INDECOMPOSABLE $\mathbb{Z}_p[G]$-LATTICES
FOR A CLASS OF METABELIAN GROUPS

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1. Introduction. Let $\mathbb{Q}_p$ be the field of $p$-adic numbers and $\mathbb{Z}_p$ its ring of integers. In this paper we determine all finitely generated and $\mathbb{Z}_p$-torsion-free $\mathbb{Z}_p[G]$-modules for a class of metabelian groups $G$ (Theorem 2.4). This generalizes Théorème II.5 of [3]. An application of this result to the study of the Galois structure of unit groups in certain real algebraic number fields will be given elsewhere (see [4]).

In the sequel $G$ will denote a non-abelian group of order $pm$ which is a non-trivial semidirect product of the cyclic group $S$ of order $p$ by an abelian group $T$ of order $m$ such that the order of every element of $T$ divides $p-1$. It is known (see [3], Proposition 1.3) that every such group $G$ is uniquely determined by $S$, $T$ and a non-trivial $p$-adic character $\chi$ of $T$. We have then $tst^{-1} = s^{\chi(t)}$ for $s \in S$, $t \in T$.

We shall apply the method of Rosen [7] which utilizes skew group rings. The same approach was earlier used by Pu [6] for classifying integral representations of metacyclic groups of order $pq$ (with prime $p$, $q$).

Let $R$ be an integral domain and let $A$ be an $R$-algebra. A left $A$-module which is finitely generated and projective as an $R$-module will be called a $A$-lattice. Observe that in the case of a Dedekind ring $R$ every finitely generated and torsion-free module is projective.

If $\Gamma$ is a finite group, $R$ is a ring and $\text{Aut}(R)$ denotes the group of all automorphisms of $R$, then the skew group ring $R \ast \Gamma$ is defined as a free left $R$-module with $\Gamma$ serving as a system of free generators, in which multiplication is defined by putting

$$ rx \cdot sy = r^\Psi_x(s)xy $$

(for $x, y \in \Gamma$, $r, s \in R$), where $\Psi$ is a fixed homomorphism of $G$ into $\text{Aut}(R)$. The image of $x$ under $\Psi$ will be denoted by $\Psi_x$.

Let $M$, $N$ be $R[\Gamma]$-modules. We treat $\text{Hom}_R(N, M)$ as an $R[\Gamma]$-module with the action of $\Gamma$ defined by

$$ (xf)(m) = xf(x^{-1}m) $$

for $x \in \Gamma$, $m \in N$ and $f \in \text{Hom}_R(N, M)$. 


Every cocycle \( F \in Z^1(\Gamma, \text{Hom}_R(N, M)) \) (i.e. a map \( \Gamma \to \text{Hom}_R(N, M) \)) satisfying \( F_{xy} = xF_y + F_x \) for all \( x, y \in \Gamma \) defines an \( R[\Gamma] \)-extension of \( M \) by \( N \) (treated both as \( R[\Gamma] \)-modules), which is defined as the \( R \)-direct sum \( M \oplus N \) on which \( \Gamma \) acts by
\[
x(m, n) = (xm + F_x(n), xn) \quad (m \in M, n \in N, x \in \Gamma).
\]

We shall denote this extension by \((M, N; F)\), and in the cases where the choice of \( F \) is obvious we write simply \((M, N)\).

We shall use the following two lemmas:

(1.3) **Lemma** ([5], Lemme III.2). Let \( \Gamma \) be a finite group, \( \Delta \) its normal subgroup and \( M \) an \( R[\Gamma] \)-module such that \( M^\Delta = 0 \). If we define the action of \( \Gamma/\Delta \) on \( H^1(\Delta, M) \) by the formula
\[
(x\Delta) F_g = x^{-1} F_{xgx^{-1}} \quad \text{for} \ g \in \Delta, \ x \in \Gamma \text{ and } F \in Z^1(\Delta, M),
\]
then the groups \( H^1(\Gamma, M) \) and \( H^1(\Delta, M)^{\Gamma/\Delta} \) are isomorphic.

(1.4) **Lemma** ([1], Corollary (3.45)). Let \( M, N \) be \( R[\Gamma] \)-lattices and let \( F, F' \) be cocycles in \( Z^1(\Gamma, \text{Hom}_R(N, M)) \). Then the extensions \((M, N; F)\) and \((M, N; F')\) are \( R[\Gamma] \)-isomorphic if and only if there exist
\[
a \in \text{Aut}_{R[\Gamma]}(M), \quad b \in \text{Aut}_{R[\Gamma]}(N), \quad c \in \text{Hom}_R(N, M)
\]
such that for all \( x \in \Gamma \)
\[
aF_x(xm) - F'_x(xbm) = xc(m) - c(xm).
\]

We shall consider skew group-rings of the form \( R \rtimes T \), where \( R \) is either \( Z_p[\xi] \) or \( Q_p(\xi) \), and the homomorphism \( \Psi \) in (1.1) is defined by \( \Psi_t(\xi) = \xi^{|t|} \) for any \( t \in T \), where \( \xi \) is a fixed primitive \( p \)-th root of unity.

Some further notation is needed. We put:
\[
L = Q_p(\xi), \quad R_L = Z_p[\xi], \quad T_0 = \text{Ker} \chi, \quad n_0 = \# T_0, \quad n = m/n_0,
\]

\( t_1 \) is a fixed representative of the generating coset in \( T/T_0 \), and \( P = (1 - \xi) R_L \).

\( \tilde{H} \) will denote the character group of an abelian group \( H \). For any \( X \in \tilde{T}_0 \) we choose a character \( \bar{X} \in \tilde{T} \), whose restriction to \( T_0 \) coincides with \( X \), and by \( \chi_1 \) and \( \bar{X}_1 \) we denote the principal characters of \( T_0 \) and \( T \), respectively. Finally, for any \( X \in \tilde{T} \) we set
\[
e_X = (\sum_{t \in T} X(t^{-1}) t)/m.
\]

2. **Indecomposable lattices over the skew group-ring** \( R_L \rtimes T \). Let
\[
T = \bigcup_{i=0}^{n-1} T_0 t_1^i
\]
be the decomposition of \( T \) into disjoint cosets with respect to \( T_0 \).
(2.1) Proposition. The left ideals $L \chi$ (where $\chi \in \hat{T}_0$) form a complete set of simple $L \ast T$-modules.

Proof. Since $L \ast T$ is obviously semi-simple, it suffices to show that all minimal ideals of $L \ast T$ are of the asserted form. First of all observe that

$$L \ast T = \bigoplus_{\chi \in \hat{T}} L \chi.$$

Since all ideals $L \chi$ are minimal, it remains to show that for any $\chi, \eta$ in $\hat{T}$ which coincide on $T_0$ we have $L \chi \simeq L \eta$. Indeed, under this condition the element $X(t_1)/Y(t_1)$ of $Q_p$ is an $n$-th root of unity, whence its norm from $Q_p$ to $Q_p(\xi)$ is 1. Using Hilbert's Theorem 90 we can write (with a suitable $c \in Q_p(\xi)$)

$$X(t_1)/Y(t_1) = \Psi_{t_1}(c)/c,$$

and this allows us to construct an $L \ast T$-isomorphism of $L \chi$ onto $L \eta$ mapping $ae \chi$ onto $ace \eta$ for $a \in L$.

We need two lemmas:

(2.2) Lemma. The ring $R_L \ast T$ is hereditary.

Proof. Since the extension $L/L^T$ is tamely ramified, the trace map $Tr = Tr_{L/L^T}$ on integers is surjective, and hence there exists $s_0 \in Z_p[\xi]$ such that

$$Tr(s_0) = \sum_{t \in T} \Psi_t(s_0) = n_0.$$ 

If now $M$ is a left ideal of $R_L \ast T$, then it is also an $R_L$-submodule of the $R_L$-lattice $R_L \ast T$. Therefore it is $R_L$-projective. Now let $F : N \rightarrow M$ be an $R_L \ast T$-surjection of an $R_L \ast T$-module $N$ on $M$. Then there exists $a \in \text{Hom}_{R_L}(M, N)$ which splits $F$, i.e. $Fa = 1_M$. Put

$$a' = \left(\sum_{t \in T} ts_0 at^{-1}\right)/n_0 \in \text{Hom}_{R_L \ast T}(M, N).$$

Then

$$Fa' = \left(\sum_{t \in T} ts_0 Fat^{-1}\right)/n_0 = \left(\sum_{t \in T} \Psi_t(s_0)/n_0\right) = 1,$$

whence $a'$ splits $F$, and this shows that $M$ is $R_L \ast T$-projective.

(2.3) Lemma ([1], 26.12 (ii)). Let $R$ be a Dedekind ring and assume that the skew group-ring $R \ast T$ is hereditary. Then an $R \ast T$-lattice $M$ is indecomposable if and only if $K \otimes M$ is a simple $K \ast T$-module, where $K$ denotes the quotient field of $R^T$ $(T$ acting on $R$ by right multiplication, as defined in (1.1)).

Now we can prove the main result of this section:

(2.4) Theorem. Every indecomposable $R_L$-lattice is isomorphic to $P^j\chi$ with a suitable $0 \leqslant j < n$ and $\chi \in \hat{T}_0$. 
Proof. From Proposition 2.1 and Lemmas 2.2 and 2.3 it follows that $M$ is an indecomposable $R_L \ast T$-lattice if and only if $L \otimes M = Lx_\chi$ with a suitable $x \in \hat{T}_0$.

The module $M$ has $R_L$-rank one, hence it is isomorphic to $R_L$ as an $R_L$-module. Consequently, in view of $M \subset L \ast T$ we arrive at $M = R_L xe_\chi$ with a suitable $x$ in $L$, and this is of the asserted form.


(3.1) Proposition. For $x \in \hat{T}$, $y \in \hat{T}_0$ and $j = 0, 1, \ldots, n-1$ let

$$M = \text{Hom}_{Z_p}(Z_p e_x, P^j e_y).$$

Then the group $H^1(G, M)$ is cyclic of $p$ elements if $x = \chi^{j-1} y$, and is zero otherwise.

Proof. Since $M^S = 0$, Lemma 1.3 reduces our task to computing $H^1(S, M)^{G/S}$.

Denote by $a$ a generator of $S$ and let $c$ be a cocycle in $Z^1(S, M)$. It is determined by $c_a(e_x)$. The fact that the class of $c$ is fixed by $G/S$ is equivalent to the existence of an element $v$ of $M$ such that, for any $t \in T$,

$$t^{-1} c_{x \cdot t} c_x - c_a = (1 - \xi) v.$$ 

Hence, with $y_i = (1 - \xi^{x(t)})/(1 - \xi)$, we have $y_i c_a - t c_a = (1 - \xi^{x(t)}) t v$ because $c_{x \cdot t} = y_i c_a$. Since $(t c_a)(x) = t c_a(t^{-1} x)$ for any $x \in Z_p e_x$, and with suitable $a \in R_L$ we have $c_a(e_x) = (1 - \xi^j a e_\bar{y})$, we get

$$y_i (1 - \xi^j a e_\bar{y} - X^{-1}(t)(1 - \xi^{x(t)}) \Psi_i(a) Y(t) e_\bar{y} \in P^{j+1} e_\bar{y}.$$ 

Using $y_i \equiv \chi(t) \pmod{P}$ and $\Psi_i(a) \equiv a \pmod{P}$ (for $t \in T$), we infer finally that for all $t$ in $T$

(3.2)

$$a X(t) Y^{-1}(t) - a \chi^{j-1}(t) \in P.$$

If now $X \not= \bar{Y} \chi^{j-1}$, then $a$ must belong to $P$, since otherwise we would have $X(t) \equiv \bar{Y} \chi^{j-1} \pmod{P}$, which gives a contradiction since all $n_0$-th roots of unity are distinct $\pmod{P}$. But $a \in P$ implies that $c$ must be the zero cocycle, and hence $H^1(G, M) = 0$ in this case.

If $X = \bar{Y} \chi^{j-1}$, then (3.2) is satisfied by all $a \in R_L$, which implies that $H^1(S, M)^{G/S} = H^1(S, M)$, so it remains to show that the last group has $p$ elements. To do this consider a cocycle $f$ which is determined by the value

$$f_a(e_x) = (1 - \xi)^j a e_\bar{y} \quad \text{with} \ a \in R_L.$$

Write $a = a + h(1 - \xi)$ with $0 \leq a < p$, $h \in R_L$ and put $g_a(e_x) = (1 - \xi^j a e_\bar{y})$. Then

$$g_a(e_x) - f_a(e_x) = (1 - \xi) w e_x,$$

where $w \in M$ and $w(e_x) = h(1 - \xi)^j$. Thus $g$ and $f$ are equivalent, and so define the same element of $H^1(S, M)$. 


If two cocycles \( f, g \) are equivalent and
\[
f_\sigma(e_\chi) = (1 - \zeta)^j a e_\gamma, \quad g_\sigma(e_\chi) = (1 - \zeta)^j a' e_\gamma
\]
with \( 0 \leq a, a' < p \) and \( a \neq a' \), then there exists \( b \in M \) such that
\[
f_\sigma(e_\chi) - g_\sigma(e_\chi) = \sigma b(e_\chi) - b(\sigma e_\chi),
\]
whence \( a \equiv a' \pmod{P} \), thus \( a = a' \), a contradiction. This shows that \( H^1(S, M) \) has \( p \) elements, and so the proposition is established.

(3.3) Lemma. For every integer \( j \) satisfying \( 0 \leq j < n \) and every \( Y \in \hat{T}_0 \) there exists exactly one (up to \( \mathbb{Z}_p[G] \)-isomorphism) non-trivial extension of \( P^j e_\gamma \) by \( \mathbb{Z}_p e_{\chi^j-1} \).

Proof. First we show that all non-trivial extensions are isomorphic. Let \( M \) be defined as in Proposition 3.1, with \( X = \chi^{j-1} Y \). Write
\[
f_\sigma(e_\chi) = (1 - \zeta)^j a e_\gamma, \quad f'_\sigma(e_\chi) = (1 - \zeta)^j a' e_\gamma
\]
with \( a, a' \in \mathbb{Z}_p \setminus p \mathbb{Z}_p \). Since \( a/a' \) is a \( p \)-adic unit, the \( \mathbb{Z}_p[G] \)-endomorphism \( \alpha: x \rightarrow (a/a') x \) of \( P^j e_\gamma \) is in fact an automorphism. Since \( f_\sigma(e_\chi) - \alpha f'_\sigma(e_\chi) = 0 \), Lemma 1.4 implies that the extensions defined by \( f \) and \( f' \) are \( \mathbb{Z}_p[G] \)-isomorphic.

It remains to prove the existence of a non-trivial extension. Let \( f_\sigma(e_\chi) = (1 - \zeta)^j e_\gamma \). If the extension defined by \( f \) were trivial, then there would exist a \( \mathbb{Z}_p[G] \)-automorphism \( \alpha \) of \( P^j e_\gamma \) and \( c \in M \) such that
\[
\alpha f_\sigma(\sigma e_\chi) = \sigma c(e_\chi) - c(\sigma e_\chi) = (\zeta - 1) c(e_\chi).
\]
Since \( c(e_\chi) \in P^j e_\gamma \), this would imply \( \alpha((1 - \zeta)^j e_\gamma) \in P^{j+1} e_\gamma \), and hence
\[
\alpha(P^j e_\gamma) \subseteq P^{j+1} e_\gamma.
\]
Thus \( \alpha \) could not be an automorphism.

Finally we prove our main result, giving a classification of indecomposable \( \mathbb{Z}_p[G] \)-lattices:

(3.4) Theorem. Every indecomposable \( \mathbb{Z}_p[G] \)-lattice is isomorphic to one of the following modules: \( \mathbb{Z}_p e_\chi \), \( P^j e_\gamma \) and \( (P^j e_\gamma, \mathbb{Z}_p e_{\chi^j-1}) \), where \( X \in \hat{T}, Y \in \hat{T}_0 \) and \( 0 \leq j < n \).

Proof. Since \( p \chi [G:S] \), the Lemma in [2] shows that every indecomposable \( \mathbb{Z}_p[G] \)-lattice is isomorphic to a direct summand of \( \mathbb{Z}_p[G] \otimes M \) for a certain indecomposable \( \mathbb{Z}_p[S] \)-module \( M \). We have three choices for \( M \), namely \( \mathbb{Z}_p, R_L \) and \( \mathbb{Z}_p[S] \).

Observe that
\[
\mathbb{Z}_p[G] \cong \bigoplus_{X \in \hat{T}} \mathbb{Z}_p[S] e_\chi,
\]
the isomorphism given by the map

\[ x \mapsto \sum_{x \in \mathcal{T}} xe_X \quad (x \in \mathbb{Z}_p[G]). \]

Since the element \( \sigma \) acts trivially on \( \mathbb{Z}_p \), we have

\[ \mathbb{Z}_p[S] e_X \otimes_{\mathbb{Z}_p[S]} \mathbb{Z}_p \simeq \mathbb{Z}_p e_X; \]

hence

\[ \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[S]} \mathbb{Z}_p \simeq \bigoplus_{x \in \mathcal{T}} (\mathbb{Z}_p[S] e_X \otimes_{\mathbb{Z}_p[S]} \mathbb{Z}_p) \simeq \bigoplus_{x \in \mathcal{T}} \mathbb{Z}_p e_X. \]

Now consider

\[ N = \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[S]} R_L \simeq \bigoplus_{t \in \mathcal{T}} (\mathbb{Z}_p[S] \otimes R_L) \simeq \bigoplus_{t \in \mathcal{T}} t \otimes R_L. \]

Clearly, the \( \mathbb{Z}_p \)-rank of \( N \) equals \((p-1)m\). Since \( S = 1 + \sigma + \sigma^2 + \ldots + \sigma^{p-1} \) annihilates \( N \), it follows that \( N \) (as a \( \mathbb{Z}_p[G] \)-module) is an \( R_L \ast T \)-lattice, because the rings \( \mathbb{Z}_p[G]/\mathcal{S}\mathbb{Z}_p[G] \) and \( R_L \ast T \) are isomorphic. Thus Theorem 2.4 implies

\[ N \simeq \bigoplus_{j=0}^{n-1} \bigoplus_{l=0}^{n_0-1} P^r e_{Y_l} \]

with suitable \( 0 \leq r < n \) and \( Y_l \in \mathcal{T}_0 \). Since \( N \) is \( \mathbb{Z}_p[G] \)-cyclic, the summands here are pairwise non-isomorphic. Hence

\[ \mathbb{Z}_p[G] \simeq \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[S]} \mathbb{Z}_p[S] \simeq \bigoplus_{j=0}^{n-1} \bigoplus_{Y \in \mathcal{T}_0} P^j e_{Y}. \]

In view of (3.5) it remains to decompose the modules \( \mathbb{Z}_p[S] e_X \) for \( X \in \mathcal{T} \). Let \( X = \chi^{-1} Y \), where \( 0 \leq j < n \) and \( Y \in \mathcal{T}_0 \), and consider the exact sequence of \( \mathbb{Z}_p[G] \)-lattices:

\[ 0 \rightarrow \mathbb{Z}_p[S] (1-\sigma) e_X \rightarrow \mathbb{Z}_p[S] e_X \rightarrow \mathcal{S}\mathbb{Z}_p e_X \rightarrow 0, \]

where the last epimorphism is the multiplication by \( \mathcal{S} \).

Now observe that the first non-zero term of this sequence is isomorphic to \( P^j e_{Y} \) and the last one to \( \mathbb{Z}_p e_X \). In fact, the second statement is obvious, and to prove the first one write, for \( j = 0, 1, \ldots, n-1 \),

\[ v_j = \left( \sum_{t \in \mathcal{T}} \chi(t^{-1}) \frac{\mathcal{S}\mathbb{Z}_p}{m} \right). \]

Using Proposition II.7 of [3] we get \( R_L v_j = P^j \) \((j = 0, 1, \ldots, n-1)\). If we put

\[ e_X(\sigma) = \left( \sum_{t \in \mathcal{T}} \chi(t^{-1}) \frac{\mathcal{S}\mathbb{Z}_p}{m} \right), \]
then by the Corollary (Scolie) to Proposition II.11 in [3] we obtain
\[ e_x(\sigma) \equiv (1 - \sigma) \mod (1 - \sigma)^2, \]
and thus \( Z_p[S] e_x(\sigma) = Z_p[S](1 - \sigma) \).

Now we obtain the asserted isomorphism from
\[ Z_p[S](1 - \sigma) e_{\chi^{-1}} = Z_p[S] e_x(\sigma) e_{\chi^{-1}} \]
to \( Z_p[S] v_j e_{\bar{\gamma}} \) by putting, for \( g \in Z_p[X] \),
\[ g(\sigma) e_x(\sigma) e_{\chi^{-1}} \mapsto g(\xi) v_j e_{\bar{\gamma}}. \]

Now (3.6) implies
\[ (3.7) \quad Z_p[S] e_x \simeq (P_j e_{\bar{\gamma}}, Z_p e_{\chi^{-1}}) \]
for any \( \chi \in \mathcal{F} \) of the form \( X = \chi^{-1} \bar{\gamma} \).

To complete the proof of the theorem it remains to show that the modules listed in its statement are indeed indecomposable.

The modules \( Z_p e_X \) are obviously indecomposable. Every decomposition of \( P_j e_{\bar{\gamma}} \), as a \( Z_p[G] \)-module, would be also an \( R_L \ast T \)-decomposition, which cannot exist by Theorem 2.4.

For any \( Z_p[G] \)-module \( M \) define
\[ \tilde{M} = \{ m \in M : \bar{S} m = 0 \}. \]

If we would have \( Z_p[S] e_X = M_1 \oplus M_2 \), then
\[ (Z_p[S] e_X)^{\sim} = Z_p[S](1 - \sigma) e_X = \tilde{M_1} \oplus \tilde{M_2}, \]
and by the indecomposability of \( Z_p[S](1 - \sigma) e_X \) as an \( R_L \ast T \)-module we have, say, \( \tilde{M_1} = Z_p[S](1 - \sigma) e_X \).

Comparing the \( Z_p \)-ranks we see that if \( M_1 \neq \tilde{M_1} \), then \( M_1 = Z_p[S] e_X \). If, however, \( M_1 = \tilde{M_1} \), then \( Z_p[S](1 - \sigma) e_X \) is a direct summand of \( Z_p[S] e_X \), but this is impossible because \( Z_p[S] e_X \) is a non-trivial extension of \( Z_p[S](1 - \sigma) e_X \) by \( Z_p e_X \). The theorem is thus proved.

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