SIMPLICIAL APPROXIMATION OF ANTIPODAL MAPS

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In this note we shall prove that each continuous antipodal map \( f: P \to R^n \) defined on a symmetric polyhedron \( P \subseteq R^n \) can be approximated by a simplicial antipodal map \( g: P \to R^n \) such that \( 0 \in R^n \) is a regular value of the map \( g \).

The result is related to the following question of Nirenberg [4]:

Let \( f: \text{Cl} \, X \to R^n, \ f(\text{Bd} \, X) \subseteq R^n \setminus \{0\}, \) be a continuous antipodal map, where \( X \) is a symmetric, open and bounded subset of \( R^n \). Is it possible to find for each \( \varepsilon > 0 \) an antipodal map

\[
f_{\varepsilon}: \text{Cl} \, X \to R^n
\]

of class \( C^1 \) such that the point \( 0 \) is a regular value of the map \( f_{\varepsilon} \) and \( \|f(x) - f_{\varepsilon}(x)\| < \varepsilon \) for each \( x \in \text{Cl} \, X \)?

The purpose of the question was to obtain a simple proof, based on the degree theory, of the Borsuk antipodal theorem. The Nirenberg question was answered in the affirmative by Ivanov [3].

The main result presented here has a simple proof and, as shown, it simplifies the proof of the Borsuk theorem.

We shall use the following terminology: A set \( X \subseteq R^n \) is said to be symmetric if \( x \in X \) implies \( -x \in X \), and a map \( f: X \to R^n \) is said to be antipodal provided that \( f(-x) = -f(x) \) for each \( x \in X \). The symbols \( \text{Cl} \, X, \text{Int} \, X, \text{Bd} \, X \) mean the closure, the interior and the boundary of the set \( X \).

1. Preliminaries. Let us recall some facts on simplicial complexes which we shall apply in this note. For details and proofs the reader is referred to [2] and [1].

A set \( \{a_0, \ldots, a_k\} \subseteq R^n \) of \( k+1 \) points is said to be affinely independent if it is not contained in any \((k-1)\)-flat. This is equivalent to the fact that the points \( a_1 - a_0, \ldots, a_k - a_0 \) are linearly independent.

Assume that the set \( \{a_0, \ldots, a_k\} \subseteq R^n \) is affinely independent. The convex hull

\[
[a_0, \ldots, a_k] := \{x \in R^n: x = \sum_{i=0}^{k} \lambda_i a_i, \ 0 \leq \lambda_i, \ \sum_{i=0}^{k} \lambda_i = 1\}
\]
is called the \textit{k-simplex} with vertices \(a_0, \ldots, a_k\). If
\[ \{a_{i_0}, \ldots, a_{i_j}\} \subseteq \{a_0, \ldots, a_k\}, \quad j \leq k, \]
then the simplex \([a_{i_0}, \ldots, a_{i_j}]\) is said to be a \textit{j-face} of \([a_0, \ldots, a_k]\).

Define a map \(L: \mathbb{R}^k \to \mathbb{R}^n\) by
\[ L(\lambda_1, \ldots, \lambda_k) := a_0 + \sum_{i=1}^{k} \lambda_i (a_i - a_0). \]
Observe that
\[ [a_0, \ldots, a_k] = L(A_k), \]
where
\[ A_k := \{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k : 0 \leq \lambda_i, \sum_{i=1}^{k} \lambda_i \leq 1 \}. \]
If \(n = k\), then the Jacobian \(\det L'(x)\) is equal to
\[ \det(a_1 - a_0, \ldots, a_n - a_0) \neq 0. \]

A \textit{simplicial complex} is a finite family \(K\) of simplexes in \(\mathbb{R}^n\) such that:
(a) If \(s \in K\), then so does every face of \(s\).
(b) If \(s, \sigma \in K\), then \(s \cap \sigma\) is either empty or a face common to both \(s\) and \(\sigma\).

The \textit{barycenter} of a \textit{k-simplex} \(s = [a_0, \ldots, a_k] \subseteq \mathbb{R}^n\) is the point
\[ b(s) := \frac{1}{k+1} \sum_{i=0}^{k} a_i. \]
The \textit{barycentric subdivision} \(K^{(1)}\) of a complex \(K\) is the set of all simplexes of the form
\[ [b(s_0), \ldots, b(s_j)], \]
where \(s_0 \subseteq s_1 \subseteq \ldots \subseteq s_j\) is a strictly increasing sequence of simplexes of \(K\).
Define the \((r+1)\text{-st barycentric subdivision}\) of a complex \(K\) by
\[ K^{(r+1)} := [K^{(r)}]^{(1)}. \]

A set \(P \subset \mathbb{R}^n\) is said to be a \textit{polyhedron} whenever \(P = |K|\) for some complex \(K\), where
\[ |K| := \bigcup \{ s : s \in K \}. \]
For any vertex \(a \in K\) the set
\[ \text{st}(a, K) := K \backslash \{ s \in K : a \notin s \} \]
is called the \textit{star} of \(a\). Put
\[ |\text{st}(a, K)| := |K| \backslash \{ s \in K : a \notin s \}. \]
The set $|\text{st}(a, K)|$ is an open subset of the compact space $K$ and the following facts hold:

$$|K^{(r)}| = |K|,$$

$$\text{diam}[a_0, \ldots, a_k] = \max\{\|a_i - a_j\| : i, j = k\},$$

$$\text{mesh } K^{(r)} \leq \left(\frac{n}{n+1}\right)^r \text{mesh } K,$$

where

$$|K| \subset \mathbb{R}^n \quad \text{and} \quad \text{mesh } K := \max\{\text{diam } s : s \in K\}.$$

Recall that if $f: U \to \mathbb{R}^n$, $U$ open in $\mathbb{R}^n$, is a map of class $C^1$, then a point $x \in U$ is said to be critical whenever the Jacobian $\det f'(x) = 0$. A point $a \in \mathbb{R}^n$ is called a regular value of the map $f$ if the set $f^{-1}(a)$ does not contain any critical point.

Each map $\varphi: V(K) \to \mathbb{R}^n$ defined on the set of all vertices of a complex $K$ induces the so-called simplicial map $|\varphi| : |K| \to \mathbb{R}^n$ defined as follows:

$$|\varphi|(x) := \sum_{i=0}^{k} \lambda_i \varphi(a_i),$$

where

$$x = \sum_{i=0}^{k} \lambda_i a_i \in s \in K, \quad 0 \leq \lambda_i, \quad \sum_{i=0}^{k} \lambda_i = 1.$$

The map $|\varphi|$ is continuous and, moreover, $|\varphi|$ is of class $C^\infty$ on the open set $U = |K| \setminus S(K)$, where $S(K)$ is the union of all $k$-simplexes, $k < n$. Extend the definition of simplicial map. If $X \subset \mathbb{R}^n$ is a compact set, then a continuous map $f: X \to \mathbb{R}^n$ is said to be simplicial whenever there exists a simplicial map $F: P \to \mathbb{R}^n$, where $P \supset X$ is a polyhedron, such that $F|X = f$.

A point $a \in \mathbb{R}^n$ is said to be a regular value of a simplicial map $f: X \to \mathbb{R}^n$ if there exists an open set $U \subset \mathbb{R}^n$, $U \subset X$, such that

(i) $a \notin f(X \setminus U)$, i.e., $f^{-1}(a) \subset U$,

(ii) $f|U$ is of class $C^1$,

(iii) $a$ is a regular value of the map $f|U$ of class $C^1$.

For example, for a given complex $K$ let $Z(K)$ be the union of all images $|\varphi|(s)$ of simplexes $s \in K$ such that $|\varphi|(s)$ is contained in an $(n-1)$-flat. It is clear that $|\varphi|[S(K)] \subset Z(K)$. Thus the map $|\varphi||U$, $U = |K| \setminus S(K)$, is of class $C^\infty$ and each point $a \in \mathbb{R}^n \setminus Z(K)$ is a regular value of the simplicial map.

2. An approximation theorem. We shall precede the main result of our note by the following

Lemma. For each continuous antipodal map $f: X \to \mathbb{R}^n$, where $X \subset \mathbb{R}^n$ is a compact symmetric set, there exists a continuous antipodal map $F: \mathbb{R}^n \to \mathbb{R}^n$ such that $F|X = f$. 

Proof. Without loss of generality we may assume that \( 0 \in X \) because, for every antipodal map, \( 0 \in X \) implies \( f(0) = 0 \). Define for each \( k = 1, \ldots, n \)
\[
R^+_k := \{(x_1, \ldots, x_n) \in \mathbb{R}^n: x_k \geq 0 \text{ and } x_i = 0 \text{ for } i > k\},
\]
\[
R^-_k := \{x \in \mathbb{R}^n: -x \in R^+_k\}, \quad R_k := R^+_k \cup R^-_k.
\]
We have \( R_1 \subset R_2 \subset \ldots \subset R_n = \mathbb{R}^n \). Now, we shall construct the map \( F \) in \( n \) steps.

\((k = 1)\) Let \( f_1: R^+_1 \to \mathbb{R}^m \) be a continuous extension of the map \( f|X \cap R^+_1 \). Then, let us extend the map \( f_1 \) onto \( R^-_1 \) defining
\[
f_1(x) := -f(-x) \quad \text{for } x \in R^-_1.
\]

\((k + 1)\) Assume that for \( k < n \) the map \( f_k: R_k \to \mathbb{R}^m \) is defined. Since
\[
f_k|X \cap R_k = f|X \cap R_k,
\]
the map
\[
g := f_k \cup f|X \cap R^+_{k+1}: R_k \cup X \cap R^+_{k+1} \to \mathbb{R}^m
\]
is continuous. According to the Tietze–Urysohn theorem the map \( g \) is extendable to a continuous map \( f_{k+1}: R^+_{k+1} \to \mathbb{R}^m \). Extend the map \( f_{k+1} \) onto \( R^-_{k+1} \) by the formula
\[
f_{k+1}(x) := -f_{k+1}(-x) \quad \text{for } x \in R^-_{k+1}.
\]
Put \( F := f_n \). This completes the proof.

Theorem (Approximation Theorem). Let \( f: \text{Bd } X \to \mathbb{R}^n \) be a continuous antipodal map defined on a compact symmetric subset \( X \subset \mathbb{R}^n \). Then for each \( \varepsilon > 0 \) there exists a simplicial antipodal map \( f_\varepsilon: P \to \mathbb{R}^n \), defined on a symmetric polyhedron \( P \), \( X \subset P \subset \mathbb{R}^n \), such that

(a) 0 is a regular value of the map \( f_\varepsilon \),
(b) \( \|f(x) - f_\varepsilon(x)\| < \varepsilon \) for each \( x \in \text{Bd } X \).

Proof. Fix a number \( M > 0 \) and let \( e_i \in \mathbb{R}^n \), \( i = 1, \ldots, n \), be points of \( \mathbb{R}^n \) defined as follows:
\[
e_1 := (M, 0, \ldots, 0), \quad e_2 := (0, 0, M, \ldots, 0), \quad \ldots, \quad e_n := (0, 0, \ldots, 0, M).
\]
Let \( K \) be a simplicial complex consisting of \( n \)-simplexes of the form
\[
[0, \pm e_1, \ldots, \pm e_n]
\]
and their \( k \)-faces, \( k < n \). The polyhedron \(|K|\) is the smallest convex set which contains the set \( \{e_1, \ldots, e_n, -e_1, \ldots, -e_n\} \). Assume that the number \( M > 0 \) is such that \( X \subset |K| \). According to the previous lemma the map \( f_\varepsilon: \text{Bd } X \to \mathbb{R}^n \) has a continuous antipodal extension \( F_\varepsilon: |K| \to \mathbb{R}^n \).

Now, fix an \( \varepsilon > 0 \). In view of the fact that the map \( F \) is uniformly continuous there exists an \( r \)-th barycentric subdivision \( K^{(r)} \) of the complex \( K \) such that
(1) \[ \text{mesh } K^{(r)} \leq \frac{\varepsilon}{36} \quad \text{and} \quad \text{diam } F(s) \leq \frac{\varepsilon}{36} \quad \text{for each } s \in K^{(r)}. \]

Let \( A := V(K^{(r)}) \) be the set of all vertices of the complex \( K^{(r)} \). Consider the following subsets of \( A \):

\[
\begin{align*}
A_1 & := \{ a \in A : a \in \text{Bd st}(0, K^{(r)}) \} \cup \{ 0 \}, \\
A_2 & := \{ b \in A \setminus A_1 : b \in \text{Bd st}(a, K^{(r)}) \setminus \{ a \}, a \in A_1 \}, \\
A_3 & := A \setminus (A_1 \cup A_2).
\end{align*}
\]

The sets \( A, A_1, A_2, A_3 \) are finite and symmetric. Since the map \( F \) is antipodal, so, in particular, the set \( E := A_2 \cup F(A_3) \) is also symmetric. Let \( Z \) be the union of all \( k \)-simplexes, \( k < n \), with vertices belonging to the set \( E \). The set \( Z \) is a compact nowhere dense symmetric subset of \( \mathbb{R}^n \). Hence there exist points

\[ c_s \in B(0, \delta) \setminus Z, \]

where

\[ B(0, \delta) := \{ x \in \mathbb{R}^n : \| x \| \leq \delta \}, \quad B(0, \delta) \subset \text{st}(0, K^{(r)}) \}, \quad 0 < \delta < \frac{\varepsilon}{36}. \]

Now, let us define an antipodal map \( \varphi : A \to \mathbb{R}^n \) in the following way: For each \( a = (a_1, \ldots, a_n) \in A_2 \cup A_3 \) put

\[ k := \max \{ i \leq n : a_i \neq 0 \}, \]

and then define

\[
\varphi(a) := \begin{cases} 
  a & \text{if } a \in A_1, \\
  a + c_i & \text{if } a \in A_2 \text{ and } a_k > 0, \\
  a - c_i & \text{if } a \in A_2 \text{ and } a_k < 0, \\
  F(a) + c_j & \text{if } a \in A_3 \text{ and } a_k > 0, \\
  F(a) - c_j & \text{if } a \in A_3 \text{ and } a_k < 0.
\end{cases}
\]

Let \( f_\varepsilon : P \to \mathbb{R}^n, P := |K^{(r)}| = |K| \), be a simplicial map induced by the map \( \varphi \), i.e.,

\[ f_\varepsilon(x) := \sum_{i=0}^{j} \lambda_i \varphi(a_i), \]

where

\[ x = \sum_{i=0}^{j} \lambda_i a_i \in s = [a_0, \ldots, a_j] \in K^{(r)}, \quad 0 \leq \lambda_i, \quad \sum_{i=0}^{j} \lambda_i = 1. \]

We verify that

\[ \| F(x) - f_\varepsilon(x) \| < \varepsilon \quad \text{for each } x \in P. \]
Indeed, first observe that

$$\varphi(a) = F(a) + \eta(a),$$

where \(\|\eta(a)\| < \varepsilon/6\) for each \(a \in A\).

From (1)–(5) we get, for \(x \in [a_0, \ldots, a_j] \in K^{(r)}\),

$$\|F(x) - f_\varepsilon(x)\| \leq \|F(x) - \varphi(a_0)\| + \|\varphi(a_0) - j_\varepsilon(x)\|$$

$$\leq \|F(x) - F(a_0)\| + \|\eta(a_0)\| + \max \{\|\varphi(a_i) - \varphi(a_j)\| : i \leq j\}$$

$$\leq \varepsilon/6 + \varepsilon/6 + \max \{\|F(a_0) - F(a_j)\| : i \leq j\} + \|\eta(a_0)\|$$

$$+ \max \{\|\eta(a_i)\| : i \leq j\} \leq 5 \cdot \varepsilon/6 < \varepsilon.$$

Since the map \(\varphi\) is antipodal, so is the map \(f_\varepsilon\). The proof will be completed if we show that 0 is a regular value of the map \(f_\varepsilon\). First, observe that \(f_\varepsilon\) is of class \(C^\infty\) on the open set

$$U := \text{Int}(0, K^{(r)}) \cup ([K^{(r)} - \{\text{Int}(0, K^{(r)})\}], K^{(r)})].$$

Next, consider a point \(x \in f_\varepsilon^{-1}(0)\). If \(x = 0\), then \(x\) is not a critical point of the map \(f_\varepsilon|U\) because \(f_\varepsilon|\text{Int}(0, K^{(r)})\) is the identity map. If \(x \neq 0\) and \(f_\varepsilon(x) = 0\), then in view of the choice of the point \(c \in R^n\) we get \(x \in \text{Int}s\) for some \(n\)-simplex \(s = [a_0, \ldots, a_n] \in K^{(r)}\) such that the set \(\{\varphi(a_0), \ldots, \varphi(a_n)\}\) is affinely independent. Hence \(\det f_\varepsilon(x) \neq 0\). The proof that 0 is a regular value is completed.

3. **On a proof of the Borsuk antipodal theorem.** In this part we would like to explain a role which the approximation theorem plays in the proof of the Borsuk theorem suggested by Nirenberg [4].

The **classical degree function** is an integer-value function \(\deg(f, X, a)\) defined for all continuous maps \(f: X \to R^n\), where \(X\) is a compact subset of \(R^n\) and \(a \notin f(\text{Bd}X)\), satisfying the following conditions:

(a) If \(\deg(f, X, a) \neq 0\), then \(a \in \text{Int}f(X)\).

(b) If \(f: X \to R^n\) is a map of class \(C^1\) and the point

$$a \in f(X) - f(\text{Bd}X)$$

is a regular value, then

$$\deg(f, X, a) = \sum \{\text{sgn} \det f'(x) : x \in f^{-1}(a)\}.$$

(c) If \(H \subseteq X\) is a closed subset and \(a \notin f(H \cup \text{Bd}X)\), then

$$\deg(f, X, a) = \deg(f, C(X \setminus H), a).$$

(d) For each continuous map \(f: X \to R^n\) and a point \(a \notin f(\text{Bd}X)\)
there exists an \( \varepsilon > 0 \) such that, for every continuous map \( g: X \to \mathbb{R}^n \), if \( \| f(x) - g(x) \| < \varepsilon \) for each \( x \in \text{Bd} \, X \), then

\[
\text{deg}(f, X, a) = \text{deg}(g, X, a).
\]

From (a)–(d) we get further properties:

(e) \( F: X \times [0, 1] \to \mathbb{R}^n \) is a continuous map such that for each \( t \in [0, 1] \) and \( x \in \text{Bd} \, X \) we have \( a \neq F(x, t) \), then

\[
\text{deg}(f_0, X, a) = \text{deg}(f_1, X, a),
\]

where \( f_0(x) = F(x, 0) \) and \( f_1(x) = F(x, 1) \).

(f) For any continuous maps \( f, g: X \to \mathbb{R}^n \) and a point \( a \notin f(\text{Bd} \, X) \),

\[
f|\text{Bd} \, X = g|\text{Bd} \, X \text{ implies } \text{deg}(f, X, a) = \text{deg}(g, X, a).
\]

(g) If \( f: X \to \mathbb{R}^n \) is a map of class \( C^1 \) and a point \( a \in \mathbb{R}^n \setminus f(\text{Bd} \, X) \) is a regular value of \( f \), then \( \text{deg}(f, X, a) \) is an odd integer if and only if the cardinality of \( f^{-1}(a) \) is an odd number.

Notice that if \( f: X \to \mathbb{R}^n \) is a simplicial map and a point \( a \in f(X) \setminus f(\text{Bd} \, X) \) is a regular value of \( f \), then there exists a closed subset \( H \subseteq X \) such that \( a \notin f(H) \). Then the map

\[
g = f|\text{Cl}(X \setminus H)
\]

is of class \( C^1 \) and the point \( a \) is a regular value of \( g \). The property (c) yields

\[
\text{deg}(f, X, a) = \text{deg}(g, \text{Cl}(X \setminus H), a).
\]

But from the above and the property (g) we infer that:

\( (g') \) If \( f: X \to \mathbb{R}^n \) is a simplicial map and a point \( a \in \mathbb{R}^n \setminus f(\text{Bd} \, X) \) is a regular value of \( f \), then \( \text{deg}(f, X, a) \) is an odd integer if and only if the cardinality of \( f^{-1}(a) \) is an odd number.

The Borsuk Theorem (see [4]). If \( f: \text{Bd} \, X \to \mathbb{R}^n \setminus \{0\} \) is a continuous antipodal map and \( X \subset \mathbb{R}^n \) is a compact symmetric set such that \( 0 \in X \), then, for each continuous extension \( f^*: X \to \mathbb{R}^n \) of the map \( f \), \( \text{deg}(f^*, X, 0) \) is an odd integer.

Proof. According to the property (f) and the Lemma we may assume that \( f^* \) is an antipodal map. The property (d) and the approximation theorem imply that there exists a simplicial antipodal map \( f_\varepsilon: X \to \mathbb{R}^n \) such that \( 0 \) is a regular value of \( f_\varepsilon \) and, by (e),

\[
\text{deg}(f_\varepsilon, X, 0) = \text{deg}(f^*, X, 0).
\]

To see that \( \text{deg}(f_\varepsilon, X, 0) \) is an odd integer it suffices to observe, in view of the property (g'), that the cardinality of the set \( f_\varepsilon^{-1}(0) \) is an odd number. But this is obvious because, since \( f_\varepsilon \) is an antipodal map and \( 0 \) is a regular value of \( f_\varepsilon \), so \( f_\varepsilon^{-1}(0) \) is a finite symmetric set which contains \( 0 \). It is clear that such a set has an odd number of elements.
THE BORSUK–ULAM THEOREM. If \( g : \text{Bd} \, X \to \mathbb{R}^m \subset \mathbb{R}^n \), \( n > m \), is a continuous map defined on the boundary of a compact symmetric set \( X \subset \mathbb{R}^n \) such that \( 0 \in X \), then, for some point \( x \in \text{Bd} \, X \), \( g(x) = g(-x) \).

Proof. Suppose that, for each \( x \in \text{Bd} \, X \), \( g(x) \neq g(-x) \). Define

\[
f(x) := g(x) - g(-x).
\]

The map \( f : \text{Bd} \, X \to \mathbb{R}^m \setminus \{0\} \) is antipodal. Hence, for an arbitrary continuous antipodal extension \( f^* : X \to \mathbb{R}^m \subset \mathbb{R}^n \), \( \deg(f^*, X, 0) \) is an odd integer (see the Lemma, property (i) and the Borsuk theorem). But from the property (a) we infer that

\[
0 \in \text{Int}_{\mathbb{R}^n} f^*(X) \subset \text{Int}_{\mathbb{R}^n} \mathbb{R}^m = \emptyset,
\]

a contradiction.

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Reçu par la Rédaction le 5.2.1988