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On a meromorphic function having few poles but not tending to infinity along a path

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Abstract. A classical theorem of Iversen states that an entire function $f(z)$ tends to ∞ as $z \to \infty$ along a suitable path. It is reasonable to ask whether a corresponding result still holds if $f(z)$ has a given sequence of poles, provided that the characteristic $T(r, f)$ grows sufficiently rapidly depending on the sequence. In this connection I proved recently [4] that if $\sum |z_n|^{-1/2} < \infty$, where z_n are the poles and if when $r \to \infty$

$$\liminf r^{-1/2} T(r, f) > 0,$$

then Iversen's theorem still holds. In this paper it is shown on the other hand that if r_n is any sequence of positive numbers tending to infinity with n and such that $\sum r_n^{-1/2} = \infty$, then $f(z)$ exists having no zeros and no poles at points other than the r_n, having essentially arbitrarily rapid growth, but not tending to infinity as z tends to infinity along any path. This example also shows that an earlier theorem of Edrei and Fuchs [2] cannot be extended to functions of infinite order.

1. Introduction and statement of results. According to a classical theorem of Iversen [5] a non-constant entire function $f(z)$ always has ∞ as an asymptotic value, i.e. there exists a path Γ going from a finite point to ∞ and such that

$$f(z) \to \infty \quad \text{as } z \to \infty \text{ along } \Gamma. \tag{1.1}$$

It is natural to ask whether Iversen's theorem remains true if $f(z)$ has sufficiently few poles compared with its growth. We use the classical notation of Nevanlinna (see e.g. [3], Chapter 1) and quote two results in this direction. The first is the following theorem of Edrei and Fuchs [2], Theorem 2:

THEOREM A. *Suppose that $f(z)$ is meromorphic of finite lower order ϱ. There exists a positive quantity $\delta(\varrho)$ depending only on ϱ such that if*

$$\overline{\lim_{r\to\infty}} \frac{N(r, \infty)+N(r, 0)}{T(r, f)} < \delta(\varrho), \tag{1.2}$$

then $f(z)$ has ∞ as an asymptotic value.

We note that in Theorem A we require $f(z)$ to have few poles and few zeros. Recently [4], Theorem 2, I proved a Theorem with a stronger hypothesis on the poles but no hypothesis on the zeros. This is

THEOREM B. *Suppose that f is meromorphic in the plane and that*

$$T(r,f)-\tfrac{1}{2}r^{1/2}\int_r^{\infty}\frac{N(t,\infty)\,dt}{t^{3/2}}\to\infty,\qquad \text{as } r\to\infty. \tag{1.3}$$

Then ∞ is an asymptotic value of $f(z)$. In particular the conclusion holds if

$$\int_{r_0}^{\infty}\frac{N(t,\infty)\,dt}{t^{3/2}}<\infty \tag{1.4}$$

and

$$\varliminf_{r\to\infty} r^{-1/2}\,T(r,f)>0. \tag{1.5}$$

If $z_n = r_n e^{i\theta_n}$ are the poles of $f(z)$ at points other than the origin, then condition (1.4) can be written in the form

$$\sum r_n^{-1/2}<\infty. \tag{1.6}$$

Thus if the poles of f satisfy condition (1.6) then, if f grows so rapidly that (1.5) holds, ∞ is an asymptotic value of $f(z)$.

In this paper we show that Theorem A fails for functions of infinite order and no condition weaker than (1.6) can ensure that if f grows sufficiently rapidly, then ∞ is an asymptotic value of f, even if we assume an addition that f has no zeros. Our result is

THEOREM C. *Suppose that r_n is a non-decreasing sequence of positive numbers tending to ∞ with n and not satisfying* (1.6). *Then there exists $f(z)$ meromorphic of infinite lower order in the plane and possessing no poles at points other than $z = r_n$ and no zeros, such that ∞ is not an asymptotic value of $f(z)$.*

The condition is to be understood in the sense that if the sequence r_n assumes a value r p-times, then $f(z)$ has at most a pole of multiplicity p at $z = r$ (and may be regular there). A refinement of the construction shows that, for a given sequence r_n, $T(r,f)$ may tend to ∞ more quickly than any preassigned function $\psi(r)$, but we shall not insist on that here.

We shall see that we may without loss of generality assume that $r_n > n^2$ in Theorem C, so that

$$N(r,\infty) = O(r^{1/2}),\qquad N(r,0) = 0, \tag{1.7}$$

while for every positive p

$$\frac{T(r,f)}{r^p}\to\infty$$

with r. Thus even a much weaker condition than (1.2) cannot ensure (1.1) for functions of infinite order, nor can sufficiently rapid growth compared with $N(r, \infty)$ ensure (1.1) if (1.4) fails.

2. An approximation lemma. We shall base our construction on the following approximation Theorem which was proved in a previous paper by Barth, Brannan and the author [1], Theorem 4.

LEMMA 1. *Suppose that we are given a harmonic polynomial $u(z)$ of degree N in x, y ($z = x+iy$), positive numbers ε, R and also a continuous function $\psi(r)$ satisfying $\psi(r) > 1$, $r > 0$ and*

$$(2.1) \qquad \int_1^\infty \frac{\psi(r)\,dr}{r^{3/2}} = \infty.$$

Then there exists a Jordan domain D containing $|z| < R$ and a function $v(z)$ harmonic in D, continuous in the closure $\bar{D}$ of D and satisfying

$$(2.2) \qquad |u(z)-v(z)| < \varepsilon, \quad |z| < R,$$

and

$$(2.3) \qquad v(z) \leqslant \psi(|z|)$$

for z on the boundary of D.

For our application we need to know a little more detail about the function $v(z)$ and the domain D in the above lemma. It turns out that D contains a sequence of sectorial regions

$$(2.4) \qquad S_\nu\colon\ R_\nu < |z| < R'_\nu, \quad |\arg z| < \pi-\eta_{\nu+1}, \quad \nu = 1 \text{ to } N$$

and the arcs

$$(2.5) \qquad |\arg z| = \pi-\eta_{\nu+1}, \quad R_\nu \leqslant |z| \leqslant R'_\nu$$

of S_ν lie outside D.

The quantities R_ν, R'_ν, η_ν are defined inductively [1], p. 17. First R_1 is chosen sufficiently large. If R_ν has been chosen, then R'_ν has to be chosen so large that $R'_\nu > 1000R_\nu$ and [1], p. 22,

$$\int_{9R_\nu}^{R'_\nu/9} \frac{\psi(t)\,dt}{t^{3/2}} > C_\nu, \quad \nu = 1 \text{ to } N,$$

where C_ν depends only on the construction so far and in particular on R_1 to R_ν, R'_1 to $R'_{\nu-1}$, and η_1 to η_ν. Then $\eta_{\nu+1}$ must be chosen sufficiently small and $R_{\nu+1}$ may be chosen arbitrarily subject to $R_{\nu+1} > R'_\nu+1$. Further $v(z) \leqslant 0$ on the whole boundary Γ of D except on the segments (2.5).

In particular it is not necessary to have given in advance a function $\psi(r)$ satisfying (2.1). It is sufficient to specify $\psi_\nu(r)$ for $R_\nu \leqslant r \leqslant R'_\nu$, possibly in dependence on the previous construction, in such a way that

$$\int_{9R_\nu}^{R'_\nu/9} \frac{\psi_\nu(r)\,dr}{r^{3/2}} > C_\nu, \tag{2.6}$$

for some $R'_\nu > 1000R_\nu$. We may then define

$$\psi(r) = \psi_\nu(r), \quad R_\nu < r < R'_\nu, \quad \nu = 1 \text{ to } N,$$
$$\psi_\nu(r) = 1,$$

otherwise and condition (2.3) is still satisfied.

3. Construction of the poles. We suppose given a sequence of numbers r_ν, such that

$$0 < r_\nu \leqslant r_{\nu+1}, \quad 1 \leqslant \nu < \infty, \tag{3.1}$$

$$r_\nu \to \infty, \quad \text{as } \nu \to \infty. \tag{3.2}$$

and

$$\sum_{\nu=1}^{\infty} r_\nu^{-1/2} = \infty. \tag{3.3}$$

We shall construct a subsequence of the r_ν which will satisfy certain conditions and so we suppose without loss of generality that in addition

$$r_\nu > \nu^2. \tag{3.4}$$

For otherwise we can select a subsequence r'_ν of the r_ν which satisfies (3.4) as wel as (3.1) to (3.3). In fact if r'_1 to r'_p have already been chosen with $r'_p = r_{\nu_p}$, we define ν_{p+1} to be the smallest integer such that $\nu_{p+1} > \nu_p$ and

$$r_{\nu_{p+1}} > (p+1)^2.$$

Such a choice is possible in view of (3.2). We then define

$$r'_{p+1} = r_{\nu_{p+1}}.$$

Evidently r'_p is a subsequence of the sequence r_ν which satisfies (3.1), (3.2) and (3.4). If for all $p \geqslant p_0$ we have $\nu_{p+1} = \nu_p + 1$, then we have

$$r'_p = r_{p+k}, \quad p \geqslant p_0,$$

where k is a constant and then

$$\sum (r'_p)^{-1/2} = \infty$$

in view of (3.3). If on the other hand we have $\nu_{p+1} > \nu_p + 1$ for some arbitrarily large p, we deduce that for such p

$$r_{\nu_p+1} \leqslant (p+1)^2,$$

so that

$$r'_p = r_{\nu_p} \leqslant r_{\nu_p+1} \leqslant (p+1)^2 \leqslant 4p^2.$$

Thus in this case $r'_p \leqslant 4p^2$ for infinitely many p and, since r'_p increases with p, it follows that

$$\sum r_p'^{-\frac{1}{2}} = \infty.$$

Thus our subsequence r'_p satisfies the analogues of (3.2) to (3.4).

We assume accordingly that we are given a sequence r_ν satisfying (3.1) to (3.4) and shall construct a subsequence ϱ_ν of the r_ν. With this subsequence we define the function

$$F(z) = \prod_{\nu=1}^{\infty} (1 - z/\varrho_\nu). \tag{3.5}$$

In view of (3.4) the product converges and represents an entire function of order $\frac{1}{2}$ mean type at most. We proceed to obtain some lower bounds for $F(z)$. Our results are contained in

LEMMA 2. *We have*

$$|F(z)| \geqslant 1 \quad \text{for } \tfrac{1}{2}\pi \leqslant |\arg z| \leqslant \pi. \tag{3.6}$$

Next if $F(z)$ has no zeros for $\frac{1}{2}r < |z| < 2r^4$, where $r > 1$, then

$$|F(z)| > e^{-6}, \quad |z| = r. \tag{3.7}$$

Finally if $r' > 2r$ we have

$$\int_r^{r'} \frac{\log |F(-t)|\, dt}{t^{3/2}} > \tfrac{1}{5} \sum \varrho_\nu^{-\frac{1}{2}}, \tag{3.8}$$

where the sum is extended over all the zeros ϱ_ν which satisfy

$$r \leqslant \varrho_\nu \leqslant \tfrac{1}{2} r'. \tag{3.9}$$

Inequality (3.6) is obvious. If $z = x + iy$, where $x \leqslant 0$, then for each ϱ_ν $|1 - z/\varrho_\nu| \geqslant 1$, and so $|F(z)| \geqslant 1$.

Next suppose that $F(z)$ has no zeros in $\frac{1}{2}r < |z| < 2r^4$. Then if $|z| = r$, we write

$$\log |F(z)| = \sum\nolimits_1 \log \left|1 - \frac{z}{\varrho_\nu}\right| + \sum\nolimits_2 \log \left|1 - \frac{z}{\varrho_\nu}\right| = \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say,}$$

where the first sum is extended over all the zeros ϱ_ν in $|z| \leqslant \frac{1}{2}r$ and the second sum over all the zeros in $|z| \geqslant 2r^4$. Clearly if $\varrho_\nu \leqslant \frac{1}{2}r$, $|z| = r$, we have

$$\left|1 - \frac{z}{\varrho_\nu}\right| \geqslant 2 - 1 = 1,$$

so that

$$\sum\nolimits_1 \geqslant 0.$$

Also in $\sum_2$ we have by hypothesis $|z|/\varrho_\nu < \frac{1}{2}$, so that

$$\log\left|1-\frac{z}{\varrho_\nu}\right| \leqslant \left|\log\left(1-\frac{z}{\varrho_\nu}\right)\right| < \frac{2|z|}{\varrho_\nu}.$$

Thus

$$\left|\sum\nolimits_2\right| < 2\sum\nolimits_2 \frac{r}{\varrho_\nu} = 2r\sum\nolimits_2 \varrho_\nu^{-1}.$$

In $\sum_2$ we have $\varrho_\nu > r^4$. From this and (3.4) we deduce

$$\sum\nolimits_2 \varrho_\nu^{-1} < r^{-1}\sum_{\nu=1}^{\infty} \varrho_\nu^{-3/4} < r^{-1}\sum_{1}^{\infty} \nu^{-3/2} < 3/r.$$

Hence

$$\left|\sum\nolimits_2\right| \leqslant 6, \quad \text{i.e.} \quad \log|F(z)| \geqslant -6,$$

and this proves (3.7).

It remains to prove (3.8). We note that for $r \leqslant t \leqslant r'$, we have

$$\log|F(-t)| = \sum \log\left(1+\frac{t}{\varrho_\nu}\right) \geqslant \sum{}' \log\left(1+\frac{t}{\varrho_\nu}\right),$$

where $\sum'$ denotes summation over all those zeros ϱ_ν, which satisfy (3.9). Thus

$$\int_r^{r'} \frac{\log|F(-t)|\,dt}{t^{3/2}} \geqslant \sum{}' \int_r^{r'} \log\left(1+\frac{t}{\varrho_\nu}\right)\frac{dt}{t^{3/2}}$$

$$\geqslant \sum{}' \int_{\varrho_\nu}^{2\varrho_\nu} \log\left(1+\frac{t}{\varrho_\nu}\right)\frac{dt}{t^{3/2}} = \sum{}' \varrho_\nu^{-1/2}\int_1^2 \log(1+x)\frac{dx}{x^{3/2}}$$

$$\geqslant \sum{}' \frac{\log 2}{2^{3/2}} \varrho_\nu^{-1/2} > \tfrac{1}{5}\sum{}' \varrho_\nu^{-1/2}.$$

This proves (3.8) and completes the proof of Lemma 2.

4. Proof of Theorem C. We now construct a sequence of harmonic polynomials $v_k(z)$ which will converge to a non-constant harmonic function $v(z)$. The function $v(z)$ will satisfy

(4.1) $$v(z) < \log|F(z)|+7$$

on a sequence of Jordan curves Γ_k, which surround the origin and tend to ∞ with k. Here $F(z)$ is given by (3.5). Thus if $g(z) = v(z)+iw(z)$ is an entire function whose real part is $v(z)$, then

$$f(z) = \frac{e^{g(z)}}{F(z)}$$

is the function whose existence is asserted in Theorem C. Evidently $f(z)$ has no zeros and the poles of $f(z)$ are the ϱ_ν, a subsequence of the r_ν, which satisfy (1.7) in view of (3.4). If Γ is a path going to ∞, then Γ meets Γ_k for all sufficiently large k at a point z_k say and, in view of (4.1)

$$|f(z_k)| < e^7.$$

Thus $f(z)$ cannot tend to ∞ as $z \to \infty$ along Γ and so ∞ is not an asymptotic value of $f(z)$. Also g cannot be a polynomial since otherwise f would have positive integral lower order, which would contradict (1.7) and Theorem A. Thus f has infinite lower order.

The construction of the $v_k(z)$ is similar to that in Section 3 of [1], starting with Lemma 1. We choose $\varepsilon_k = 2^{-k}$, set $t_1 = 1$, and $v_1(z) = x$. Suppose that t_k, $v_k(z)$ have been defined. We then construct a Jordan domain D_k containing the disk $|z| < t_k$, and a harmonic polynomial $v_{k+1}(z)$, such that

$$(4.2) \qquad |v_{k+1}(z) - v_k(z)| < \varepsilon_k, \quad |z| \leqslant t_k,$$

and

$$(4.3) \qquad v_{k+1}(z) < \log|F(z)| + 6,$$

on the boundary Γ_k of D_k. We next choose t_{k+1} so large that $t_{k+1} > 2t_k$, and that $\bar{D}_k$ lies in $|z| < t_{k+1}$ and continue with the inductive process. We now check that it is possible to choose $v_{k+1}(z)$ and the function $F(z)$ so that (4.2) and (4.3) are satisfied. For this we need Lemmas 1 and 2.

We apply Lemma 1 with $u(z) = v_k(z)$, $\varepsilon = \frac{1}{2}\varepsilon_k$ and $R = t_k$. The domain D_k will have the properties of D asserted after Lemma 1 with $N = N_k$, where N_k is the degree of $v_k(z)$. In order to succeed with our construction we shall have to make some further restrictions on the sectors S_ν given by (2.4) and the zeros ϱ_p of $F(z)$. We asume that the ϱ_p lying in $|z| < t_k$ have already been defined and all lie in $|z| < \frac{1}{2}t_k$. Next we choose the zeros ϱ_p in $t_k < |z| < t_{k+1}$. Let S_ν be the sectors given by (2.4) for $\nu = 1$ to N_k, and suppose that

$$(4.4) \qquad R'_\nu > 200 R_\nu^4, \quad \nu = 1 \text{ to } N_k.$$

We then define the ϱ_p in $t_k < \varrho_p < t_{k+1}$ to be all those numbers r_n which satisfy

$$(4.5) \qquad 9R_\nu^4 < r_n < \tfrac{1}{9}R'_\nu,$$

for some $\nu \leqslant N_k$. We set

$$(4.6) \qquad \psi_\nu(r) = \tfrac{1}{2}\log|F(-r)|$$

and note that we can satisfy (4.4) and (2.6).

In the first instance it follows from Lemma 2, (3.8) that

$$\int_{9R_\nu}^{R'_\nu/9} \frac{\psi_\nu(r)\,dr}{r^{3/2}} > \tfrac{1}{10}\sum r_n^{-1/2},$$

where the sum is extended over all those r_n which satisfy

$$9R_\nu^4 < r_n < \tfrac{1}{18}R'_\nu.$$

We deduce from (3.3) that (2.6) will be satisfied provided that R'_ν is chosen sufficiently large. We must also make sure that (4.4) is satisfied. Thus if R_1 is sufficiently large, $R_\nu > R'_{\nu-1}+1$, and R'_ν is sufficiently large compared with R_ν all the conditions for the construction of Lemma 1 will be satisfied and we deduce the existence of the harmonic function $v_{k+1}(z)$ satisfying (4.2). Since v_{k+1} is harmonic in D_k and continuous in $\bar{D}_k$ we may suppose without loss of generality that v_{k+1} is a polynomial, since by a classical theorem (see e.g. [6], p. 299), v_{k+1} can be uniformly approximated in $\bar{D}_k$ by harmonic polynomials.

Next we check that (4.3) holds on Γ_k. Suppose first that $z = te^{i\varphi}$ is a point on Γ_k which lies in an annulus

(4.7) $$R_\nu < t < R'_\nu,$$

for some $\nu \leqslant N_k$. Then since D_k contains the sector S_ν given by (2.4) we deduce that

(4.8) $$\pi-\eta_{\nu+1} < |\varphi| \leqslant \pi.$$

In view of our construction and (4.6) we have

$$v_{k+1}(z) < \tfrac{1}{2}\log|F(-t)|, \quad R_\nu < t < R'_\nu$$

for $z = te^{i\varphi}$. For $t = R_\nu$, R'_ν the inequality continues to hold since then the right-hand side is positive while the left-hand side is not. We have not so far made any requirement of the quantity $n_{\nu+1}$, but we now chose $n_{\nu+1}$ so small that

$$\log|F(te^{i\varphi})| > \tfrac{1}{2}\log|F(-t)|,$$

in the range (4.7), (4.8) and deduce (4.3) in this case.

Next we suppose that $z = te^{i\varphi}$ lies on Γ_k but not in any of the ranges (4.7). Suppose first that

$$R'_\nu \leqslant t \leqslant R_{\nu+1}$$

for some ν, such that $1 \leqslant \nu \leqslant N_k-1$. Then by our construction $F(z)$ has no zeros ϱ_p such that $\tfrac{1}{2}t < \varrho_p \leqslant 2t^4$ and so

$$\log|F(z)| > -6$$

in view of (3.7). Also by construction

$$v_{k+1}(z) \leqslant 0$$

in this case. Thus we have

$$v_{k+1}(z) \leqslant \log|F(z)|+6$$

in this case so that (4.3) still holds. If

$$t_k \leqslant |z| \leqslant R_1 \quad \text{or} \quad R_{N_k} \leqslant |z| \leqslant t_{k+1},$$

the conclusion is similar. Thus (4.3) holds on the whole of Γ_k, and our inductive step is justified.

We can now complete the proof of Theorem C. It follows from (4.2) that $v_k(z)$ converges locally uniformly in the plane to a harmonic function $v(z)$. Also for $|z| \leqslant t_k$ we have

$$|v(z)-v_k(z)| \leqslant \sum_{\nu=k}^{\infty} |v_{\nu+1}-v_\nu| < \sum_{\nu=k}^{\infty} \varepsilon_\nu = 2^{1-k} \leqslant 1.$$

In particular

$$|v(z)-x| < 1, \quad |z| \leqslant 1,$$

so that $v(z)$ is not constant. Also since $\bar{D}_k$ lies in $|z| < t_{k+1}$, we have on Γ_k

$$v(z) \leqslant v_{k+1}(z)+1 < \log|F(z)|+7$$

in view of (4.3). This proves (4.1) and completes the proof of Theorem C.

References

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