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# THE $G_\delta$-TOPOLOGY AND $K$-ANALYTIC SPACES WITHOUT PERFECT COMPACT SETS

BY

JOSE L. BLASCO (VALENCIA)

**Introduction.** The word *space* will refer to Tychonov spaces. A space $X$ is called *functionally countable* if each real-valued continuous function on $X$ has countable image. In the first part of this work we prove that all Baire functions on a functionally countable space (not necessarily Lindelöf) have countable image, herewith solving a question raised by Levy and Rice ([10], Question 3, P 1207). In fact, a slightly stronger result is proved in Theorem 1.

The second part deals with $K$-analytic spaces. A space $X$ is called $K$-*analytic* if it is the image of the Baire 0-dimensional product space $N^N$ under an upper-semicontinuous compact-valued map. If this map is disjoint, then $X$ is called a $K$-*Lusin space*. The main result is Theorem 5 where we add some new equivalent conditions to those already known which characterize $K$-analytic spaces without compact perfect subsets ([18], p. 107). This result simultaneously generalizes the Jayne Theorem ([9], Theorem 6) as well as some characterizations of compact dispersed spaces due to Meyer [13], Pełczyński and Semadeni [17], Levy and Rice [10].

As an application of our results we prove that a perfectly normal $K$-analytic space is either the countable union of dispersed compact subspaces or contains a compact perfect set. Finally, we also characterize the $K$-analytic spaces $X$ without compact perfect sets such that $C^*(X)$ is isometric to $B_\alpha^*(C(Y))$, $1 \leqslant \alpha \leqslant \omega_1$, for some space $Y$.

**The results.** As usual, $C(X)$ will denote the ring of all continuous real-valued functions on a space $X$. The set of all bounded functions in $C(X)$ is denoted by $C^*(X)$. The set of points of $X$ where a member of $C(X)$ is equal to zero is called a *zero-set*. We shall denote by $Z(X)$ the collection of all zero-sets of $X$.

By an *algebra on X* is meant a subring $A$ of $C(X)$ which separates points and closed sets, contains the real constants and is closed under uniform convergence and inversion (i.e., any function in $A$ which has no zeros is invertible in $A$). In general, an algebra on $X$ need not coincide with $C(X)$, e.g., the Baire functions on the real line $R$, viewed as an algebra on $R$ with the discrete topology. But when $X$ is Lindelöf, the only algebra on $X$ is $C(X)$ ([15], Corollary 4.7).

Given an algebra $A$ on $X$, let $B_0(A) = A$, and define $B_\alpha(A)$ inductively for each ordinal $\alpha$ to be the space of pointwise limits of sequences of functions in $\bigcup\{B_\sigma(A)\colon \sigma < \alpha\}$. If $\omega_1$ is the first uncountable ordinal, then $B(A) = B_{\omega_1}(A) = B_{\omega_1+1}(A)$, and the members of $B(A)$ are called the *Baire functions generated by A*.

1. THEOREM. *Let A be an algebra on a space X. If each function in A has countable image, then $B(A) = B_1(A)$ and each function in $B(A)$ has countable image.*

Proof. Let $f \in B_2(A)$, let $\{f_n\}_{n=1}^\infty$ be a sequence from $B_1(A)$ converging pointwise to $f$ and let $\{f_{nm}\}_{m=1}^\infty$ be a sequence from $A$ converging pointwise to each $f_n$, $n = 1, 2, \ldots$ We define

$$\varphi\colon X \to R^{N\times N}$$

by

$$\varphi(x) = \{f_{nm}(x)\}_{n,m=1}^\infty.$$

Clearly, $\varphi$ is continuous. For each positive integer $n$ define a function $\tilde{f}_n$ on $\varphi(X)$ by $\tilde{f}_n \circ \varphi = f_n$. Then the sequence of projections $\{\pi_{nm}\}_{m=1}^\infty$ converges pointwise to $\tilde{f}_n$ on $\varphi(X)$, and therefore

$$\tilde{f}_n \in B_1(C(\varphi(X))).$$

If $\tilde{f}$ is the function defined on $\varphi(X)$ by $\tilde{f}\circ\varphi = f$, it follows that $\{\tilde{f}_n\}_{n=1}^\infty$ converges pointwise to $\tilde{f}$ on $\varphi(X)$, hence

$$\tilde{f} \in B_2(C(\varphi(X))).$$

Since $A$ is closed under composition with continuous functions defined on $\varphi(X)$ ([15], Theorem 4.9), the metric space $\varphi(X)$ is functionally countable. By Remark (iii) in [10], $\varphi(X)$ is countable, therefore $f(X) = \tilde{f}(\varphi(X))$ is countable and

$$B(C(\varphi(X))) = B_1(C(\varphi(X))).$$

Then there exists a sequence $\{h_k\}_{k=1}^\infty$ in $C(\varphi(X))$ converging pointwise on $\varphi(X)$ to $\tilde{f}$. Consequently, $\{h_k \circ \varphi\}_{k=1}^\infty$ is a sequence in $A$ which converges pointwise to $f$, and hence $f \in B_1(A)$.

Remark. As we shall see below, if $X$ is $K$-analytic and

$$B(C(X)) = B_1(C(X)),$$

then $X$ is functionally countable. Assuming the continuum hypothesis, we can see that there exists an uncountable subset $S$ of the real line such that

$$B(C(S)) = B_1(C(S))$$

([12], Example 5.2).

A *P-space* is a (Tychonov) space in which each $G_\delta$-set (or zero-set) is open. The *$G_\delta$-topology* of a space $X$ is the topology having for a basis the family of all $G_\delta$-sets (or zero-sets) of $X$. Write $bX$ for $X$ provided with the associated $G_\delta$-topology. The collection $Ba(X)$ of all Baire sets in $X$ is the $\sigma$-algebra generated by the zero-sets of $X$. A collection $\mathscr{F}$ of subsets of $X$ is *completely Baire-additive* if the union of each subfamily of $\mathscr{F}$ is a Baire set in $X$.

The following results are easily established and will be needed in the sequel:

(F1) *If $S$ is a subset of $X$, then the $G_\delta$-topology on $S$ is the restriction to $S$ of the $G_\delta$-topology on $X$.*

(F2) *If $X = \bigcup \{S_n: n = 1, 2, \ldots\}$ and for each $n$ the space $bS_n$ is Lindelöf, then $bX$ is Lindelöf.*

(F3) *If $X$ is Lindelöf and each point $x$ in $X$ has a neighborhood $V(x)$ such that $bV(x)$ is Lindelöf, then $bX$ is Lindelöf.*

A space is *realcompact* if it is homeomorphic to a closed subset of a product of real lines.

2. LEMMA. *Let $X$ be a realcompact space. If $bX$ is not Lindelöf, then there is a Baire set $G$ in $X$ such that $bG$ and $b(X\backslash G)$ are not Lindelöf.*

Proof. The collection $Ba(X)$ is a base for the closed sets of $bX$. Therefore, if $bX$ is not Lindelöf, then there exists a $Ba(X)$-filter $\mathscr{F}$ which is closed under countable intersections and $\bigcap \mathscr{F} = \emptyset$. As $X$ is realcompact, every $Ba(X)$-ultrafilter closed under countable intersections has non-empty intersection (see [1], Theorem 4, and [5], Theorem 8.4). Therefore $\mathscr{F}$ is not a $Ba(X)$-ultrafilter. Consequently, there is a Baire set $G$ which does not belong to $\mathscr{F}$ meeting every member of $\mathscr{F}$. As $\bigcap \mathscr{F} = \emptyset$ and every Baire set in $X$ is closed and open (clopen) in $bX$, it follows that $bG$ and $b(X\backslash G)$ are not Lindelöf.

3. LEMMA. *Let $S$ be a subset of a space $Y$ such that every Baire set in $S$ is Lindelöf. If $bS$ is not Lindelöf, then in $Y$ there exist disjoint closed sets $F_1$ and $F_2$ such that $b(F_1 \cap S)$ and $b(F_2 \cap S)$ are not Lindelöf.*

Proof. By Lemma 2 there is a Baire set $G$ in $S$ such that $bG$ and $b(S\backslash G)$ are not Lindelöf. By our assumption, $G$ and $S\backslash G$ are Lindelöf, and by (F3) there exist two points $p \in G$ and $q \in S\backslash G$ such that if $V_1$ and $V_2$ are neighborhoods in $Y$ of $p$ and $q$, respectively, then $b(V_1 \cap G)$ and $b(V_2 \cap (S\backslash G))$ are not Lindelöf. Let $F_1$ and $F_2$ be disjoint zero-set-neighborhoods in $Y$ of $p$ and $q$, respectively.

The set $F_1 \cap G$ is clopen in $b(F_1 \cap S)$ because it is a Baire set in $F_1 \cap S$. As $b(F_1 \cap G)$ is not Lindelöf, $b(F_1 \cap S)$ is not Lindelöf. We conclude the proof by applying a similar argument to the set $F_2 \cap S$.

We say that a map $K$ from a space $E$ to the power set of a space $X$ is *upper-semicontinuous* if for each $x$ in $E$ and each open set $U$ of $X$ containing

$K(x)$ there is a neighborhood $V$ of $x$ with $K(V) \subset U$. A space $X$ is called *$K$-analytic* if it is of the form

$$X = K(N^N) = \bigcup \{K(\sigma)\colon \sigma \in N^N\},$$

where $K$ is an upper-semicontinuous map from $N^N$ to the compact sets of $X$.

We use $N^{(N)}$ to denote the set of finite sequences of positive integers. For each $s = \{s_1, \ldots, s_n\}$ in $N^{(N)}$ we will use $I(s)$ to denote the set of points $\sigma$ of $N^N$ with $\sigma_i = s_i$, $1 \leqslant i \leqslant n$. In the proof of Theorem 3.5.1 in [18], p. 63, the following result is established:

4. THEOREM. *Let $X$ be a $K$-analytic space with a representation $X = K(N^N)$. Suppose that for each finite sequence $\{\eta_1, \ldots, \eta_n\}$, $\eta_i = 0$ or 1, there are defined a closed set $H(\eta_1, \ldots, \eta_n)$ and a positive integer $s(\eta_1, \ldots, \eta_n)$ satisfying*

(a) *$H(\eta_1, \ldots, \eta_n, j) \subset H(\eta_1, \ldots, \eta_n)$ for each $j \geqslant 1$;*

(b) *$K\big(I(s(\eta_1), \ldots, s(\eta_1, \ldots, \eta_n))\big) \cap H(\eta_1, \ldots, \eta_n)$ is non-empty;*

(c) *for fixed $n$, the sets $H(\eta_1, \ldots, \eta_n)$ are all disjoint.*

*Then $X$ contains a compact perfect set.*

The following is the main result:

5. THEOREM. *The following conditions are equivalent for a $K$-analytic space $X$:*

(1) *$X$ contains no compact perfect sets.*

(2) *$bX$ is Lindelöf.*

(3) *If $F$ is a subring of $C^*(bX)$ which contains all the real constants and separates the points of $X$, then $C(bX) = B_1(F)$.*

(4) *$C(bX) = B\big(C(X)\big)$.*

(5) *$bX$ is pseudo-$\aleph_1$-compact (i.e., each locally finite family of open sets in $bX$ is countable).*

(6) *$X$ is functionally countable.*

(7) *$B\big(C(X)\big) = B_1\big(C(X)\big)$.*

(8) *$B\big(C(X)\big) = B_\alpha\big(C(X)\big)$ for some $\alpha < \omega_1$.*

(9) *If $F$ is a subring of $C^*(X)$ which is closed under bounded inversion (i.e., $f, g \in F$, $f/g$ bounded and $g^2 > 0$ imply $f/g \in F$) and contains a function which has no zeros, then $F$ is not isometric to $C([0, 1])$.*

Proof. non(2)$\Rightarrow$non(1). Let $X = K(N^N)$ be a representation of the $K$-analytic space $X$. For each $s$ in $N^{(N)}$, write $A(s) = K\big(I(s)\big)$ and note that

$$(*) \qquad A(s) = \bigcup \{A(s, n)\colon n = 1, 2, \ldots\}.$$

Suppose that $bX$ is not Lindelöf. Since every Baire set in $X$ is $K$-analytic ([18], p. 23) and therefore Lindelöf ([18], p. 36), by Lemma 3 there exist disjoint closed sets $H(0)$ and $H(1)$ in $X$ such that $bH(0)$ and $bH(1)$ are not Lindelöf. By (F2) and (*) we can choose positive integers $s(0)$ and $s(1)$ such that the sets $H(0) \cap A\big(s(0)\big)$ and $H(1) \cap A\big(s(1)\big)$ are not Lindelöf in its $G_\delta$-topology.

For a fixed finite sequence $\{\eta_1, \ldots, \eta_n\}$, $\eta_i = 0$ or 1, suppose we have defined the closed set in $X$, $H(\eta_1, \ldots, \eta_n)$, and the positive integer $s(\eta_1, \ldots, \eta_n)$ so that the $K$-analytic set

$$H(\eta_1, \ldots, \eta_n) \cap A(s(\eta_1), \ldots, s(\eta_1, \ldots, \eta_n))$$

is not Lindelöf in its $G_\delta$-topology.

By Lemma 3 there exist disjoint closed sets in $X$, $F(\eta_1, \ldots, \eta_n, 0)$ and $F(\eta_1, \ldots, \eta_n, 1)$ such that

$$F(\eta_1, \ldots, \eta_n, j) \cap H(\eta_1, \ldots, \eta_n) \cap A(s(\eta_1), \ldots, s(\eta_1, \ldots, \eta_n)), \quad j = 0, 1,$$

is not Lindelöf in its $G_\delta$-topology.

Put

$$H(\eta_1, \ldots, \eta_n, j) = F(\eta_1, \ldots, \eta_n, j) \cap H(\eta_1, \ldots, \eta_n).$$

By (F2) and (*) we can choose positive integers

$$s(\eta_1, \ldots, \eta_n, 0) \quad \text{and} \quad s(\eta_1, \ldots, \eta_n, 1)$$

such that

$$H(\eta_1, \ldots, \eta_n, j) \cap A(s(\eta_1), \ldots, s(\eta_1, \ldots, \eta_n, j)), \quad j = 0, 1,$$

is not Lindelöf in its $G_\delta$-topology.

By an induction process, for each finite sequence $\{\eta_1, \ldots, \eta_n\}$, $\eta_i = 0$ or 1, we have defined a closed set $H(\eta_1, \ldots, \eta_n)$ in $X$ and a positive integer $s(\eta_1, \ldots, \eta_n)$ satisfying the conditions of Theorem 4. Therefore $X$ contains a compact perfect set.

(2)$\Rightarrow$(3). Let $X_0$ be the set $X$ provided with the smallest topology such that all functions in $F$ are continuous. Since $F$ separates the points of $X$, $X_0$ is a Hausdorff space and, by Theorem 3.7 in [4], $X_0$ is Tychonov. The identity from $bX$ onto $X_0$ is continuous because $F \subset C^*(bX)$, and since the $G_\delta$-topology of $X_0$ is the coarsest $P$-space topology finer than the original topology of $X_0$, the identity from $bX$ onto $bX_0$ is continuous. By assumption, $bX$ is Lindelöf and, since every Lindelöf subspace of a $P$-space is closed, we infer that the identity from $bX$ onto $bX_0$ is a closed map, and therefore a homeomorphism. Then $C(bX_0) = C(bX)$.

On the other hand, $F$ is a subring of $C^*(X_0)$ which contains all the real constants (i.e., $F$ is an algebra of bounded functions on $X_0$ in the terminology of [2]) and the topology of $X_0$ is the smallest topology such that all functions in $F$ are continuous. Since $X_0$ is Lindelöf, we have $C(X_0) \subset B_1(F)$ ([2], Theorem 2,E) and, consequently, $B_1(F) = B_1(C(X_0))$. Now then $bX_0$ is Lindelöf, therefore $C(bX_0)$ is the only algebra on $bX_0$ ([15], Corollary 4.7), and since $B_1(C(X_0))$ is an algebra on $bX_0$ ([12], Theorem 3.1), it follows that

$$C(bX_0) = B_1(C(X_0)).$$

Then

$$C(bX) = C(bX_0) = B_1(C(X_0)) = B_1(F).$$

(3)⇒(4) is immediate.

(4)⇒(5). We will employ the technique used in [10], Proposition 5.6. By Theorem 2.6 in [20] it is enough to show that each discrete family of non-empty open sets in $bX$ is countable. (Recall that a family $\mathcal{U}$ of subsets of a space is *discrete* if each point of the space has a neighborhood meeting at most one element of $\mathcal{U}$.)

Let $\{U_i\colon i \in I\}$ be a discrete family of non-empty clopen sets in $bX$ and, for each $i \in I$, let $h_i$ be the characteristic function of $U_i$. If $J \subset I$, then the function

$$f_J = \sum \{h_i\colon i \in J\}$$

is continuous on $bX$, so, by assumption,

$$f_J \in B(C(X)) \quad \text{and} \quad \bigcup \{U_i\colon i \in J\} \in Ba(X).$$

Then $\{U_i\colon i \in I\}$ is a disjoint completely Baire-additive family on $X$. As $X$ is $K$-analytic, by Lemma 2 in [3] the set $I$ is countable.

(5)⇒(6). This follows from the fact that every pseudo-$\aleph_1$-compact $P$-space is functionally countable.

(6)⇒(7). This follows immediately from Theorem 1.

(7)⇒(8) is immediate.

(8)⇒(1). This implication is proved in [9], Theorem 1.

Thus we have established the equivalence of (1)–(8).

(9)⇒(1). Suppose $X$ contains a compact perfect subset $K$. Then there exists a continuous map $\varphi$ from $K$ onto the unit interval [0, 1] ([17], p. 214) and by the Tietze Extension Theorem we can extend $\varphi$ to a continuous map $\tilde{\varphi}$ from $X$ onto [0, 1]. Then

$$F = \{g \circ \tilde{\varphi}\colon g \in C([0, 1])\}$$

is a subring of $C^*(X)$ containing all the real constants, which is closed under bounded inversion and isometric to $C([0, 1])$.

non(9)⇒non(6). Suppose $F$ is a subring of $C^*(X)$ isometric to $C([0, 1])$, which is closed under bounded inversion and contains a function $f$ which vanishes nowhere. If $m$ and $n$ are integers, $n \neq 0$, then $m/n = mf/nf$ belongs to $F$, hence $F$ contains all the rational constants. By our assumption, $F$ is complete in the uniform norm, therefore $F$ is closed and contains all the real constant functions.

Consider the following equivalence relation in $X$: $x \equiv x'$ if $g(x) = g(x')$ for every $g \in F$. Let $Y$ be the set of all equivalence classes and let $\varphi$ be the mapping from $X$ onto $Y$ such that $\varphi(x)$ is the equivalence class that contains $x$. For each $g \in F$, define a function $\tilde{g}$ on $Y$ by $\tilde{g}(\varphi(x)) = g(x)$, $x \in X$, and set $\tilde{F} = \{\tilde{g}\colon g \in F\}$. Now endow $Y$ with the smallest topology such that all functions in $\tilde{F}$ are continuous. Then $Y$ is Tychonov, $\varphi$ is continuous and the mapping $g \to \tilde{g}$ is a ring isomorphism from $F$ onto $\tilde{F}$ ([4], 3I). Since $X$ is $K$-analytic and $\varphi$ is continuous, the space $Y$ is $K$-analytic ([18], p. 23).

It is clear that $\tilde{F}$ is a subring of $C^*(Y)$ which separates points and closed sets in $Y$, contains all the real constants and is closed under uniform convergence and bounded inversion. If $A(\tilde{F})$ is the algebra on $Y$ generated by $\tilde{F}$, then the set of all bounded functions in $A(\tilde{F})$ coincides with $\tilde{F}$ ([16], Corollary 3.6). Since each $K$-analytic space is Lindelöf, $Y$ is Lindelöf, and therefore $C(Y)$ is the only algebra on $Y$. Thus $C(Y) = A(\tilde{F})$ and $C^*(Y) = \tilde{F}$.

The mapping $\tilde{g} \to \tilde{g} \circ \varphi$ is a ring isomorphism from $\tilde{F}$ onto $F$, therefore $\tilde{F} = C^*(Y)$ is isometric to $C([0, 1])$ ([4], 1J). It is known that $C^*(Y)$ is ring isomorphic with $C(\beta Y)$, where $\beta Y$ denotes the Stone–Čech compactification of $Y$. Thus $C(\beta Y)$ is isometric to $C([0, 1])$ and, by the Banach–Stone Theorem, $\beta Y$ is homeomorphic to $[0, 1]$. Since a compact metric space cannot be the Stone–Čech compactification of another space ([4], Corollary 9.6), it follows that $Y = \beta Y$. Then $Y$ is homeomorphic to $[0, 1]$ and $X$ is not functionally countable.

Remarks. (a) The equivalence (1)$\Leftrightarrow$(2) in Theorem 5 says that $X$ does not contain a perfect compact subset if and only if every covering of $X$ consisting of $G_\delta$-sets contains a countable subcover. This in a natural way refers to the result that an analytic set does not contain a perfect compact set if and only if it is countable.

(b) The statement (3) in Theorem 5 can be considered as a Stone–Weierstrass theorem for $C(bX)$ (see [2]).

6. COROLLARY. *If $X$ is a $K$-analytic space which contains no compact perfect sets and $\varphi$ is a Baire-measurable map from $X$ onto a space $Y$, then $Y$ contains no compact perfect sets. If, in addition, every point of $Y$ is a $G_\delta$-set, then $Y$ is countable.*

Proof. Let us prove first that if a (Tychonov) space $E$ contains a compact perfect set $K$, then $bE$ is not Lindelöf. In fact, let $g$ be a continuous mapping from $K$ onto $[0, 1]$. Since $E$ is Tychonov, $g$ has an extension to a function $h$ in $C(E)$ ([4], p. 43). The family $\{h^{-1}(t): t \in h(E)\}$ is an uncountable discrete open cover of $bE$, therefore $bE$ is not Lindelöf.

By our hypothesis, $bX$ is Lindelöf and the map $\varphi$ from $bX$ onto $bY$ is continuous. Then $bY$ is Lindelöf, and therefore $Y$ contains no compact perfect subsets. If every point of $Y$ is a $G_\delta$-set, then the family $\{\{x\}: x \in Y\}$ is an open cover of $bY$. Since $bY$ is Lindelöf, there is a countable subcover, and therefore $Y$ is countable.

Remark. A different proof of Corollary 6 was given in [18], p. 106, under the assumption that $\varphi$ is continuous by applying Choquet's Capacitability Theorem.

In [9], Theorem 4, it is proved that a $K$-Lusin space is either the countable union of compact dispersed subspaces or contains a compact perfect set.

However, there exist $K$-analytic spaces without compact perfect subsets which are not $\sigma$-compact ([19], [18], p. 103). In fact, we have the following result:

7. COROLLARY. *Let $X$ be a perfectly normal $K$-analytic space. Then one of the following conditions holds*:

(1) *$X$ contains a compact perfect subset*;

(2) *$X$ is the countable union of dispersed compact subspaces.*

*In addition, these conditions are mutually exclusive.*

Proof. First, suppose that $X$ contains a compact perfect set. Then there exists a continuous map from $X$ onto $[0, 1]$, and therefore $X$ is not functionally countable. In this case $X$ is not the countable union of dispersed compact spaces, since every dispersed compact space is functionally countable.

Now suppose that $X$ does not contain compact perfect subsets. Thus, by Theorem 5, $bX$ is Lindelöf. If $X = K(N^N)$ is a representation of $X$, then $K(\sigma)$ is a compact zero-set for each $\sigma \in N^N$ and the family $\{K(\sigma)\colon \sigma \in N^N\}$ is an open cover of $bX$. Since $bX$ is Lindelöf, there is a countable subcover and, consequently, $X$ is $\sigma$-compact.

If $X$ and $Y$ are infinite dispersed compact spaces, then $C^*(X)$ is not isometric to $B_1^*(C(Y))$ ([8], Theorem 5). However, this result does not hold for $K$-analytic spaces without compact perfect subsets. For example, $C^*(N)$ is ring isomorphic with the ring of bounded Baire functions of class 1 in the one-point compactification of the discrete space $N$.

8. THEOREM. *Let $X$ be an infinite $K$-analytic space without compact perfect subsets. Then $C^*(X)$ is isometric to $B_\alpha^*(C(Y))$, $1 \leqslant \alpha \leqslant \omega_1$, for some space $Y$ if and only if $X$ is countable and contains a dense $C^*$-embedded copy of $N$.*

For the proof of this theorem we need some results.

9. LEMMA. *If $X$ is a $K$-analytic space in which each compact subset is finite, then $X$ is countable.*

Proof. We denote by $kX$ the set $X$ with the topology whose closed sets are precisely the sets meeting each compact subset of $X$ in a compact set. As each compact subset of $X$ is finite, all points of $kX$ are isolated.

Since $X$ is $K$-analytic, so is $kX$ ([18], p. 104), and hence $kX$ is Lindelöf. As $\{\{x\}\colon x \in X\}$ is an open cover of $kX$, it follows that $kX$ is countable.

Recall that a subspace $S$ of a space $E$ is *$C^*$-embedded in $E$* if every function in $C^*(S)$ can be extended to a function in $C(E)$. A space is called *basically disconnected* if the complement of every zero-set has an open closure. We write $vE$ for the smallest realcompact subspace of $\beta E$ which contains $E$. A point $x$ of $E$ is called a *$P$-point* if every $G_\delta$-set (or zero-set) in $E$ containing $x$ is a neighborhood of $x$. Thus $E$ is a $P$-space if and only if every point of $E$ is a $P$-point.

10. LEMMA. *For any space $E$ the following holds*:

(a) *If $E$ is dense in $F$, then every $P$-point in $E$ is a $P$-point in $F$.*

(b) *Every $P$-point in $\beta E$ belongs to $vE$.*

Proof. (a) Suppose that $x$ is a $P$-point in $E$ and let $f \in C(F)$. There is an open set $U$ in $F$ such that $x \in U$ and the restriction of $f$ to $U \cap E$ is constant ([4], 4L). Then the restriction of $f$ to $\mathrm{cl}_F(U \cap E)$ is constant, and since $\mathrm{cl}_F U = \mathrm{cl}_F(U \cap E)$, it follows that $f$ is constant in a neighborhood of $x$ in $F$. Hence $x$ is a $P$-point in $F$.

(b) If $x$ is a point in $\beta E \backslash vE$, there is a zero-set $Z$ in $\beta E$ such that $x \in Z$ and $Z \cap vE = \emptyset$ ([14], p. 947). Since $vE$ is dense in $\beta E$, it follows that $Z$ is not a neighborhood of $x$ in $\beta E$. Then $x$ is not a $P$-point in $\beta E$.

The Baire sets of a space $Y$ of *multiplicative class* $\alpha$, denoted by $Z_\alpha(Y)$, are defined to be the zero-sets of functions in $B_\alpha^*(C(Y))$ (the set of bounded functions in $B_\alpha(C(Y))$). Those of *additive class* $\alpha$, denoted by $CZ_\alpha(Y)$, are defined as the complements of sets in $Z_\alpha(Y)$. The sets in

$$A_\alpha(Y) = Z_\alpha(Y) \cap CZ_\alpha(Y)$$

are called the sets of *ambiguous class* $\alpha$. With the set-theoretic operations of union and intersection, $A_\alpha(Y)$ is a Boolean algebra for each $\alpha \leqslant \omega_1$.

It is known (see the proof of Theorem 1 in [8]) that there exists a compactification $Y_\alpha$ of $bY$, which is homeomorphic to the Stone space $\Omega_\alpha$ of the Boolean algebra $A_\alpha(Y)$, such that

$$B_\alpha^*(C(Y)) = \{f|_{bY}\colon f \in C(Y_\alpha)\}.$$

Therefore the spaces $B_\alpha^*(C(Y))$ and $C(Y_\alpha)$ are norm isometric. In the case $\alpha = \omega_1$ the Boolean algebra of clopen sets in $\Omega_\alpha$ is $\sigma$-complete, therefore the compact spaces $\Omega_{\omega_1}$ and $Y_{\omega_1}$ are basically disconnected.

Proof of Theorem 8. The sufficiency of the condition is clear. Let us see the necessity. Suppose that $C^*(X)$ is isometric to $B_\alpha^*(C(Y))$, $1 \leqslant \alpha \leqslant \omega_1$, for some space $Y$. Then $C(\beta X)$ is isometric to $C(Y_\alpha)$ and, according to the Banach–Stone Theorem, there exists a homeomorphism $\varphi$ from $Y_\alpha$ onto $\beta X$.

By Lemma 10, each point $z$ in $bY$ is a $P$-point in $Y_\alpha$, hence $\varphi(z)$ is a $P$-point in $\beta X$ and $\varphi(z) \in vX$. Since every Lindelöf space is realcompact, $X = vX$. Thus $\varphi(bY)$ is a dense subspace of $X$. We do not distinguish notationally between $bY$ and $\varphi(bY)$. Therefore, each function in $B_\alpha^*(C(Y))$ can be continuously extended to $\beta X$. In particular, each function in $C^*(Y)$ admits a continuous extension to $X$.

Let us see that $Y$ is functionally countable. Let $f \in C(Y)$ and suppose that $f(Y)$ is not countable. Then there is a positive integer $m$ such that the set $f(Y) \cap [-m, m]$ is not countable. The function $g = \min(|f|, m)$ is in $C^*(Y)$ and $g(Y)$ is not countable. If $h$ is the continuous extension of $g$ to $X$, it follows that $h(X)$ is not countable, which is a contradiction, because $X$ is functionally countable by Theorem 5.

Then $Y$ is functionally countable and, by Theorem 1,

$$B_1(C(Y)) = B(C(Y)).$$

Therefore $Z_\alpha(Y)$ is the $\sigma$-algebra of the Baire sets in $Y$ and the space $Y_\alpha$ is basically disconnected.

Since $\beta X$ is homeomorphic to $Y_\alpha$, we see that $\beta X$ is basically disconnected, and therefore every infinite compact set in $\beta X$ contains a copy of $\beta N$ ([4], 14N.5). By assumption, $X$ contains no compact perfect sets, therefore each compact subset of $X$ is finite. Since $X$ is $K$-analytic, by Lemma 9 we see that $X$ is countable.

Then $Y$ is countable, $bY$ is homeomorphic to $N$ and

$$B_\alpha^*(C(Y)) = C^*(N).$$

Since each function in $B_\alpha^*(C(Y))$ admits a continuous extension to $X$, it follows that $bY$ is a dense $C^*$-embedded copy of $N$.

**11.** Corollary. *If $K$ is an infinite dispersed compact space, then $C(K)$ is not isometric to $B_\alpha^*(C(Y))$, $1 \leqslant \alpha \leqslant \omega_1$, for any space $Y$.*

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## *REFERENCES*

[1] Z. Frolík, *Realcompactness is a Baire measurable property*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), pp. 617–621.

[2] — *Stone–Weierstrass theorems for $C(X)$ with the sequential topology*, Proc. Amer. Math. Soc. 27 (1971), pp. 486–494.

[3] — and P. Holický, *Decomposability of completely Suslin-additive families*, ibidem 82 (1981), pp. 359–365.

[4] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton 1960.

[5] H. Gordon, *Rings of functions determined by zero-sets*, Pacific J. Math. 36 (1971), pp. 133–157.

[6] P. R. Halmos, *Lectures on Boolean Algebras*, Van Nostrand, Princeton 1963.

[7] F. Hausdorff, *Set Theory*, Chelsea, New York 1957.

[8] J. E. Jayne, *The space of class $\alpha$ Baire functions*, Bull. Amer. Math. Soc. 80 (1974), pp. 1151–1156.

[9] — *Spaces of Baire functions. I*, Ann. Inst. Fourier (Grenoble) 24 (1974), pp. 47–76.

[10] R. Levy and M. D. Rice, *Normal P-spaces and the $G_\delta$-topology*, Colloq. Math. 44 (1981), pp. 227–240.

[11] R. D. Mauldin, *On the Baire system generated by a linear lattice of functions*, Fund. Math. 68 (1970), pp. 51–59.

[12] — *Baire functions, Borel sets and ordinary function systems*. Adv. in Math. 12 (1974), pp. 418–450.

[13] P. R. Meyer, *The Baire order problem for compact spaces*, Duke Math. J. 33 (1966), pp. 33–40.

[14] S. Mrówka, *Some properties of Q-spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 5 (1957), pp. 947–950.

[15] — *On some approximation theorems*, Nieuw Arch. Wisk. (3) 16 (1968), pp. 94–111.

[16] — *Characterization of classes of functions by Lebesgue sets*, Czechoslovak Math. J. 19 (1969), pp. 738–744.

[17] A. Pełczyński and Z. Semadeni, *Spaces of continuous functions. III (Spaces C(Ω) for Ω without perfect subsets)*, Studia Math. 18 (1959), pp. 211–222.

[18] C. A. Rogers, J. E. Jayne, C. Dellacherie, F. Topsøe, J. Hoffman-Jørgensen, D. A. Martin, A. S. Kechris and A. H. Stone, *Analytic Sets*, Academic Press, London 1980.

[19] M. Talagrand, *Sur les convexes compacts dont l'ensemble des points extrêmaux est K-analytique*, Bull. Soc. Math. France 107 (1979), pp. 49–53.

[20] M. R. Wiscamb, *The discrete countable chain condition*, Proc. Amer. Math. Soc. 23 (1969), pp. 608–612.

UNIVERSIDAD DE VALENCIA
FACULTAD DE CIENCIAS MATEMÁTICAS
DOCTOR MOLINER 50
BURJASOT (VALENCIA), ESPAÑA

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