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On polynomial mappings into spheres

by J. Bochnak (Amsterdam) and W. Kucharz* (Albuquerque)

Zdzisław Opial in memoriam

1. Introduction. Let $X \subset \boldsymbol{R}^n$ and $Y \subset \boldsymbol{R}^p$ be real algebraic sets. A mapping

$$f = (f_1, \ldots, f_p)\colon\ X \to Y$$

is said to be a *polynomial* (resp. *regular*) *mapping* if for each $i = 1, \ldots, p$, there exists a polynomial φ_i (resp. there exist polynomials φ_i and ψ_i) in $\boldsymbol{R}[T_1, \ldots, T_n]$ such that $f_i(x) = \varphi_i(x)$ (resp. $\psi_i(x) \neq 0$ and $f_i(x) = \varphi_i(x)/\psi_i(x)$) for all x in X. Clearly, polynomial mappings depend on the embeddings of X and Y in $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$, respectively. Therefore while studying polynomial mappings from X to Y, these embeddings will always be explicitly specified or it will be obvious from the context how to specify them. For example, if $X_i \subset \boldsymbol{R}^{n_i}$, $i = 1, 2$, are algebraic sets, then talking about polynomial mappings from $X_1 \times X_2$ to Y, we mean the polynomial mappings corresponding to the natural embedding of $X_1 \times X_2$ in $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} = \boldsymbol{R}^{n_1+n_2}$ and to the given embedding of Y in $\boldsymbol{R}^p$. On the order hand, regular mappings, sometimes also called real algebraic morphisms or entire rational mappings, are independent of the embeddings of X and Y in affine spaces (cf. [2], Chapter 3).

In this paper we study polynomial mappings with values in the unit sphere $S^d = \{x \in \boldsymbol{R}^{d+1} \mid \|x\| = 1\}$. Our aim is to generalize the following theorem of Loday.

Theorem 1.1 ([11]). *If $d \geq 2$, then each polynomial mapping from $S^1 \times \ldots \times S^1$ (d factors) into S^d is null homotopic.*

Using Loday's idea we shall generalize this result as follows. Given a real algebraic set X, let $\mathscr{R}(X)$ denote the ring of regular functions from X to $\boldsymbol{R}$, let $\mathscr{R}(X, \boldsymbol{C}) = \mathscr{R}(X) \otimes_{\boldsymbol{R}} \boldsymbol{C}$ and let $\tilde{K}_0(\mathscr{R}(X, \boldsymbol{C}))$ be the projective class group of $\mathscr{R}(X, \boldsymbol{C})$ (also called the reduced Grothendieck K-group, or the reduced Grothendieck group of the category of finitely generated projective $\mathscr{R}(X, \boldsymbol{C})$-modules; cf. [1], [12], [15]).

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THEOREM 1.2. *Let X be a compact connected nonsingular orientable real algebraic subset of $\boldsymbol{R}^n$ of dimension $d-1$, $d \geqslant 2$. If the group $\tilde{K}_0(\mathscr{R}(X, \boldsymbol{C}))$ is finite, then each polynomial mapping from $X \times S^1$ to S^d is null homotopic.*

The interesting point is that in our earlier paper [3] we were able to show that the assumption of $\tilde{K}_0(\mathscr{R}(X, \boldsymbol{C}))$ being finite is satisfied for a very large class of algebraic sets. For example, "most" nonsingular algebraic hypersurfaces X of $\boldsymbol{R}^d$ have the group $\tilde{K}_0(\mathscr{R}(X, \boldsymbol{C}))$ finite (the meaning of "most" is made precise in [3] Remark 5.11 and Theorem 8.1).

Theorem 1.2 contains an immediate generalization of Loday's result.

THEOREM 1.3. *Let $k_1, \ldots, k_p$ be positive odd integers, $p \geqslant 1$, and let $d = k_1 + \ldots + k_p + 1$. Then each polynomial mapping from $S^{k_1} \times \ldots \times S^{k_p} \times S^1$ to S^d is null homotopic.*

We conjecture that if $k_1, \ldots, k_{p+1}$, $p \geqslant 1$, are positive odd integers then each polynomial mapping from $S^{k_1} \times \ldots \times S^{k_{p+1}}$ to S^d, $d = k_i + \ldots + k_{p+1}$, is null homotopic. The assumption of dealing in Theorem 1.3 with odd-dimensional spheres is, however, important in view of the following example.

EXAMPLE 1.4 ([11]). For each positive integer n and each integer k, there is a polynomial mapping from $S^{2n} \times S^1$ into S^{2n+1} of topological degree k.

Theorem 1.3 is in fact a particular case of our next result. A real algebraic set $X \subset \boldsymbol{R}^n$ is said to be *admissible* if it is biregularly isomorphic to a projective algebraic subset of $\boldsymbol{RP}^k$, for some k, whose Zariski (complex) closure in $\boldsymbol{CP}^k$ is a nonsingular complete intersection (we regard $\boldsymbol{RP}^k$, in the obvious way, as a subset of $\boldsymbol{CP}^k$). Admissible sets are nonsingular and they are very common. It is rather easy to see that most compact algebraic hypersurfaces in $\boldsymbol{R}^n$ are admissible (cf. [3], Theorem 8.1, for more information).

THEOREM 1.5. *Let X be an admissible real algebraic set in $\boldsymbol{R}^n$, orientable, connected, of odd dimension k. Let $k_1, \ldots, k_p$ be positive odd integers, $p \geqslant 0$, and let $d = k + k_1 + \ldots + k_p + 1$. Then each polynomial mapping from $X \times S^{k_1} \times \ldots \times S^{k_p} \times S^1$ to S^d is null homotopic.*

Since S^k is, of course, an admissible algebraic set, Theorem 1.3 follows from the last statement. To prove Theorem 1.5 we show (in Section 3) that for X and k_i satisfying its assumptions, the group $\tilde{K}_0(\mathscr{R}(X \times S^{k_1} \times \ldots \times S^{k_p}, \boldsymbol{C}))$ is finite, and then we apply Theorem 1.2.

We also have a result of a different nature.

THEOREM 1.6. *Let M be a C^∞ connected, orientable submanifold of $\boldsymbol{R}^n$ such that $M = H_1 \cap \ldots \cap H_c$, where $c = \operatorname{codim} M$ and $H_1, \ldots, H_c$ are compact C^∞ hypersurfaces of $\boldsymbol{R}^n$ which are in general position at each point of M. Let $k_1, \ldots, k_p$ be odd positive integers, $p \geqslant 0$, and let $d = \dim M + k_1 + \ldots + k_p + 1$. Then there is a C^∞ embedding $h\colon M \to \boldsymbol{R}^n$, arbitrarily close in the C^∞ topology to the* *inclusion mapping $M \hookrightarrow \mathbf{R}^n$, such that $X = h(M)$ is a nonsingular real algebraic subset of $\mathbf{R}^n$, and each polynomial mapping from $X \times S^{k_1} \times \ldots \times S^{k_p} \times S^1$ to S^d is null homotopic.*

We suggest the reader to compare the special case of Theorem 1.6 where $M = S^{n-1}$ and $p = 0$ with Example 1.4.

To put the above results into the right perspective we should mention two of our earlier theorems.

(i) *If X is a compact connected nonsingular real algebraic set of odd dimension $d-1$, then each regular mapping $X \times S^1 \to S^d$ is null homotopic.*

Statement (i) implies that in the proof of Theorem 1.2 we need to be concerned only with the case where the dimension of X is even. (For the proof of (i) see [5], or [4] for X orientable. The polynomial case of (i) with X orientable was settled in [11].)

(ii) *If Y is a compact connected nonsingular orientable real algebraic set of odd dimension d, then for each even integer k there is a regular mapping from Y into S^d of topological degree k* (cf. [4]).

Statement (ii) stress the difference between the behavior of regular and polynomial mappings (if in Theorem 1.2 the dimension of X is even, the conclusion is always false for regular mappings).

We prove Theorem 1.2 in Section 2 and then, in Section 3, we deduce from it Theorems 1.5 and 1.6.

For definitions and notions of real algebraic geometry used in this paper we refer the reader to the book [2]. More information about regular or polynomial mappings between real algebraic sets can be found in [2], [4]–[7], [11].

2. Proof of Theorem 1.2. First we need a topological lemma.

Let X be a compact topological space with base point x_0. Denote, as usual, by $K(X)$ the Grothendieck group of the category of topological complex vector bundles on X and let $\tilde{K}(X)$ be the kernel of the homomorphism $K(X) \to K(\{x_0\})$ induced by the inclusion $\{x_0\} \hookrightarrow X$ (cf. [8], [1]). Let SX be the reduced suspension of X, i.e., SX is obtained from $X \times S^1$ by collapsing $(X \times \{q\}) \cup (\{x_0\} \times S^1)$ to a point (the base point of SX); here $q = (1, 0) \in S^1$. Let us denote by $p\colon X \times S^1 \to SX$ the canonical projection and set

$$S_-X = \{p(x, z) \in SX \mid x \in X,\ z = (t_1, t_2) \in S^1,\ t_2 \leqslant 0\},$$
$$S_+X = \{p(x, z) \in SX \mid x \in X,\ z = (t_1, t_2) \in S^1,\ t_2 \geqslant 0\},$$
$$S_0X = S_-X \cap S_+X.$$

Since S_-X and S_+X are contractible, the Mayer–Vietoris sequence of $(SX; S_-X, S_+X)$ determines the canonical isomorphism

$$\Delta^q\colon H^q(S_0X, \mathbf{Q}) \to H^{q+1}(SX, \mathbf{Q})$$

for $q \geqslant 1$ (cf. [13]). Let

$$\Delta\colon H^{\mathrm{odd}}(S_0X, Q) \to \bigoplus_{k>0} H^{2k}(SX, Q)$$

be the direct sum of the Δ^q for q odd.

We write $K^{-1}(X) = \tilde{K}(SX)$ and define the natural homomorphism

$$Ch^{\mathrm{odd}}\colon K^{-1}(X) \to H^{\mathrm{odd}}(X, Q)$$

by $Ch^{\mathrm{odd}} = \Phi \circ \Delta^{-1} \circ (Ch|K^{-1}(X))$, where

$$Ch\colon K(SX) \to H^{\mathrm{even}}(SX, Q)$$

is the Chern character (cf. [8]), $Ch|K^{-1}(X)$ is the restriction of Ch to $K^{-1}(X) = \tilde{K}(SX)$, and

$$\Phi\colon H^{\mathrm{odd}}(S_0X, Q) \to H^{\mathrm{odd}}(X, Q)$$

is the isomorphism induced by the homeomorphism $X \to S_0X$, defined by $x \to p(x, q')$ where $q' = (-1, 0) \in S^1$.

LEMMA 2.1. *Let M be a compact connected orientable C^∞ manifold of odd dimension m. Let $f\colon M \to S^m$ be a continuous mapping (of pointed spaces). If the image of the induced homomorphism $K^{-1}(f)\colon K^{-1}(S^m) \to K^{-1}(M)$ is a finite group, then f is null homotopic.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} Z \simeq K^{-1}(S^m) & \xrightarrow{Ch^{\mathrm{odd}}} & H^{\mathrm{odd}}(S^m, Q) \simeq Q \\ \downarrow{\scriptstyle K^{-1}(f)} & & \downarrow{\scriptstyle H^*(f)} \\ K^{-1}(M) & \xrightarrow{Ch^{\mathrm{odd}}} & H^{\mathrm{odd}}(M, Q) \simeq Q^l \end{array}$$

Since $K^{-1}(f)$ has a finite image in $K^{-1}(M)$, and by [8] (p. 280, Theorem 9.6) $Ch^{\mathrm{odd}}(K^{-1}(S^n)) = H^{\mathrm{odd}}(S^m, Z)$, it follows that $H^m(f) = 0$. Hence the topological degree of f is equal to zero and f is null homotopic. □

Let $X \subset R^n$ be a compact algebraic set. Denote by $\mathscr{P}(X)$ the ring of polynomial R-valued functions on X and set $\mathscr{P}(X, C) = \mathscr{P}(X) \otimes_R C$. Let $\mathscr{C}(X, C)$ be the ring of continuous C-valued functions on X. We shall consider $\mathscr{P}(X, C)$ as a subring of $\mathscr{C}(X, C)$.

In our proof of Theorem 1.2 we shall use the functors K_0 (eventually $\tilde{K}_0$) and K_1 of algebraic K-theory (cf. [1], [12], [15]), and relationships between K_1 and K^{-1}. In particular, we shall use the natural epimorphism

$$\varphi(X, \mathscr{C})\colon K_1(\mathscr{C}(X, C)) \to K^{-1}(X)$$

defined in [1] (pp. 742–743, 750). Denote by

$$\varphi(X, \mathscr{P})\colon K_1(\mathscr{P}(X, C)) \to K^{-1}(X)$$

the composition of the homomorphism $K_1(\mathscr{P}(X, \boldsymbol{C})) \to K_1(\mathscr{C}(X, \boldsymbol{C}))$, induced by the inclusion $\mathscr{P}(X, \boldsymbol{C}) \subsetneq \mathscr{C}(X, \boldsymbol{C})$, with $\varphi(X, \mathscr{C})$.

Given a polynomial mapping $f\colon X \to Y$, let $\mathscr{P}(f)\colon \mathscr{P}(Y, \boldsymbol{C}) \to \mathscr{P}(X, \boldsymbol{C})$ denote the induced ring homomorphism.

Proof of Theorem 1.2. As mentioned in the introduction, we may assume that d is odd.

Let $p_1\colon X \times S^1 \to X$ and $p_2\colon X \times S^1 \to S^1$ be the canonical projections and let $t\colon S^1 \to \boldsymbol{C}$ be the inclusion mapping. Observe that the monomorphism

$$\mathscr{P}(p_1)\colon \mathscr{P}(X, \boldsymbol{C}) \to \mathscr{P}(X \times S^1, \boldsymbol{C})$$

gives rise to the isomorphism

$$(1) \qquad \mathscr{P}(X, \boldsymbol{C})[T, T^{-1}] \to \mathscr{P}(X \times S^1, \boldsymbol{C}),$$

where the indeterminate T is sent to $t \circ p_2$ (cf. for example [4]).

We may assume that $\mathscr{P}(X, \boldsymbol{C})$ is a regular ring. Indeed, there exists an algebraic set $Y \subset \boldsymbol{R}^{n+1}$ such that the Zariski (complex) closure $Y_{\boldsymbol{C}}$ of Y in $\boldsymbol{C}^{n+1}$ is nonsingular and the canonical projection $\pi\colon \boldsymbol{R}^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n$ induces a biregular isomorphism from Y onto X (cf. [4] or [2], Lemma 12.6.6). If $g\colon X \times S^1 \to S^d$ is a polynomial mapping and if $g \circ ((\pi | Y) \times e)\colon Y \times S^1 \to S^d$, where $e\colon S^1 \to S^1$ is the identity mapping, is null homotopic, then g is also null homotopic. Since $Y_{\boldsymbol{C}}$ is nonsingular, the ring $\mathscr{P}(Y, \boldsymbol{C})$, which is isomorphic to the coordinate ring of $Y_{\boldsymbol{C}}$, is regular. Thus, replacing possibly X by Y, we may assume that $\mathscr{P}(X, \boldsymbol{C})$ is a regular ring.

Recall that for every regular ring A, there is a natural group isomorphism

$$(2) \qquad \alpha^A = (\alpha_1^A, \alpha_2^A)\colon K_1(A[T, T^{-1}]) \to K_1(A) \oplus K_0(A),$$

where α_1^A is the group homomorphism induced by the ring homomorphism $A[T, T^{-1}] \to A$, which is the identity on A and maps T onto 1 (cf. [1], p. 663, Theorem 7.4, or [15], Corollary 16.5).

Fix points x_0 in X and y_0 in S^1, and let $i_1\colon X \to X \times S^1$, $i_2\colon S^1 \to X \times S^1$ be the mappings defined by $i_1(x) = (x, y_0)$ for x in X, $i_2(y) = (x_0, y)$ for y in S^1, respectively.

Combining (1) and (2), and using the naturality of (2), we obtain the natural isomorphism

$$\alpha = (\alpha_1, \alpha_2, \alpha_3)\colon K_1(\mathscr{P}(X \times S^1, \boldsymbol{C})) \to K_1(\mathscr{P}(X, \boldsymbol{C})) \oplus K_1(\mathscr{P}(S^1, \boldsymbol{C})) \oplus \tilde{K}_0(\mathscr{P}(X, \boldsymbol{C})),$$

where $\alpha_1 = K_1(\mathscr{P}(i_1))$ and $\alpha_2 = K_1(\mathscr{P}(i_2))$.

One has also the following purely topological result: there exists the natural isomorphism

$$\beta = (\beta_1, \beta_2, \beta_3)\colon K^{-1}(X \times S^1) \to K^{-1}(X) \oplus K^{-1}(S^1) \oplus \tilde{K}(X),$$

where $\beta_1 = K^{-1}(i_1)$ and $\beta_2 = K^{-1}(i_2)$ (this is a suitable formulation of Bott's periodicity; cf. [1], pp. 749–750; [15], Theorem 17.2).

Let $f\colon X \times S^1 \to S^d$ be a polynomial mapping. In order to prove that f is null homotopic it suffices to show (applying Lemma 2.1) that the image of $K^{-1}(S^d)$ by the homomorphism

$$K^{-1}(f)\colon K^{-1}(S^d) \to K^{-1}(X \times S^1)$$

is a finite group. To show this fact let us consider the diagram

$$(*)\qquad \begin{array}{ccc} K_1(\mathscr{P}(S^d, C)) & \overset{u_1}{\to} & K^{-1}(S^d) \\ \downarrow \gamma & & \downarrow K^{-1}(f) \\ K_1(\mathscr{P}(X \times S^1), C) & \overset{u_2}{\to} & K^{-1}(X \times S^1) \\ \downarrow \alpha & & \downarrow \beta \\ K_1(\mathscr{P}(X, C)) \oplus K_1(\mathscr{P}(S^1, C)) \oplus \tilde{K}_0(\mathscr{P}(X, C)) & \overset{v}{\to} & K^{-1}(X) \oplus K^{-1}(S^1) \oplus \tilde{K}(X), \end{array}$$

where $u_1 = \varphi(S^d, \mathscr{P})$, $u_2 = \varphi(X \times S^1, \mathscr{P})$, $\gamma = K_1(\mathscr{P}(f))$ and v is the direct sum $v = \varphi(X, \mathscr{P}) \oplus \varphi(S^1, \mathscr{P}) \oplus \psi$ with $\psi\colon \tilde{K}_0(\mathscr{P}(X, C) \to \tilde{K}(X)$ being the composition of the group homomorphism $\tilde{K}_0(\mathscr{P}(X,C)) \to \tilde{K}_0(\mathscr{C}(X,C))$, induced by the inclusion $\mathscr{P}(X, C) \subsetneq \mathscr{C}(X, C)$, and the canonical isomorphism $\tilde{K}_0(\mathscr{C}(X, C)) \to \tilde{K}(X)$ (cf. [14] or [1], p. 735, Theorem 3.1).

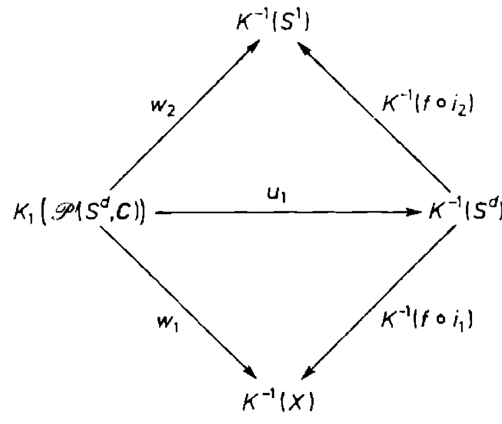
Diagram (*) is commutative. The commutativity of the upper rectangle follows from the naturality of the transformation $\varphi(\cdot, \mathscr{P})$ mentioned earlier, while the commutativity of the bottom rectangle is a consequence of [1], pp. 749-750.

Since u_1 is surjective (cf. [11], Lemma 5) and β is an isomorphism, in order to prove that $K^{-1}(f)$ has a finite image, it suffices to show that the image of $w = v \circ \alpha \circ \gamma$ is finite.

First observe that

$$w_1 = \varphi(X, \mathscr{P}) \circ \alpha_1 \circ \gamma \quad \text{and} \quad w_2 = \varphi(S^1, \mathscr{P}) \circ \alpha_2 \circ \gamma$$

are the zero homomorphisms. Indeed, the commutativity of (*) implies trivially the commutativity of the following diagram:



Since $f \circ i_1$ and $f \circ i_2$ are null homotopic, $K^{-1}(f \circ i_1) = 0$ and $K^{-1}(f \circ i_2) = 0$. Thus $w_1 = 0$ and $w_2 = 0$.

We claim now that the image of $w_3 = \psi \circ \alpha_3 \circ \gamma$ is a finite group. Indeed, we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{K}_0(\mathscr{P}(X,C)) & \xrightarrow{\ \delta\ } & \tilde{K}_0(\mathscr{R}(X,C)) \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\varepsilon} & \\ & \tilde{K}(X) & \end{array}$$

where δ is induced by the inclusion $\mathscr{P}(X, C) \subsetneq \mathscr{R}(X, C)$ and ε is the composition of the homomorphism $\tilde{K}_0(\mathscr{R}(X, C)) \to \tilde{K}_0(\mathscr{C}(X, C))$, induced by the inclusion $\mathscr{R}(X, C) \subsetneq \mathscr{C}(X, C)$, with the canonical isomorphism $\tilde{K}_0(\mathscr{C}(X, C)) \to \tilde{K}(X)$ (cf. [1], p. 735, Theorem 3.1). Since, by assumption, $\tilde{K}_0(\mathscr{R}(X, C))$ is finite, and Image $w_3 \subset$ Image $\psi \subset$ Image ε, it follows that the image of w_3, and hence of $w = (w_1, w_2, w_3) = (0, 0, w_3)$, is a finite group. This completes the proof of Theorem 1.2. □

3. Proofs of Theorems 1.5 and 1.6. Let $X \subset \boldsymbol{R}^n$ be a real algebraic set and let $X_{\boldsymbol{C}}$ be its Zariski closure in $\boldsymbol{C}^n$. Denote by $A(X_{\boldsymbol{C}})$ the coordinate ring $\boldsymbol{C}[X_1, \ldots, X_n]/(\text{ideal of } X_{\boldsymbol{C}})$ of $X_{\boldsymbol{C}}$. We shall identify $\mathscr{P}(X, C)$ with $A(X_{\boldsymbol{C}})$. Moreover, we shall identify $\mathscr{R}(X, C)$ with the localization of $A(X_{\boldsymbol{C}})$ with respect to the multiplicatively closed set

$$S = \{f \in A(X_{\boldsymbol{C}}) \mid f(X) \subset \boldsymbol{R} \setminus \{0\}\}$$

(cf. [4]).

LEMMA 3.1. *Let $X \subset \boldsymbol{R}^n$ be a nonsingular real algebraic set. If k is a positive odd integer, then the canonical projection $X \times S^k \to X$ induces an isomorphism of $K_0(\mathscr{R}(X, C))$ onto $K_0(\mathscr{R}(X \times S^k, C))$.*

Proof. Without loss of generality we may assume that the Zariski closure $X_{\boldsymbol{C}}$ of X in $\boldsymbol{C}^n$ is nonsingular (cf. [4] or [2], Lemma 12.6.6).

Consider the following commutative diagram

$$\begin{array}{ccc} K_0(A(X_{\boldsymbol{C}})) \otimes K_0(A(S^k_{\boldsymbol{C}})) & \xrightarrow{\mu} & K_0(A(X_{\boldsymbol{C}} \times S^k_{\boldsymbol{C}})) \\ \downarrow & & \downarrow \\ K_0(\mathscr{R}(X, C)) \otimes K_0(\mathscr{R}(S^k, C)) & \xrightarrow{\nu} & K_0(\mathscr{R}(X \times S^k, C)) \end{array}$$

where μ and ν are the obvious homomorphisms obtained from the homomorphisms induced by the canonical projections $X \times S^k \to X$, $X_{\boldsymbol{C}} \times S^k_{\boldsymbol{C}} \to X_{\boldsymbol{C}}$, etc., and where the vertical arrows are induced by the localization homomorphisms $A(X_{\boldsymbol{C}}) \to \mathscr{R}(X, C)$, $A(S^k_{\boldsymbol{C}}) \to \mathscr{R}(S^k, C)$, $A(X_{\boldsymbol{C}} \times S^k_{\boldsymbol{C}}) \to \mathscr{R}(X \times S^k, C)$. Since $X_{\boldsymbol{C}}$ and $S^k_{\boldsymbol{C}}$ are nonsingular, it follows from [1] (p. 499, Theorem 6.5) that the vertical homomorphisms are surjective. By [9] (cf. also [10] for a detailed proof) μ is an isomorphism and hence ν is surjective.

For each odd-dimensional sphere S^k the group $\tilde{K}_0(\mathscr{C}(S^k, \boldsymbol{C})) \simeq \tilde{K}(S^k) = 0$ (cf. [8] p. 109), and hence also $\tilde{K}_0(\mathscr{R}(S^k, \boldsymbol{C})) = 0$ (cf. [2] Theorem 12.3.6 and Section 12.6).

It follows that $K_0(\mathscr{R}(S^k, \boldsymbol{C})) \simeq \boldsymbol{Z}$ which, together with the surjectivity of ν, implies that the homomorphism

$$\alpha\colon\ K_0(\mathscr{R}(X,\ \boldsymbol{C})) \to K_0(\mathscr{R}(X \times S^k,\ \boldsymbol{C})),$$

induced by the ring homomorphism $\mathscr{R}(X, \boldsymbol{C}) \to \mathscr{R}(X \times S^k)$ corresponding to the canonical projection $X \times S^k \to X$, is surjective. Since α is clearly injective, the lemma follows. □

COROLLARY 3.2. *Let X be a nonsingular real algebraic set and let $k_1, \ldots, k_p$ be odd positive integers. If the group $\tilde{K}_0(\mathscr{R}(X,\ \boldsymbol{C}))$ is finite, then the group $\tilde{K}_0(\mathscr{R}(X \times S^{k_1} \times \ldots \times S^{k_p},\ \boldsymbol{C}))$ is finite too.*

Proof. It follows directly from Lemma 3.1 that both groups are isomorphic. □

Proof of Theorem 1.5. Since for an admissible odd-dimensional real algebraic set X the group $\tilde{K}_0(\mathscr{R}(X,\ \boldsymbol{C}))$ is finite (cf. [3], Theorem 5.10, where this group is denoted by $\tilde{K}_{\boldsymbol{C}-\mathrm{alg}}(X)$), Theorem 1.5 follows from Corollary 3.2, applying Theorem 1.2. □

Remark 3.3. As mentioned earlier, Theorem 1.5 is false, in general, for even-dimensional X (cf. Example 1.4). Nevertheless, since for most even-dimensional admissible algebraic sets X the group $\tilde{K}_0(\mathscr{R}(X,\ \boldsymbol{C}))$ is finite (cf. [3], Theorem 8.1), it follows that Theorem 1.5 still remains true in most even-dimensional cases.

Proof of Theorem 1.6. By [3], Theorem 7.1, there exists a C^∞ embedding $h\colon\ M \to \boldsymbol{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $M \subsetneq \boldsymbol{R}^n$, such that $X = h(M)$ is a nonsingular algebraic subset of $\boldsymbol{R}^n$ with $\tilde{K}_0(\mathscr{R}(X, \boldsymbol{C}))$ finite. Applying Corollary 3.2 and Theorem 1.2 one concludes the proof. □

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DEPARTMENT OF MATHEMATICS, VRIJE UNIVERSITEIT
AMSTERDAM, THE NETHERLANDS
and
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, USA

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