że znalezione tam strategie są optymalne.