INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

DISSERTATIONES MATHEMATICAE
(ROZPRAWY MATEMATYCZNE)

LXXIV

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On weak automorphisms of universal algebras

WARSZAWA 1970
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

Introduction*

The concept of weak isomorphism was introduced by A. Goetz and E. Marczewski [4]. A weak isomorphism of an algebraic structure onto itself will be called a weak automorphism. Goetz has shown [4] that if G is a group whose center contains the square of every element of G then every weak automorphism of G is either an automorphism or the product of an automorphism and the weak automorphism $x \mapsto x^{-1}$. The same result is true also for free groups but not for groups in general (see also [4], p. 166). The weak automorphisms of Boolean and Post algebras have been computed by Traczyk [9]. For Boolean algebras, the only weak automorphism which is not an automorphism is the product of an automorphism with the weak automorphism $x \mapsto x'$.

However, the set of all weak automorphisms of an algebra forms a group under composition, and the general theory of weak automorphism groups, unlike the theory of automorphism groups, has remained undeveloped. We advance here some methods for investigating these groups.

The first main observation in this study is that the automorphism group of an algebra is a normal subgroup of its weak automorphism group; this observation is utilized in much of what follows. The main results of Section 2 include a description of the structure of weak automorphism groups of algebras having a finite basis; and, as an application of these results, we find in Section 3 that all groups occur as weak automorphism groups of finitary algebras. At the same time, we establish that all groups occur as the factor groups of the weak automorphism group of an algebra by its automorphism group.

Since the weak automorphism group of an algebra $\mathfrak{A}$ with carrier A is a subgroup of the group of all permutations of A and contains the automorphism group of $\mathfrak{A}$ as a normal subgroup, it is natural to wonder if there exist algebras having the largest possible weak automorphism group, that is, if there are algebras whose weak automorphism groups are the normalizers of their automorphism groups in the symmetric group

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* Presented as a dissertation, under the direction of Abraham Goetz, to the University of Notre Dame in partial fulfillment of the requirements for the degree of doctor of philosophy.

on their carriers. We have termed such algebras pseudo-saturated. The main result of Section 4 provides us with a very positive answer to this query. Indeed, it is shown that if a subgroup H of the permutation group, $P(A)$, of a set A is the automorphism group of an algebra on A, then there is a pseudo-saturated algebra on A having H for its automorphism group. A simple sufficient condition for pseudo-saturation is also obtained. Section 5 contains a computation of the weak automorphism groups of unary algebras constructed from groups, whose automorphism groups were determined and studied by G. H. Wenzel [10], and a necessary and sufficient condition for pseudo-saturation for these algebras.

Weak isomorphisms have more "freedom" than isomorphisms in their usual sense. For the notion of isomorphism is relative to a *fixed* set of fundamental operations and to a *prescribed* correspondence between the operations of each algebra in question (usually effected by naming the operations of each algebra with a common set of names and allowing only operations having the same name to correspond). In Section 6 the concept of restricted weak automorphism is introduced. It is shown that many of the results of Sections 1 and 2 hold for groups of restricted weak automorphisms and that a stronger representation theorem holds for groups of restricted weak automorphisms of finite algebras. The concept of restricted weak isomorphism is very useful in understanding the differences and similarities between the usual notion of isomorphism and that of weak isomorphism.

Finally, Section 6 includes the generalization of weak automorphisms to finitary relational systems, and the representation theorem that for every pair of groups H, G with $H \lhd G$, H can be realized as the automorphism group, and G as the weak automorphism group of some finitary relational system.

SECTION 1

The group of weak automorphisms

1.1. Our working definition of a universal algebra (or simply, algebra) is the following. A *universal algebra* is a pair $\mathfrak{A} = (A, F)$ where A is a non-empty set and F is a set of finitary operations on A, that is, each $f \in F$

is a mapping f: $A^n \to A$ where n is a non-negative integer called the *arity* of f.

For $n \geqslant 1$, the *n-ary projections* on A are the n operations $e_1^n, \ldots, e_n^n$ defined by

$$x_1 \ldots x_n e_i^n = x_i$$

for all $x_1, \ldots, x_n \in A$. Given n-ary operations $p_1, \ldots, p_k$ on A and a k-ary operation q on A, $(p_1 \ldots p_k)q$ will denote the n-ary operation q on A defined by

$$x_1 \ldots x_n[(p_1 \ldots p_k) q] = (x_1 \ldots x_n p_1) \ldots (x_1 \ldots x_n p_k) q$$

for all $x_1, \ldots, x_n \in A$, and is called the *composition* of $p_1, \ldots, p_k$ by q.

An n-ary operation h on A is called an *n-ary derived operation* of $\mathfrak{A} = (A, F)$ if there exists a finite sequence $f_1, \ldots, f_m$ with $f_m = h$ and where each f_i is either an n-ary projection on A or, for some non-negative integer k, f_i is a composition of k of the preceding f's by a k-ary element of F ([5], p. 37; [7]). The set of all n-ary derived operations of $\mathfrak{A}$ will be denoted by $\boldsymbol{D}_n F$, and $\boldsymbol{D}F = \bigcup_{n=0}^{\infty} \boldsymbol{D}_n F$ is called the set of derived operations of $\mathfrak{A}$. It is clear that if $g_1, \ldots, g_k \in \boldsymbol{D}_n F$ and $g \in \boldsymbol{D}_k F$, then $(g_1 \ldots g_k) g \in \boldsymbol{D}_n F$. We shall call a set G of finitary operations on A a *fundamental set of operations* for $\mathfrak{A}$ if $\boldsymbol{D}G = \boldsymbol{D}F$; in particular, F is a fundamental set of operations for $\mathfrak{A}$.

An *endomorphism* of the algebra $\mathfrak{A} = (A, F)$ is a mapping φ: $A \to A$ such that for every operation $f \in F$,

$$(x_1 \ldots x_n f)\varphi = x_1\varphi \ldots x_n \varphi f$$

for all $x_1, \ldots, x_n \in A$, where n is the arity of f. An *automorphism* of $\mathfrak{A}$ is a bijective endomorphism. The set of all automorphisms of $\mathfrak{A}$, $A(\mathfrak{A})$, forms a group under composition; this group will be denoted by $\boldsymbol{A}(\mathfrak{A}) = (A(\mathfrak{A}), \{\circ\})$. We note that $A(\mathfrak{A}) = A((A, \boldsymbol{D}F))$.

By $\boldsymbol{A}(\boldsymbol{D}F) = (A(\boldsymbol{D}F), \{\circ\})$ we will mean the group on the set $A(\boldsymbol{D}F)$ of all bijective mappings σ: $\boldsymbol{D}F \to \boldsymbol{D}F$ satisfying

$$(1.1) \qquad (\mathbf{D}_n F)\sigma = \mathbf{D}_n F \quad \text{for} \quad n = 0, 1, \ldots$$

and

(1.2) for all non-negative integers n and k, if $f_1, \ldots, f_k \in \boldsymbol{D}_n F$ and $f \in \boldsymbol{D}_k F$, then

$$((f_1 \ldots f_k)f)\sigma = (f_1\sigma \ldots f_k\sigma)(f\sigma),$$

with composition, $\circ$, as the group operation. An element of $A(\boldsymbol{D}F)$ will be called a *clone automorphism* of $\boldsymbol{DF}$ ([3], p. 127). We note that if $\sigma \in \boldsymbol{A}(\boldsymbol{DF})$, then $e_i^n \sigma = e_i^n$ for all n and i with $1 \leqslant i \leqslant n$.

Following A. Goetz [4], a *weak automorphism* of the algebra $\mathfrak{A} = (A, F)$ is a bijective mapping $\Phi\colon A \to A$ such that the induced mapping $\bar{\Phi}\colon f \mapsto f\bar{\Phi}$ given by

$$f\bar{\Phi} = g \quad \text{if and only if} \quad (x_1 \ldots x_n f)\Phi = x_1\Phi \ldots x_n\Phi g$$

for all $x_1, \ldots, x_n \in A$ maps $\boldsymbol{D}_n F$ onto $\boldsymbol{D}_n F$ for $n = 0, 1, \ldots$

1.2. It is clear that every automorphism of $\mathfrak{A}$ is also a weak automorphism of $\mathfrak{A}$. Many algebras have weak automorphisms which are not automorphisms, however. For example, every non-trivial Boolean algebra $(B, \{\wedge, ', 0, 1\})$ has the proper weak automorphism $b \to b'$. Similarly, every non-abelian group has the proper weak automorphism $x \mapsto x^{-1}$. Along these lines one can generally say the following:

THEOREM 1.1. *Let $\mathfrak{A} = (A, F)$ be an algebra. If f is an invertible unary derived operation of $\mathfrak{A}$, and if $f^{-1} \in \boldsymbol{DF}$, then f is a weak automorphism of $\mathfrak{A}$.*

Proof. If $g \in \boldsymbol{D}_n F$, then $g\bar{f} \in \boldsymbol{D}_n F$ because $f, f^{-1} \in \boldsymbol{D}_1 F$ and $x_1 \ldots x_n(g\bar{f}) = (x_1 f^{-1} \ldots x_n f^{-1} g)f$. Similarly, $h = g(\overline{f^{-1}}) \in \boldsymbol{D}_n F$, and $h\bar{f} = g$. Therefore $\bar{f}$ is a bijection of $\boldsymbol{D}_n F$ onto $\boldsymbol{D}_n F$ proving f is weak automorphism of $\mathfrak{A}$.

COROLLARY 1.2. *If A is finite, then every invertible unary derived operation of $\mathfrak{A}$ is a weak automorphism of $\mathfrak{A}$.*

Proof. If $f \in \boldsymbol{D}_1 F$ is invertible, then because A is finite, for some $k > 0$, $f^{-1} = f^k \in \boldsymbol{D}_1 F$ and the theorem now applies.

1.3. It is immediate from the definitions that the set of all weak automorphisms of an algebra $\mathfrak{A} = (A, F)$, denoted by $W(\mathfrak{A})$, forms a group under composition, $\circ$, denoted by $\boldsymbol{W}(\mathfrak{A}) = (W(\mathfrak{A}), \{\circ\})$. It is also clear that if $\Phi, \Psi \in W(\mathfrak{A})$, then $\overline{\Phi\circ\Psi} = \bar{\Phi}\circ\bar{\Psi}$. Furthermore, as noted in [4], the bijection $\bar{\Phi}$ of $\boldsymbol{DF}$ is in fact a clone automorphism of $\boldsymbol{DF}$. Finally if 1_A denotes the identity mapping on A, then $\overline{1_A} = 1_{\boldsymbol{DF}}$. Thus the mapping $\Phi \mapsto \bar{\Phi}$ is a group homomorphism of $\boldsymbol{W}(\mathfrak{A})$ into $\boldsymbol{A}(\boldsymbol{DF})$ and the automorphisms of $\mathfrak{A}$ are precisely those weak automorphisms φ for which $\bar{\varphi} = 1_{\boldsymbol{DF}}$. Collecting these facts we have the following:

THEOREM 1.3. *The mapping $\Phi \mapsto \bar{\Phi}$ defined above is a homomorphism of $\boldsymbol{W}(\mathfrak{A})$ into $\boldsymbol{A}(\boldsymbol{DF})$ with kernel $\boldsymbol{A}(\mathfrak{A})$.*

COROLLARY 1.4. *$\boldsymbol{A}(\mathfrak{A})$ is a normal subgroup of $\boldsymbol{W}(\mathfrak{A})$.*

COROLLARY 1.5. *$\boldsymbol{W}(\mathfrak{A})/\boldsymbol{A}(\mathfrak{A})$ is isomorphic to a subgroup of $\boldsymbol{A}(\boldsymbol{DF})$.*

SECTION 2

Weak automorphisms of finitely generated free algebras

2.1. To deal with finitely generated free algebras we use the concept of independence of E. Marczewski [7]. The elements $a_1, \ldots, a_n$ of an algebra $\mathfrak{A} = (A, F)$ are called *independent* in $\mathfrak{A}$ if the identity

$$a_1 \ldots a_n f = a_1 \ldots a_n g$$

for some n-ary derived operations f and g of $\mathfrak{A}$ implies that f is identically equal to g on A.

The elements $a_1, \ldots, a_n$ are said to *generate* $\mathfrak{A}$ if for every element $a \in A$ there is an n-ary derived operation f of $\mathfrak{A}$ such that $a_1 \ldots a_n f = a$.

The elements $a_1, \ldots, a_n$ are called a *basis* of $\mathfrak{A}$ if they are independent in $\mathfrak{A}$ and generate $\mathfrak{A}$. Marczewski proved that $a_1, \ldots, a_n$ is a basis of $\mathfrak{A}$ if and only if every mapping of $\{a_1, \ldots, a_n\}$ into A is extendable to an endomorphism of $\mathfrak{A}$. Therefore the elements of a basis of $\mathfrak{A}$ can be considered as free generators of $\mathfrak{A}$ in the smallest variety containing $\mathfrak{A}$.

2.2. We examine now the structure of $W(\mathfrak{A})$ for algebras having a basis of finitely many elements. We use a generalization of a matrix to describe an automorphism of such an algebra.

Let $\mathfrak{A} = (A, F)$. A vertical n-tuple,

$$\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \quad \text{where } f_i \in \boldsymbol{D}_n F \text{ for } i = 1, \ldots, n$$

is called an *n-ary matrix of operations* of $\mathfrak{A}$. An n-ary matrix of operations

$$\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

of $\mathfrak{A}$ is called *non-singular* if there is an n-ary matrix of operations

$$\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

of $\mathfrak{A}$ such that

$$(f_1 \cdots f_n) g_i = e_i^n$$

and

$$(g_1 \cdots g_n) f_j = e_j^n$$

for $i, j = 1, \ldots, n$. We define a multiplication on the set $M_n(\mathfrak{A})$, of n-ary matrices of operations of $\mathfrak{A}$ by

$$\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} (g_1 \cdots g_n)f_1 \\ \vdots \\ (g_1 \cdots g_n)f_n \end{bmatrix}.$$

This operation gives the set $M_n(\mathfrak{A})$ the structure of a monoid with identity element

$$I_n = \begin{bmatrix} e_1^n \\ \vdots \\ e_n^n \end{bmatrix}.$$

We shall denote this monoid by $\boldsymbol{M}_n(\mathfrak{A})$.

It is easy to see now that $X \in M_n(\mathfrak{A})$ is non-singular if and donly if there is a $Y \in M_n(\mathfrak{A})$ such that

$$XY = YX = I_n. \tag{2.1}$$

Using now (2.1), it is easy to see the following is true.

PROPOSITION 2.1. *The set of non-singular elements of $M_n(\mathfrak{A})$ forms a group under the given multiplication.*

We shall denote the set of non-singular n-ary matrices of operations of $\mathfrak{A}$ by $G_n(\mathfrak{A})$, and the group formed by this set under the given multiplication of n-ary matrices of operations by $\boldsymbol{G}_n(\mathfrak{A})$.

If the algebra $\mathfrak{A}$ has a basis of n elements, we have the following equivalent form of (2.1):.

PROPOSITION 2.2. *Let $\mathfrak{A} = (A, F)$ have a basis $a_1, \ldots, a_n$. The n-ary matrix of operations*

$$X = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

of $\mathfrak{A}$ is non-singular if and only if $a_1 \ldots a_n f_1, \ldots, a_1 \ldots a_n f_n$ is a basis for $\mathfrak{A}$.

Proof. Let $b_i = a_1 \ldots a_n f_i$ for $i = 1, \ldots, n$. If X is non-singular, there is an n-ary matrix of operations

$$Y = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

of $\mathfrak{A}$ such that $XY = YX = I_n$. To show $b_1, \ldots, b_n$ is independent, suppose $p, q \in \boldsymbol{D}_n F$ and $b_1 \ldots b_n p = b_1 \ldots b_n q$. Then

$$\begin{aligned} a_1 \ldots a_n[(f_1 \ldots f_n)p] &= a_1 \ldots a_n f_1 \ldots a_1 \ldots a_n f_n p = b_1 \ldots b_n p = b_1 \ldots b_n q \\ &= a_1 \ldots a_n f_1 \ldots a_1 \ldots a_n f_n q = a_1 \ldots a_n[(f_1 \ldots f_n) q], \end{aligned}$$

and since $a_1, \ldots, a_n$ is an independent set, we conclude that $(f_1 \ldots f_n)p = (f_1 \ldots f_n)q$ on A. Therefore,

$$\begin{aligned} p &= (e_1^n \ldots e_n^n)p = \big((g_1 \ldots g_n)f_1 \ldots (g_1 \ldots g_n)f_n\big)p \\ &= (g_1 \ldots g_n)[(f_1 \ldots f_n)p] = (g_1 \ldots g_n)[(f_1 \ldots f_n)q] \\ &= \big((g_1 \ldots g_n)f_1 \ldots (g_1 \ldots g_n)f_n\big)q = (e_1^n \ldots e_n^n)q = q, \end{aligned}$$

proving $b_1, \ldots, b_n$ independent. To see that $b_1, \ldots, b_n$ is also a generating set for $\mathfrak{A}$, let $c \in A$ be arbitrary. Since $a_1, \ldots, a_n$ generates A, there is a $f \in \boldsymbol{D}_n F$ such that $a_1 \ldots a_n f = c$. But then, $b_1 \ldots b_n[(g_1 \ldots g_n)f] = c$ also.

Conversely, suppose $b_1, \ldots, b_n$ is a basis for $\mathfrak{A}$. Then there exist n-ary derived operations $g_1, \ldots, g_n$ such that $b_1 \ldots b_n g_i = a_i$ for $i = 1, \ldots, n$; that is,

$$a_1 \ldots a_n[(f_1 \ldots f_n)g_i] = a_1 \ldots a_n e_i^n \quad \text{for} \quad i = 1, \ldots, n.$$

By independence, $(f_1 \ldots f_n)g_i = e_i^n$ for $i = 1, \ldots, n$. On the other hand,

$$b_1 \ldots b_n[(g_1 \ldots g_n)f_j] = a_1 \ldots a_n f_j = b_j = b_1 \ldots b_n e_j^n,$$

so by the independence of $b_1, \ldots, b_n$ we conclude $(g_1 \ldots g_n)f_j = e_j^n$ for $j = 1, \ldots, n$. Therefore, X is non-singular.

As mentioned in 2.1, if $\mathfrak{A} = (A, F)$ has a basis $a_1, \ldots, a_n$, then each n-tuple of elements of A, $(c_1, \ldots, c_n)$, uniquely determines an endomorphism φ of $\mathfrak{A}$ such that $a_i\varphi = c_i$ for $i = 1, \ldots, n$. Again because $a_1, \ldots, a_n$ is a basis for $\mathfrak{A}$, for each c_i there is a unique $\varphi_i \in \boldsymbol{D}_n F$ such that $c_i = a_1 \ldots a_n \varphi_i$ for $i = 1, \ldots, n$. Thus the mapping

$$\varphi \mapsto \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} \tag{2.2}$$

where each φ_i is the unique element of $\boldsymbol{D}_n F$ satisfying

$$a_i\varphi = a_1 \ldots a_n \varphi_i \tag{2.3}$$

is a bijection from the set of endomorphisms of $\mathfrak{A}$ onto $M_n(\mathfrak{A})$. Moreover, $1_A \mapsto I_n$. If φ and ψ are both endomorphisms of $\mathfrak{A}$, we have

$$\begin{aligned} a_i(\varphi\psi) &= (a_i\varphi)\psi = (a_1 \ldots a_n\varphi_i)\psi = a_1\psi \ldots a_n\psi\varphi_i \\ &= a_1 \ldots a_n\psi_1 \ldots a_1 \ldots a_n\psi_n\varphi_i = a_1 \ldots a_n[(\psi_1 \ldots \psi_n)\varphi_i], \end{aligned}$$

so that $(\varphi\psi)_i = (\psi_1 \ldots \psi_n)\varphi_i$ for $i = 1, \ldots, n$. Therefore the mapping defined by (2.2) and (2.3) is a homomorphism of the monoid of endomorphisms of $\mathfrak{A}$ onto $\boldsymbol{M}_n(\mathfrak{A})$. Finally, it is clear that an endomorphism φ of $\mathfrak{A}$ is an automorphism of $\mathfrak{A}$ if and only if $a_1\varphi, \ldots, a_n\varphi$ is basis for $\mathfrak{A}$. Applying Proposition 2.2 and summarizing, we have:

THEOREM 2.3. *If $\mathfrak{A} = (A, F)$ has a basis $a_1, \ldots, a_n$, then the mapping which assigns to each endomorphism φ of $\mathfrak{A}$ the n-ary matrix of operations*

$$\begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix}$$

satisfying $a_i\varphi = a_i \ldots a_n\varphi_i$ for $i = 1, \ldots, n$ is an isomorphism of the endomorphism monoid of $\mathfrak{A}$ onto $\boldsymbol{M}_n(\mathfrak{A})$. The restriction of this mapping to $A(\mathfrak{A})$ is an isomorphism of $\boldsymbol{A}(\mathfrak{A})$ onto $\boldsymbol{G}_n(\mathfrak{A})$.

We are now prepared to describe the structure of weak automorphism groups of algebras having a finite basis.

THEOREM 2.4. *Let $\mathfrak{A} = (A, F)$ be an algebra with basis $a_1, \ldots, a_n$. Then $\boldsymbol{W}(\mathfrak{A})$ is a semidirect product of $\boldsymbol{A}(\mathfrak{A})$ by $\boldsymbol{A}(\boldsymbol{D}F)$. More precisely, $\boldsymbol{W}(\mathfrak{A})$ is isomorphic to the group on the set $A(\boldsymbol{D}F) \times G_n(\mathfrak{A})$ with multiplication*

$$\left(\sigma, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}\right)\left(\tau, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right) = \left(\sigma\tau, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \tau \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right) \quad \textit{where} \quad \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}\tau = \begin{bmatrix} f_1\tau \\ \vdots \\ f_n\tau \end{bmatrix}.$$

Proof. Let $\Phi \in W(\mathfrak{A})$. Then as we have seen, $\overline{\Phi} \in A(\boldsymbol{D}F)$. Putting $b_i = a_i\Phi$ for $i = 1, \ldots, n$, we show $b_1, \ldots, b_n$ is a basis for $\mathfrak{A}$.

If $c \in A$, then because Φ is surjective, there is a $b \in A$ with $b\Phi = c$. Now $a_1, \ldots, a_n$ generates A so for some $h \in \boldsymbol{D}_nF$, $b = a_1 \ldots a_nh$. Then

$$c = b\Phi = (a_1 \ldots a_nh)\Phi = a_1\Phi \ldots a_n\Phi(h\overline{\Phi}) = b_1 \ldots b_n(h\overline{\Phi})$$

showing $b_1, \ldots, b_n$ generates A. To check independence, suppose $b_1 \ldots b_nf_1 = b_1 \ldots b_nf_2$ for some $f_1, f_2 \in \boldsymbol{D}_nF$. Because $\Phi \in W(\mathfrak{A})$, there exist $g_1, g_2 \in \boldsymbol{D}_nF$ such that $g_j\overline{\Phi} = f_j$ for $j = 1, 2$. Then,

$$(a_1 \ldots a_ng_1)\Phi = b_1 \ldots b_nf_1 = b_1 \ldots b_nf_2 = (a_1 \ldots a_ng_2)\Phi$$

whence

$$a_1 \ldots a_ng_1 = a_1 \ldots a_ng_2$$

by the injectivity of Φ. The independece of $a_1, \ldots, a_n$ now gives $g_1 = g_2$ showing $f_1 = g_1\overline{\Phi} = g_2\overline{\Phi} = f_2$. Therefore $b_1, \ldots, b_n$ is a basis for $\mathfrak{A}$.

Consequently, there exist uniquely determined $\Phi_1, \ldots, \Phi_n \in \boldsymbol{D}_nF$ such that

$$a_i\Phi = a_1 \ldots a_n\Phi_i$$

for $i = 1, \ldots, n$, and by Proposition 2.2, we have

$$\begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_n \end{bmatrix} \in G_n(\mathfrak{A}).$$

We therefore have a mapping

$$J\colon \Phi \mapsto \left(\bar{\Phi}, \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_n \end{bmatrix}\right)$$

of $W(\mathfrak{A})$ into $A(\boldsymbol{D}F) \times G_n(\mathfrak{A})$. Now every element $a \in A$ has an expression as $a = a_1 \dots a_n f$ for a unique $f \in \boldsymbol{D}_n F$. Thus, if $\Phi \in W(\mathfrak{A})$.

$$a\Phi = (a_1 \dots a_n f)\Phi = a_1\Phi \dots a_n\Phi(f\bar{\Phi}) = (a_1 \dots a_n \Phi_1) \dots (a_1 \dots a_n \Phi_n)(f\bar{\Phi})$$

from which the injectivity of J is clear. J is also surjective, for if

$$\left(\sigma, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}\right) \in A(\boldsymbol{D}F) \times G_n(\mathfrak{A}),$$

again using the unique expression of an element $a \in A$ as $a_1 \dots a_n f$, we define a map $\Phi\colon A \to A$ by

$$(2.4) \qquad (a_1 \dots a_n f)\Phi = (a_1 \dots a_n f_1) \dots (a_1 \dots a_n f_n)(f\sigma).$$

Because

$$\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in G_n(\mathfrak{A}),$$

Proposition 2.2 shows $a_1 \dots a_n f_1, \dots, a_1 \dots a_n f_n$ is a basis for $\mathfrak{A}$ from which it follows that Φ defined by (2.4) is bijective.

We now show (2.4) defines a weak automorphism of $\mathfrak{A}$. Let $g \in \boldsymbol{D}_m F$; it will be shown that

$$(x_1 \dots x_m g)\Phi = x_1\Phi \dots x_m\Phi(g\sigma)$$

for all $x_1, \dots, x_m \in A$. For given $x_1, \dots, x_m$ there exist unique $g_1, \dots, g_m \in \boldsymbol{D}_n F$ such that $x_i = a_1 \dots a_n g_i$ for $i = 1, \dots, m$. Therefore,

$$x_1 \dots x_m g = a_1 \dots a_n[(g_1 \cdots g_m)g]$$

with $(g_1 \cdots g_m)g \in \boldsymbol{D}_n F$. By (2.4), writing $c_i = a_1 \dots a_n f_i$ for $i = 1, \dots, n$, we have

$$\begin{aligned} (x_1 \dots x_m g)\Phi &= \big(a_1 \dots a_n[(g_1 \cdots g_m)g]\big)\Phi \\ &= (a_1 \dots a_n f_1) \dots (a_1 \dots a_n f_n)\big([(g_1 \cdots g_m)g]\sigma\big) \\ &= (a_1 \dots a_n f_1) \dots (a_1 \dots a_n f_n)[(g_1\sigma \dots g_m\sigma)(g\sigma)] \\ &= \big(c_1 \dots c_n(g_1\sigma)\big) \dots \big(c_1 \dots c_n(g_m\sigma)\big)(g\sigma) \\ &= x_1\Phi \dots x_m\Phi(g\sigma). \end{aligned}$$

Because, by the above, $\bar{\Phi} = \sigma$ have that $\Phi \in W(\mathfrak{A})$. Finally

$$a_i\Phi = (a_1 \dots a_n e_i^n)\Phi = (a_1 \dots a_n f_1) \dots (a_1 \dots a_n f_n)(e_i^n \sigma)$$
$$= (a_1 \dots a_n f_1) \dots (a_1 \dots a_n f_n) e_i^n = a_1 \dots a_n f_i$$

for $i = 1, \dots, n$ so that $\Phi_i = f_i$ for $i = 1, \dots, n$. This establishes the surjectivity of J.

At this point, the group structure of $W(\mathfrak{A})$ can be transferred to $A(DF) \times G_n(\mathfrak{A})$ in such a way that J is an isomorphism. Let W temporarily denote the group obtained in this way. First of all, we see the identity of W is

$$\left(1_{DF}, \begin{bmatrix} e_1^n \\ \vdots \\ e_n^n \end{bmatrix}\right).$$

To compute the product

$$\left(\sigma, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}\right)\left(\tau, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right)$$

in W, we need only find the image in W under J of the product of the pre-images in $W(\mathfrak{A})$. Assume therefore, that

$$\Phi J = \left(\sigma, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}\right) \quad \text{and} \quad \Psi J = \left(\tau, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right).$$

As in the proof above, we have

$$(a_1 \dots a_n f)\Phi = (a_1 \dots a_n f_1) \dots (a_1 \dots a_n f_n)(f\sigma)$$

and similarly for Ψ. Now

$$\begin{aligned}(a_1 \dots a_n f)(\Phi\Psi) &= \big(a_1 \dots a_n [(f_1 \cdots f_n)(f\sigma)]\big)\Psi \\ &= (a_1 \dots a_n g_1) \dots (a_1 \dots a_n g_n)\big([(f_1 \cdots f_n)(f\sigma)]\tau\big) \\ &= (a_1 \dots a_n g_1) \dots (a_1 \dots a_n g_n)[(f_1\tau \dots f_n\tau)(f\sigma\tau)] \\ &= a_1 \dots a_n[(g_1 \dots g_n)(f_1\tau)] \dots a_1 \dots a_n[(g_1 \dots g_n)(f_n\tau)](f\sigma\tau)\end{aligned}$$

whence

$$(\Phi\Psi)J = \left(\sigma\tau, \begin{bmatrix} f_1\tau \\ \vdots \\ f_n\tau \end{bmatrix}\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right).$$

Finally, since the image under J of $A(\mathfrak{A})$ is the normal subgroup $\{1_{DF}\} \times G_n(\mathfrak{A})$ of W, we have that $W(\mathfrak{A})$ is a semidirect product of $A(\mathfrak{A})$ by $A(DF)$.

2.3. As an example, we shall apply Theorem 2.4 to the computation of the weak automorphism group of a finite dimensional vector space $\boldsymbol{V}$ over a field K. Thus $\boldsymbol{V} = (V, F)$ where F consists of the binary operation, $+$, of addition, and unary operations $\{s_a \mid a \in K\}$ of scalar multiplication, that is, $xs_a = ax$ for $a \in K$ and $x \in V$. We shall assume that $\boldsymbol{V}$ is non-trivial, that is, that V is not a one-element set.

We shall compute first $\boldsymbol{A}(\boldsymbol{D}F)$, the group of clone automorphisms of $\boldsymbol{D}F$. It follows from the vector space laws that if $g \in \boldsymbol{D}_n F$, then there exist $a_1, \ldots, a_n \in K$ such that

$$x_1 \ldots x_n g = \sum_{i=1}^{n} a_i x_i \tag{2.5}$$

for all $x_1, \ldots, x_n \in V$; moreover since $\boldsymbol{V}$ is non-trivial $a_1, \ldots, a_n$ are uniquely determined by g. In particular, all the derived unary operations s_a are distinct. Hence if $\sigma \in A(\boldsymbol{D}F)$, then because σ permutes the unary derived operations, σ induces a permutation φ_σ of K by

$$a\varphi_\sigma = b \quad \text{if and only if} \quad s_a\sigma = s_b.$$

Since σ preserves compositions and $(s_a)s_b = s_{ba}$, we have

$$(ab)\varphi_\sigma = (a\varphi_\sigma)(b\varphi_\sigma) \tag{2.6}$$

for all $a, b \in K$. Since $e_1^1\sigma = e_1^1$ we also have

$$1\varphi_\sigma = 1 \tag{2.7}$$

where 1 is the multiplicative identity of the field K. Since $(s_0)s_a = s_0$, $0\varphi = (0\varphi_\sigma)(a\varphi_\sigma)$ for all a; taking a so that $s_a\sigma = s_0$, we obtain

$$0\varphi_\sigma = 0. \tag{2.8}$$

Suppose $g \in \boldsymbol{D}_2 F$ is such that $+\sigma = g$. By (2.5) we may write

$$x_1 x_2 g = a_1 x_1 + a_2 x_2$$

for some $a_1, a_2 \in K$. But then,

$$s_1 = s_1\sigma = \big((s_0 s_1)+\big)\sigma = (s_0 s_1)g = s_{a_2}$$

so that $a_2 = 1$. Similarly, $a_1 = 1$ and we have $+\sigma = +$. And now,

$$s_{(a+b)}\sigma = \big((s_a s_b)+\big)\sigma = (s_a\sigma s_b\sigma)(+\sigma) = (s_a\sigma s_b\sigma)+$$

whence

$$(a+b)\varphi_\sigma = a\varphi_\sigma + b\varphi_\sigma \tag{2.9}$$

for all $a, b \in K$. By (2.6), (2.7), (2.8) and (2.9), φ_σ is an automorphism of the field K.

Since the operations in F are fundamental for $\boldsymbol{V}$, and since σ is a clone automorphism, we see that if the n-ary derived operation g is given by (2.5), then $g\sigma$ is given by

$$x_1 \ldots x_n(g\sigma) = \sum_{i=1}^{n} a_i \varphi_\sigma x_i. \tag{2.10}$$

Conversely, if φ is an automorphism of K, then defining σ_φ on the derived operations of $\boldsymbol{V}$ as in (2.10), we have σ_φ is a clone automorphism of $\boldsymbol{D}F$, with $\varphi_{\sigma_\varphi} = \varphi$ and $\sigma_{\varphi_\sigma} = \sigma$. Moreover, the mapping

$$\sigma \mapsto \varphi_\sigma$$

is easily seen to be an isomorphism of $\boldsymbol{A}(\boldsymbol{D}F)$ onto the group of automorphisms of the field K; we shall denote here the set of automorphisms of K by A_K.

Suppose now that V has finite dimension n. By (2.5) an n-ary matrix of operations

$$\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

can be uniquely represented by the n by n matrix $[a_{ij}]$ where

$$x_1 \ldots x_n g_i = \sum_{j=1}^{n} a_{ij} x_j$$

for $i = 1, \ldots, n$. In fact, it is easily seen that the mapping

$$\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \mapsto [a_{ij}]$$

s an isomorphism of $G_n(V)$ onto $GL_n(K)$. By Theorem 2.4 we conclude that $\boldsymbol{W}(\boldsymbol{V})$ is isomorphic to the group on $A_K \times GL_n(K)$ with multiplication given by

$$(\varphi, A)(\Psi, B) = \big(\varphi\Psi, (A\Psi)B\big)$$

where $A\Psi$ is the matrix obtained from A by applying Ψ to each entry. In particular we see that $\boldsymbol{W}(\boldsymbol{V})/\boldsymbol{A}(\boldsymbol{V})$ is isomorphic to the group of automorphisms of the field K.(1)

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(1) The author has learned that J. Dudek, in a forthcoming paper has independently shown that the quotient of $W(V)$ by $A(V)$ is isomorphic to the automorphism group of the field K.

SECTION 3

Representation of groups as weak automorphism groups of algebras

3.1. The example of Section 2.3 above shows that every finite cyclic group is isomorphic to $\boldsymbol{W}(\mathfrak{A})/\boldsymbol{A}(\mathfrak{A})$ for some finitary algebra $\mathfrak{A}$ for, as is well known, every finite cyclic group is isomorphic to the automorphism group of some field. We now take up the general problem of when a given group is isomorphic to $\boldsymbol{W}(\mathfrak{A})/\boldsymbol{A}(\mathfrak{A})$ for some $\mathfrak{A}$, and when it is isomorphic to $\boldsymbol{W}(\mathfrak{A})$ for some $\mathfrak{A}$.

LEMMA 3.1. *Let* $\boldsymbol{M} = (M, \{\cdot\})$ *be a monoid with identity element* 1. *Let* $L(\boldsymbol{M})$ *denote the set of left translations of* $\boldsymbol{M}$, *that is*, $L(\boldsymbol{M}) = \{l_s \mid s \in M\}$ *where* l_s *is the mapping of* M *into* M *given by* $xl_s = s\cdot x$. *If* $\mathfrak{A} = (M, L(\boldsymbol{M}))$, *then* $\boldsymbol{A}(\boldsymbol{D}L(\boldsymbol{M}))$ *is isomorphic to* $\boldsymbol{A}(\boldsymbol{M})$.

Proof. Let $F = L(\boldsymbol{M})$. Because $\boldsymbol{M}$ has an identity, F is in one-to-one correspondence with M by $s \leftrightarrow l_s$. Moreover, because $l_s l_t = l_{t\cdot s}$ and $l_1 = e_1^1$, we have $F = \boldsymbol{D}_1 F$. Consequently, if $\sigma \in A(\boldsymbol{D}F)$, σ determines a permutation π_σ of M by

$$s\pi_\sigma = t \quad \text{if and only if} \quad l_s\sigma = l_t.$$

Again using the identities $l_s l_t = l_{t\cdot s}$ and $l_1 = e_1^1$, we find that $(s\cdot t)\pi_\sigma = (s)\pi_\sigma\cdot(t)\pi_\sigma$ and $1\pi_\sigma = 1$ and therefore $\pi_\sigma \in A(\boldsymbol{M})$.

We have now a mapping K: $\sigma \to \pi_\sigma$ of $A(\boldsymbol{D}F)$ into $A(\boldsymbol{M})$ such that

$$l_s\sigma = l_{s\pi_\sigma}.$$

In fact it is clear that K is a group homomorphism of $\boldsymbol{A}(\boldsymbol{D}F)$ into $\boldsymbol{A}(\boldsymbol{M})$, and is injective because F is fundamental. We now show that K is surjective.

Since $F = \boldsymbol{D}_1 F$ is fundamental, every n-ary derived operation of $\mathfrak{A}$ is of the form $(e_i^n)l_s$, that is, if $g \in \boldsymbol{D}_n F$, then there exists on $s \in M$ and an integer i such that

$$x_1 \ldots x_n g = s\cdot x_i \quad . \tag{3.1}$$

for all $x_1, \ldots, x_n \in M$. Thus, given $\pi \in A(\boldsymbol{M})$, define a mapping σ_π on $\boldsymbol{D}F$ by

$$x_1 \ldots x_n(g\sigma_\pi) = (s\pi)\cdot x_i$$

where g is given by (3.1). Clearly then $\sigma_\pi \in A(\boldsymbol{D}F)$ and

$$(\sigma_\pi)K = \pi_{\sigma_\pi} = \pi.$$

Therefore, K is an isomorphism of $\boldsymbol{A}(\boldsymbol{D}F)$ onto $\boldsymbol{A}(\boldsymbol{M})$ completing the proof.

LEMMA 3.2. *Let* $\boldsymbol{M} = (M, \{\cdot\})$ *be a monoid with identity* 1. *Let* N *be the set of all elements of* M *having a two-sided inverse relative to* 1. *If* $\mathfrak{A} = (M, L(\boldsymbol{M}))$, *then* $\boldsymbol{A}(\mathfrak{A})$ *is isomorphic to* $\boldsymbol{N} = (N, \{\cdot\})$. *Moreover,* $\boldsymbol{W}(\mathfrak{A})$ *is isomorphic to the group on* $A(\boldsymbol{M}) \times N$ *with multiplication*

$$(\sigma, s)(\tau, t) = (\sigma\tau, (s)\tau\cdot t).$$

Proof. Since $\boldsymbol{M}$ has a unit 1, $\mathfrak{A}$ has the one element basis $\{1\}$. By Theorem 2.3, the automorphisms of $\mathfrak{A}$ are representable by the non-singular unary matrices of operations of $\mathfrak{A}$. As in the proof of Lemma 3.1 above, $\boldsymbol{D}_1 L(\boldsymbol{M}) = L(\boldsymbol{M})$. But $[l_s] \in G_1(\mathfrak{A})$ if and only if $s \in N$. Moreover, it is clear that $\boldsymbol{G}_1(\mathfrak{A})$ is isomorphic to $\boldsymbol{N}$ under the map $[l_s] \mapsto s$ since

$$[l_{s.t}] = [l_t l_s] = [l_s][l_t]$$

and $e_1^1 = l_1$.

We apply now Theorem 2.4 using s in place of $[l_s]$, together with Lemma 3.1 using $\pi_\sigma \in A(\boldsymbol{M})$ in place of $\sigma \in A(\boldsymbol{D}L(\boldsymbol{M}))$ to obtain the last assertion of the lemma.

THEOREM 3.3. *For any group* G *there is a unary algebra* $\mathfrak{A}$ *such that* $\boldsymbol{W}(\mathfrak{A})$ *is isomorphic to* G *and* $\mathfrak{A}$ *has no non-trivial automorphisms.*

Proof. Looking at Lemma 3.2 above, having been given G, we only need find a monoid $\boldsymbol{M}$ such that $A(\boldsymbol{M})$ is isomorphic to G and such that $\boldsymbol{M}$ has no invertible elements other than the identity.

But G. Birkhoff [2] has shown that given G there is a distributive lattice $\boldsymbol{L} = (L, \{\wedge, \vee\})$ with a 0 and a 1 such that $\boldsymbol{A}(\boldsymbol{L})$ is isomorphic to G. Now the automorphism group of a lattice is the same as the automorphism group of the semilattice obtained by discarding one of the fundamental binary operations $\wedge, \vee$ ([8], p. 41); that is, if $\boldsymbol{L}' = (L, \{\wedge\})$, then $A(\boldsymbol{L}') = A(\boldsymbol{L})$. Now the semilattice $\boldsymbol{L}'$ has an identity 1 and no other elements of $\boldsymbol{L}'$ have two-sided inverses relative to 1. For if $a \in L$ is invertible, then there is an element $b \in L$ such that $a \wedge b = 1$. Then

$$a = a \wedge 1 = a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b = 1.$$

Thus, $\boldsymbol{M} = \boldsymbol{L}'$ is the required monoid, completing the proof of the theorem.

3.2. Theorem 3.3 also shows that for each group G, there is an algebra $\mathfrak{A}$ such that $\boldsymbol{W}(\mathfrak{A})/\boldsymbol{A}(\mathfrak{A})$ is isomorphic to G, but in a rather unpleasing manner—since $\boldsymbol{A}(\mathfrak{A})$ is trivial. The next result shows that $\boldsymbol{A}(\mathfrak{A})$ can be made to be isomorphic to any group H, without changing the quotient $\boldsymbol{W}(\mathfrak{A})/\boldsymbol{A}(\mathfrak{A})$, by taking a slightly different algebra for $\mathfrak{A}$.

THEOREM 3.4. *Let* G *and* H *be any groups. Then there is an algebra* $\mathfrak{B}$ *such that* $\boldsymbol{W}(\mathfrak{B})$ *is isomorphic to* $G \times H$ *by an isomorphism which maps the subgroup* $\boldsymbol{A}(\mathfrak{B})$ *onto* $\{1\} \times H$.

Proof. Let G and H be given. Again employing Birkoff's result [2] let $\boldsymbol{L} = (L, \{\wedge, \vee\})$ be a distributive lattice with 0 and 1 such that $\boldsymbol{A}(\boldsymbol{L})$ is isomorphic to G, and as in the proof of Theorem 3.3 above, let $\boldsymbol{K} = (K, \{\vee\})$ be a semilattice with 0 and 1 such that H is isomorphic to $\boldsymbol{A}(\boldsymbol{K})$.

Now if G is the trivial group, then the algebra $\mathfrak{B} = \boldsymbol{K}$ shows the conclusion of the theorem to be true. Therefore, we may assume for the remainder of the proof that G is non-trivial. Take

$$\mathfrak{B} = (L \times K, \{\wedge_a \mid a \in L\} \cup \{\vee\})$$

where the operations $\wedge_a$ are the unary operations given by

$$(x, t) \wedge_a = (a \wedge x, t)$$

and $\vee$ is the binary operation defined by

$$(x, t) \vee (y, s) = (x \vee y, t \vee s).$$

First, the set of derived unary operations of $\mathfrak{B}$ is simply $\{\wedge_a \mid a \in L\}$. Next, the binary derived operations of $\mathfrak{B}$ are functions of one of the following forms:

$$(x, t)(y, s)f = \big((a \wedge x) \vee (b \wedge y), t \vee s\big)$$

or

$$(x, t)(y, s)g = (a \wedge x, t)$$

or

$$(x, t)(y, s)h = (b \wedge y, s)$$

for some $a, b \in L$.

Now let $\Phi \in W(\mathfrak{B})$. Noting that the unary derived operations of $\mathfrak{B}$ are in one-to-one correspondence, $a \leftrightarrow \wedge_a$, with the elements of L, Φ induces a permutation σ_Φ of L given by

$$a\sigma_\Phi = b \quad \text{if and only if} \quad \wedge_a \overline{\Phi} = \wedge_b$$

so that

$$\big((x, t) \wedge_a\big)\Phi = (x, t)\Phi \wedge_{a\sigma_\Phi}$$

for all $(x, t) \in L \times K$.

Because $\wedge_a \circ \wedge_b = \wedge_{a \wedge b}$, we have

$$(a \wedge b)\sigma_\Phi = a\sigma_\Phi \wedge b\sigma_\Phi \tag{3.2}$$

for all $a, b \in L$. Since $(x, t)\Phi = \big((x, t)\Phi\big) \wedge_{1\sigma_\Phi}$ for all $(x, t) \in L \times K$, and since Φ is surjective,

$$1\sigma_\Phi = 1. \tag{3.3}$$

Next,

$$(x, t)\Phi = \big((1, t) \wedge_x\big)\Phi = (1, t)\Phi \wedge_{x\sigma_\Phi} \tag{3.4}$$

Define mappings $\Phi_1\colon K \to L$ and $\Phi_2\colon K \to K$ by

$$(1, t)\Phi = (t\Phi_1, t\Phi_2).$$

By (3.4),

$$(x, t)\Phi = (x\sigma_\Phi \wedge t\Phi_1, t\Phi_2) \tag{3.5}$$

for all $(x, t) \in L \times K$. In particular,

$$(0, t)\Phi = (0\sigma_\Phi \wedge t\Phi_1, t\Phi_2).$$

But since $(0, t)\wedge_x = (0, t)$ for all $x \in L$,

$$(0, t)\Phi = (0, t)\Phi \wedge_{x\sigma_\Phi} = (x\sigma_\Phi \wedge 0\sigma_\Phi \wedge t\Phi_1, t\Phi_2)$$

so that

$$x\sigma_\Phi \wedge 0\sigma_\Phi \wedge t\Phi_1 = 0\sigma_\Phi \wedge t\Phi_1$$

for all $x \in L$ and $t \in K$. σ_Φ is surjective, so choosing x so that $x\sigma_\Phi = 0$, we have $0\sigma_\Phi \wedge t\Phi_1 = 0$ for all $t \in K$. Thus

$$(0, t)\Phi = (0, t\Phi_2). \tag{3.6}$$

Now Φ_2 is surjective because Φ is, and (3.6) implies that Φ_2 is injective; for if $s\Phi_2 = t\Phi_2$ then $(0, s)\Phi = (0, t)\Phi$ by (3.6), whence $(0, s) = (0, t)$ by the injectivity of Φ, showing $s = t$.

Therefore, by (3.5), for each $t \in K$, Φ maps $L \times \{t\}$ bijectively onto $L \times \{t\Phi_2\}$. Hence, for each $x \in K$, there is a $y \in L$ such that

$$(y\sigma_\Phi \wedge t\Phi_1, t\Phi_2) = (y, t)\Phi = (x, t\Phi_2).$$

Taking $x = 1$, we find

$$t\Phi_1 = 1 \tag{3.7}$$

for all $t \in K$. (3.5), (3.6) and (3.7) now show

$$0\sigma_\Phi = 0, \tag{3.8}$$

and

$$(x, t)\Phi = (x\sigma_\Phi, t\Phi_2) \tag{3.9}$$

for all $(x, t) \in L \times K$.

Let g be a derived binary operation such that $\vee\bar{\Phi} = g$. We will show $g = \vee$. First we show g must depend upon both variables. For if

$$(x, t)(y, s)g = (a \wedge x, t)$$

for some $a \in L$ and all $(x, t), (y, s) \in L \times S$, then

$$\big((x \vee y)\sigma_\Phi, (t \vee s)\Phi_2\big) = \big((x, t) \vee (y, s)\big)\Phi$$
$$= \big((x, t)\Phi, (y, s)\Phi\big)g = (a \wedge x\sigma_\Phi, t\Phi_2),$$

so that

$$(x \vee y)\sigma_\Phi = a \wedge x\sigma_\Phi$$

for all $x, y \in L$. Because G is non-trivial, L has an element $x_0 \neq 0, 1$. Taking $y = 1$ and $x = x_0$ we have by (3.3), $1 = 1\sigma_\Phi = a \wedge x_0\sigma_\Phi$ so that $x_0\sigma_\Phi = 1$. This contradiction shows g must be of the form

$$(x, t)(y, s)g = \big((a \wedge x) \vee (b \wedge y), t \vee s\big)$$

for some $a, b \in L$, that is,

$$(x, t)(y, s)g = \big((x, t) \wedge_a\big) \vee \big((y, s) \wedge_b\big).$$

Because $\vee\overline{\Phi} = g$ we have

$$[(x, t) \vee (y, s)]\Phi = \big((x, t)\Phi \wedge_a\big) \vee \big((y, s)\Phi \wedge_b\big)$$

form which we obtain

$$(x \vee y)\sigma_\Phi = (a \wedge x\sigma_\Phi) \vee (b \wedge y\sigma_\Phi)$$

and

$$(t \vee s)\Phi_2 = t\Phi_2 \vee s\Phi_2.$$

Thus, Φ_2 is an automorphism of K. Now taking $x = 0$, we find by (3.8)

$$y\sigma_\Phi = b \wedge y\sigma_\Phi$$

for all $y \in L$ whence $b = 1$. Similarly, taking $y = 0$, we find $a = 1$. Therefore

$$(x \vee y)\sigma_\Phi = x\sigma_\Phi \vee y\sigma_\Phi,$$

and this combined with (3.2) shows σ_Φ is an automorphism of L.

We established the existence of a mapping T: $W(\mathfrak{B}) \to A(L) \times A(K)$ such that if $\Phi T = (\sigma_\Phi, \Phi_2)$, then

$$(x, t)\Phi = (x\sigma_\Phi, t\Phi_2).$$

This immediately implies T is injective. It is easily seen that T is surjective also, for having been given $(\sigma, \tau) \in A(L) \times A(K)$, define Φ: $L \times K \to L \times K$ by

$$(x, t)\Phi = (x\sigma, t\tau).$$

Then Φ is bijective and

$$\vee\overline{\Phi} = \vee \tag{3.10}$$

and for each $a \in L$,

$$\wedge_a\overline{\Phi} = \wedge_{a\sigma}. \tag{3.11}$$

Because by (3.10) and (3.11) $\overline{\Phi}$ permutes the fundamental operations of $\mathfrak{B}$, it follows that Φ is a weak automorphism of $\mathfrak{B}$. Clearly, $\sigma_\Phi = \sigma$ and $\Phi_2 = \tau$ so that $\Phi T = (\sigma, \tau)$.

If $\Phi, \Psi \in W(\mathfrak{B})$, then

$$(x, t)\Phi\Psi = (x\sigma_\Phi, t\Phi_2)\Psi = (x\sigma_\Phi\sigma_\Psi, t\Phi_2\Psi_2)$$

so that taking coordinatewise composition in $A(L)\times A(K)$,

$$(\Phi\Psi)T = (\Phi T)(\Psi T).$$

Trivially, $(1_{L\times K})T = (1_L, 1_K)$, so that T is an isomorphism of $W(\mathfrak{B})$ onto $A(L)\times A(K)$, the direct product of the groups $A(L)$ and $A(K)$.

Now $A(\mathfrak{B})$ consists of all those $\Phi\in W(\mathfrak{B})$ such that $\overline{\Phi}$ fixes the fundamental operations of $\mathfrak{B}$. But

$$(x, t) \wedge_a \Phi = (x, t)\Phi \wedge_a$$

for all $(x, t)\in L\times K$ and all $a\in L$ implies

$$a\sigma_\Phi \wedge x\sigma_\Phi = (a \wedge x)\sigma_\Phi = a \wedge x\sigma_\Phi$$

for all $a, x \in L$. But then, $a\sigma_\Phi = a$ for all $a \in L$, that is, $\sigma_\Phi = 1_L$. Conversely, if $\sigma_\Phi = 1_L$ for some $\Phi\in W(\mathfrak{B})$, then $\Phi\in A(\mathfrak{B})$. Therefore, $A(\mathfrak{B})$ is isomorphic under T to $\{1\}\times A(K)$. Since $A(L)$ is isomorphic to G, and since $A(K)$ is isomorphic to H, the proof of the theorem is completed.

3.3. In all of the examples and constructions mentioned above, $W(\mathfrak{A})$ was always found to be a direct or semidirect product of $A(\mathfrak{A})$ and a subgroup of $W(\mathfrak{A})$. This is not true in general, however, as will be shown by an example in the following section.

Section 4

Saturated and pseudo-saturated algebras

4.1. Let G be a group acting on a set S, that is, each $x\in G$ determines a permutation $s \mapsto sx$ of S such that

$$(sx)y = s(x\cdot y)$$

and

$$s1 = s$$

for all $x, y\in G$ and all $s\in S$ where $\cdot$ is the group multiplication in G and 1 is the identity element of G. For any subset T of S, we define

$$N(T) = \{x \in G|\ Tx = T\}$$

and

$$C(T) = \{x\in G|\ tx = t \text{ for all } t\in T\}.$$

For any subset H of G, we define

$$M(H) = \{s\in S|\ sh = s \text{ for all } h\in H\}.$$

Finally, recall the definitions of the normalizer and centralizer of a subset H of G in G:

$$N_G(H) = \{x \in G \mid xHx^{-1} = H\}$$

and

$$C_G(H) = \{x \in G \mid xhx^{-1} = h \text{ for all } h \in H\}.$$

LEMMA 4.1. *For any subset T of S, $C(T)$ and $N(T)$ are subgroups of G, and $C(T)$ is a normal subgroup of $N(T)$.*

Proof. $C(T)$ and $N(T)$ are clearly subgroups of G. To see that $C(T) \triangleleft N(T)$, let $x \in C(T)$, $y \in N(T)$, and $t \in T$. Then

$$t(yxy^{-1}) = ((ty)x)y^{-1} = (ty)y^{-1} = t1 = t$$

so that $yxy^{-1} \in C(T)$.

LEMMA 4.2. *$N(T)$ is a subgroup of $N_G(C(T))$.*

Proof. The normalizer of $C(T)$ in G is the largest subgroup of G in which $C(T)$ is normal, so the lemma follows from Lemma 4.1.

LEMMA 4.3. *$CMC(T) = C(T)$ for every subset T of S.*

Proof. If $x \in C(T)$, then for every $r \in MC(T)$, $rx = r$ so that $x \in CMC(T)$. Therefore $C(T) \subset CMC(T)$. On the other hand, $T \subset MC(T)$, which implies $CMC(T) \subset C(T)$.

We note that the relation $R \subset G \times S$ given by

$$(x, s) \in R \quad \text{if and only if} \quad sx = s$$

establishes a Galois connection between G and S inducing the operators C and M, and Lemma 4.3 follows immediately from this fact.

LEMMA 4.4. *$N_G(C(T)) \subset N(MC(T))$ for every subset T of S.*

Proof. Let $x \in N_G(C(T))$; it must be shown that

$$[MC(T)]x = MC(T).$$

Let $t \in MC(T)$. Now if $y \in C(T)$, then $xyx^{-1} \in C(T)$ because $x \in N_G(C(T))$. But then, $t(xyx^{-1}) = t$, or

$$(tx)y = tx. \tag{4.1}$$

Since (4.1) holds for all $y \in C(T)$, we conclude that $tx \in MC(T)$, which shows $[MC(T)]x \subset MC(T)$.

Because $x^{-1} \in N_G(C(T))$ also, the argument above shows $[MC(T)]x^{-1} \subset MC(T)$ or $MC(T) \subset [MC(T)]x$, completing the proof of the lemma.

THEOREM 4.5. *For any subset T of S, $N_G(C(T)) = N(MC(T))$.*

Proof. Lemma 4.2 and 4.4 show for any subset T of S,

$$N(T) \subset N_G(C(T)) \subset N(MC(T)).$$

By Lemma 4.2,

$$N\big(MC(T)\big) \subset N_G\big(CMC(T)\big).$$

Using Lemma 4.3, we conclude from the above that

$$N_G\big(C(T)\big) \dot{\subset} N\big(MC(T)\big) \subset N_G\big(C(T)\big),$$

completing the proof.

4.2. We now apply the above results to algebras. Let A be a non-empty set. $\boldsymbol{O}_A$ will denote the set of all finitary operations on A. The group, $P(A)$, of all permutations on A acts on $\boldsymbol{O}_A$ as defined in Section 1.1, namely, $\Phi \in P(A)$ determines the bijection

$$f \mapsto f\overline{\Phi}$$

of $\boldsymbol{O}_A$, where, if f is say n-ary,

$$x_1 \dots x_n(f\overline{\Phi}) = (x_1\Phi^{-1} \dots x_n\Phi^{-1}f)\Phi \tag{4.2}$$

for all $x_1, \dots, x_n \in A$.

For an algebra $\mathfrak{A}$, the definitions of automorphisms and weak automorphisms can be expressed in terms of the operators C and N as

$$A(\mathfrak{A}) = C(F) = C(\boldsymbol{D}F) \tag{4.3}$$

and

$$W(\mathfrak{A}) = N(\boldsymbol{D}F). \tag{4.4}$$

We shall call an algebra $\mathfrak{A} = (A, F)$ *saturated* if for every $f \in \boldsymbol{O}_A - \boldsymbol{D}F$, $A(\mathfrak{A}) \neq A\big((A, F \cup \{f\})\big)$. Equivalently, $\mathfrak{A}$ is saturated if under the action defined above, $\boldsymbol{D}F = MC(F)$.

THEOREM 4.6. *If $\mathfrak{A} = (A, F)$ is saturated then*

$$W(\mathfrak{A}) = N_{P(A)}\big(A(\mathfrak{A})\big).$$

Proof. Using Theorem 4.5, identities (4.3) and (4.4), together with the hypothesis that $\mathfrak{A}$ is suturated, we have

$$W(\mathfrak{A}) = N(\boldsymbol{D}F) = N\big(MC(F)\big) = N_{P(A)}\big(C(F)\big) = N_{P(A)}\big(A(\mathfrak{A})\big).$$

If $\mathfrak{A} = (A, F)$, we shall call the algebra $M\mathfrak{A} = (A, MCF)$ the *saturation of* $\mathfrak{A}$. Since $\boldsymbol{D}MC(F) = MC(F) = M\big(CMC(F)\big)$, $M\mathfrak{A}$ is always saturated, and hence by the above theorem and Lemma 4.3,

$$W(M\mathfrak{A}) = N_{P(A)}\big(A(\mathfrak{A})\big).$$

We have, therefore,

COROLLARY 4.7. *If the subgroup H of $P(A)$ is the automorphism group of an algebra on A, then there is an algebra on A with automorphism group H and with weak automorphism group $N_{P(A)}(H)$.*

B. Jónsson [6] has characterized those subgroups H of $P(A)$ for which there is a finitary algebra $\mathfrak{A} = (A, F)$ such that $A(\mathfrak{A}) = H$. In particular, it follows from his results that if A is finite, then all subgroups of $P(A)$ are automorphism groups of finitary algebras on A, a result which had been previously obtained by Armbrust and Schmidt [1].

4.3. It is an easy matter now to provide an example of an algebra $\mathfrak{A}$ for which $\boldsymbol{W}(\mathfrak{A})$ is not a semidirect extension of $\boldsymbol{A}(\mathfrak{A})$. Let $A = \{1, 2, 3, 4\}$ and $F = \{f_1, f_2\}$ where

$$1f_1 = 1, \quad 2f_1 = 1, \quad 3f_1 = 4, \quad \text{and} \quad 4f_1 = 4$$

and

$$1f_2 = 2, \quad 2f_2 = 2, \quad 3f_2 = 3, \quad \text{and} \quad 4f_2 = 3.$$

Then letting $\mathfrak{A} = (A, F)$, it is easy to check that $A(\mathfrak{A}) = \{1_A, \alpha\}$ where 1_A is the identity mapping on A, and $\alpha = (14)(23)$. Letting $\mathfrak{B} = M\mathfrak{A}$, we have $A(\mathfrak{B}) = \{1_A, \alpha\}$ and $W(\mathfrak{B}) = N_{P(A)}(\{1_A, \alpha\})$. Now $\beta = (1243)$ $\epsilon W(\mathfrak{B})$, and since $\beta^2 = \alpha$, $\boldsymbol{W}(\mathfrak{B})$ cannot be a semidirect extension of $\boldsymbol{A}(\mathfrak{B})$ by a subgroup of $\boldsymbol{W}(\mathfrak{B})$.

4.4 As might be expected, an algebra need not be saturated in order that its group of weak automorphisms be the normalizer of its automorphism group. For example, let $\boldsymbol{G} = (G, \{\cdot\})$ be a group and consider the algebra $\mathfrak{A} = (G, \mathrm{F})$ where $F = \mathrm{L}(G)$ is the set of left translations of G. As Birkhoff has shown, $A(\mathfrak{A}) = R(G)$, the set of right translations of $\boldsymbol{G}$. By (4.4), $W(\mathfrak{A}) = N(\boldsymbol{D}F)$. Because F is fundamental for $\boldsymbol{D}F$, we have $N(F) \subset N(\boldsymbol{D}F)$. Since $F \subset P(G)$, we have $N(F) = N_{P(G)}(F)$ $= N_{P(G)}(L(\boldsymbol{G}))$. Now $L(\boldsymbol{G})$ is a subgroup of $P(G)$ satisfying

$$C_{P(G)}(L(\boldsymbol{G})) = R(\boldsymbol{G})$$

and

$$C_{P(G)}(R(\boldsymbol{G})) = L(\boldsymbol{G}).$$

Therefore, $N_{P(G)}(L(\boldsymbol{G})) = N_{P(G)}(R(\boldsymbol{G}))$. Thus, $W(\mathfrak{A}) \supset N_{P(G)}(A(\mathfrak{A}))$, and since the reverse inclusion always holds, we have $W(\mathfrak{A}) = N_{P(G)}(A(\mathfrak{A}))$.

Now G. H. Wenzel [10] has shown that the binary operations ψ_σ given by

$$x_1 x_2 \psi_\sigma = \left((x_2 \cdot x_1^{-1})\sigma\right) \cdot x_1$$

where σ is any mapping of G into G can be added to F without changing the automorphism group of $\mathfrak{A}$. All of these operations cannot be derived from F because no essentially binary operation can be derived from unary operations. Therefore, $\mathfrak{A}$ is not saturated.

We shall call an algebra $\mathfrak{A} = (A, F)$ *pseudo-saturated* if $W(\mathfrak{A})$ $= N_{P(A)}(A(\mathfrak{A}))$.

The proof used above to show $\mathfrak{A} = (G, L(G))$ is pseudo-saturated can be slightly generalized:

PROPOSITION 4.8. *Let $\mathfrak{A} = (A, F)$ be a unary algebra with F a subgroup of $P(A)$. If $C_{P(A)}(C_{P(A)}(F)) = F$ then $\mathfrak{A}$ is pseudo-saturated.*

Another simple criterion for pseudo-saturation, which is also applicable to the above example is the following.

PROPOSITION 4.9. *Let $\mathfrak{A} = (A, F)$ be an algebra such that $\boldsymbol{D}_n F$ is a fundamental set of operations for $\mathfrak{A}$. If no new operations of arity n can be added to F without changing $A(\mathfrak{A})$, then $\mathfrak{A}$ is pseudo-saturated.*

Proof. Let $G = MC(F)$. By hypothesis, $\boldsymbol{D}_n G = \boldsymbol{D}_n F$. If $\tau \in W(M\mathfrak{A})$, then $(\boldsymbol{D}_n G)\overline{\tau} = \boldsymbol{D}_n G$, that is $(\boldsymbol{D}_n F)\overline{\tau} = \boldsymbol{D}_n F$. Since $\boldsymbol{D}(\boldsymbol{D}_n F) = \boldsymbol{D}F$, we conclude $(\boldsymbol{D}F)\overline{\tau} = \boldsymbol{D}F$, that is, $\tau \in W(\mathfrak{A})$. Therefore, $W(M\mathfrak{A}) \subset W(\mathfrak{A})$. Since $W(\mathfrak{A}) \subset W(M\mathfrak{A})$ always, we find

$$W(\mathfrak{A}) = W(M\mathfrak{A}) = N_{P(A)}\big(A(M\mathfrak{A})\big) = N_{P(A)}\big(A(\mathfrak{A})\big).$$

Proposition 4.9 shows how to construct, in certain cases, pseudo-saturated algebras without necessarily completely saturating the algebra.

COROLLARY 4.10. *If the algebra $\mathfrak{A} = (A, F)$ has $\boldsymbol{D}_n F$ as a fundamental set of operations, and if $G = \{g \in MC(F) \mid g$ is n-ary$\}$, then the algebra $\mathfrak{B} = (A, G)$ is pseudo-saturated.*

In particular, Corollary 4.10 applies whenever $\mathfrak{A}$ has a finite set of fundamental operations.

The following example shows the converse of Proposition 4.9 is false, namely, an algebra $\mathfrak{A} = (A, F)$ can have a fundamental set of operations of arities n and be pseudo-saturated without satisfying $\boldsymbol{D}_n F = \{g \in MC(F) \mid g$ is n-ary$\}$. We consider again the algebra $\mathfrak{A} = (A, F)$ where $A = \{1, 2, 3, 4\}$ and $F = \{f_1, f_2\}$ given in section 4.3 above. We find that $G = \{g \in MCF \mid g$ is unary$\}$ consists of the sixteen operations given in the table:

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
1	1	2	1	2	4	3	4	3	1	1	2	2	3	3	4	4
2	1	2	4	3	1	2	4	3	2	3	1	4	1	4	2	3
3	4	3	1	2	4	3	1	2	3	2	4	1	4	1	3	2
4	4	3	4	3	1	2	1	2	4	4	3	3	2	2	1	1

According to Corollary 4.10, $\mathfrak{B} = (A, G)$ is pseudosaturated, so that $W(\mathfrak{B}) = N_{S_4}\big(\{1, (14)(23)\}\big)$. In fact we find $W(\mathfrak{B}) = \{f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}\, f_{16}\}$.

Let now $G_0 = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9\}$. G_0 is closed under composition and $F \subset G_0 \subset G$. Letting $\mathfrak{C} = (A, G_0)$, we have $A(\mathfrak{C}) = A(\mathfrak{B})$ and, clearly, $W(\mathfrak{C}) = W(\mathfrak{B})$. Thus $\mathfrak{C}$ is pseudo-saturated but $G_0 \neq G$.

SECTION 5

Weak automorphisms of unary algebras on groups

5.1. G. H. Wenzel [10] has determined and studied the automorphism groups of unary algebras constructed on groups, that is, of algebras $\mathfrak{A} = (G, \boldsymbol{F})$ where $\boldsymbol{G} = (G, \{\cdot\})$ is a group and $\boldsymbol{F} \subset L(\boldsymbol{G})$. In this section we compute the weak automorphism groups of algebras of this type, and determine when such an algebra is pseudo-saturated.

First we establish some notation. If $F \subset G$, $\bar{F}$ will denote the set of elements of the submonoid of $\boldsymbol{G}$ generated by F, that is, the set of all finite products of elements of F together with 1; and F_* will denote the set of elements of the subgroup of $\boldsymbol{G}$ generated by F, that is, the set of all finite products of elements of F or inverses of elements of F. We will slightly abuse the notation established earlier in the interest of brevity by using $A(F_*)$ to denote the set of all automorphisms of the subgroup $(F_*, \{\cdot\})$ of $\boldsymbol{G}$. We also introduce the notation $A_{\bar{F}}(F_*) = \{\sigma \in A(F_*) \mid \bar{F}\sigma = \bar{F}\}$.

We begin with the lemma:

LEMMA 5.1. *Let* $\boldsymbol{G} = (G, \{\cdot\})$ *be a group and* $F \subset G$. *Let* π *be an endomorphism of* $(\bar{F}, \{\cdot\})$ *and suppose* Φ *is a mapping of* G *into* G *such that*

$$(5.1)\qquad (b\cdot z)\Phi = (b)\pi\cdot(z)\Phi$$

for all $b \in \bar{F}$ *and all* $z \in G$. *Then* π *has a unique extension to an endomorphism* $\hat{\pi}$ *of* $(F_*, \{\cdot\})$ *and* $\hat{\pi}$ *satisfies*

$$(5.2)\qquad (c\cdot z)\Phi = (c)\hat{\pi}\cdot(z)\Phi$$

for all $c \in F_*$ *and all* $z \in G$. *Moreover, if* Φ *is injective,* $\hat{\pi}$ *is injective, and if* π *is surjective, so also is* $\hat{\pi}$.

Proof. If π is extendable to F_*, then it must be that for any word

$$b_1^{\varepsilon_1} \ldots b_n^{\varepsilon_n}$$

with $b_1, \ldots, b_n \in F$ and $\varepsilon_i = \pm 1$ for $i = 1, \ldots, n$, we have

$$(5.3)\qquad (b_1^{\varepsilon_1} \ldots b_n^{\varepsilon_n})\hat{\pi} = (b_1\pi)^{\varepsilon_1} \ldots (b_n\pi)^{\varepsilon_n}.$$

Thus $\hat{\pi}$ must be defined by (5.3), and conversely, if (5.3) does define a mapping $\hat{\pi}$, then because $1\pi = 1$, $\hat{\pi}$ will be a homomorphism of $(F_*, \{\cdot\})$ into itself. Thus we need show that (5.3) does in fact define a mapping, and that (5.2) is satisfied.

By (5.1), for all $b \in F$ and all $z \in G$, we have

$$(z)\Phi = (b\cdot b^{-1}\cdot z)\Phi = b\pi\cdot(b^{-1}\cdot z)\Phi,$$

so that

$$(b^{-1}\cdot z)\Phi = (b\pi)^{-1}\cdot(z)\Phi.$$

Therefore, combining this with (5.1), we obtain

$$(5.4)\qquad (b_1^{\varepsilon_1}\cdot z)\Phi = (b_1\pi)^{\varepsilon_1}\cdot(z)\Phi$$

for all $b_1 \in F$, all $z \in G$ and $\varepsilon_1 = \pm 1$.

Applying (5.4) repeatedly yields

$$(5.5)\qquad (b_1^{\varepsilon_1}\ldots b_n^{\varepsilon_n}\cdot z)\Phi = (b_1\pi)^{\varepsilon_1}\ldots(b_n\pi)^{\varepsilon_n}\cdot(z)\Phi$$

for all $b_1,\ldots,b_n \in F$, all $z \in G$ and $\varepsilon_i = \pm 1$ for $i = 1,\ldots,n$.

Now if $b_1^{\varepsilon_1}\ldots b_n^{\varepsilon_n} = c_1^{\delta_1}\ldots c_m^{\delta_m}$ where $b_1,\ldots,b_n, c_1,\ldots,c_m \in F$ and $\varepsilon_i, \delta_j = \pm 1$, then for any $z \in G$,

$$b_1\pi^{\varepsilon_1}\ldots b_n\pi^{\varepsilon_n}\cdot(z\Phi) = (b_1^{\varepsilon_1}\ldots b_n^{\varepsilon_n}z)\Phi$$
$$= (c_1^{\delta_1}\ldots c_m^{\delta_m}z)\Phi = c_1\pi^{\delta_1}\ldots c_m\pi^{\delta_m}\cdot(z\Phi),$$

whence $b_1\pi^{\varepsilon_1}\ldots b_n\pi^{\varepsilon_n} = c_1\pi^{\delta_1}\ldots c_m\pi^{\delta_m}$. Therefore, (5.3) defines a mapping $\hat{\pi}$ which is an endomorphism of $(F_*, \{\cdot\})$, and satisfies (5.2).

If now Φ is injective, and if $b, c \in F_*$ and $b\hat{\pi} = c\hat{\pi}$, then by (5.2)

$$(bz)\Phi = b\hat{\pi}\cdot z\Phi = c\hat{\pi}\cdot z\Phi = (cz)\Phi$$

whence $bz = cz$ so that $b = c$. Therefore $\hat{\pi}$, and π, are injective.

Finally, it is clear that if π is surjective then so is $\hat{\pi}$. This completes the proof of the lemma.

THEOREM 5.2. *Let $\boldsymbol{G} = (G, \{\cdot\})$ be a group and $\boldsymbol{F} \subset L(\boldsymbol{G})$. Let*

$$\overline{F} = \{b \in G \mid l_b \in \boldsymbol{F}\}.$$

Decompose G into a disjoint union of left cosets of F_* in G:

$$G = \bigcup_{\gamma\in Y} F_* x_\gamma.$$

If $\mathfrak{A} = (G, \boldsymbol{F})$, then $W(\mathfrak{A})$ is isomorphic to the group on

$$A_{\overline{F}}(F_*)\times F_*^Y \times P(Y)$$

with multiplication

$$(\pi_1, \tau_1, \sigma_1)(\pi_2, \tau_2, \sigma_2) = \big(\pi_1\circ\pi_2, (\tau_1\circ\pi_2)\cdot(\sigma_1\circ\tau_2), \sigma_1\circ\sigma_2\big).$$

Proof. Because $e_1^1 = l_1$ and $l_a \circ l_b = l_{b\cdot a}$, the derived unary operations of $\mathfrak{A}$ are all those left translations l_b for which $b \in \overline{F}$.

Let $\Phi \in W(\mathfrak{A})$. Then $\overline{\Phi}$ induces a bijection π_Φ of $\overline{F}$ such that

$$b\pi_\Phi = c \quad\text{if and only if}\quad (b\cdot z)\Phi = c\cdot(z\Phi)$$

for all $z \in G$ (that is $b\pi_\Phi = c$ if and only if $l_b\overline{\Phi} = l_c$). Moreover, $1\pi_\Phi = 1$

and a simple calculation shows $(b\cdot c)\pi_\Phi = (b\pi_\Phi)\cdot(c\pi_\Phi)$. Thus π_Φ is an automorphism of the monoid $(\bar{F}, \{\cdot\})$, and we have

$$(b\cdot z)\Phi = (b\pi_\Phi)\cdot(z\Phi)$$

for all $z\in G$ and all $b\in\bar{F}$.

By Lemma 5.1, π_Φ has a unique extension $\hat{\pi}_\Phi$ to an automorphism of $(F_*, \{\cdot\})$, and

$$(f\cdot z)\Phi = (f\hat{\pi}_\Phi)\cdot(z\Phi)$$

for all $f\in F_*$ and all $z\in G$. We note that $\hat{\pi}_\Phi\in A_{\bar{F}}(F_*)$. In particular, for all $\gamma\in Y$ and all $f\in F_*$,

$$(f\cdot x_\gamma)\Phi = (f\hat{\pi}_\Phi)\cdot(x_\gamma\Phi).$$

It follows that

$$(F_* x_\gamma)\Phi = F_*(x_\gamma\Phi)$$

for all $\gamma\in Y$, and since Φ is bijective we may express G as the disjoint union

$$G = \bigcup_{\gamma\in Y} F_*(x_\gamma\Phi).$$

Therefore, Φ determines a bijection σ_Φ of Y by

$$\gamma\sigma_\Phi = \delta \quad \text{if and only if} \quad F_*(x_\gamma)\Phi = F_* x_\delta.$$

In other words,

$$\gamma\sigma_\Phi = \delta \quad \text{if and only if} \quad x_\gamma\Phi\in F_* x_\delta,$$

and therefore Φ also determines a mapping $\tau_\Phi\colon Y\to F_*$ by

$$\gamma\tau_\Phi = f \quad \text{if and only if} \quad x_\gamma\Phi = f x_\delta.$$

In view if this, we may write

$$(5.6)\qquad (f\cdot x_\gamma)\Phi = (f\hat{\pi}_\Phi)\cdot(\gamma\tau_\Phi)\cdot x_{\gamma\sigma_\Phi}$$

for all $f\in F_*$ and all $\gamma\in Y$.

We now show that the mapping

$$\Delta\colon \Phi\to(\hat{\pi}_\Phi, \tau_\Phi, \sigma_\Phi)$$

is a bijection between $W(\mathfrak{A})$ and $A_{\bar{F}}(F_*)\times F_*^Y\times P(Y)$. Because of (5.6), Δ is injective. Given (π, τ, σ) in $A_{\bar{F}}(F_*)\times F_*^Y\times P(Y)$, define Φ on G by

$$(f x_\gamma)\Phi = (f\pi)\cdot(\gamma\tau)\cdot x_{\gamma\sigma}.$$

Then it is easily verified that $\Phi\in W(\mathfrak{A})$ and $\Phi\Delta = (\pi, \tau, \sigma)$. Hence Δ is bijective.

Accordingly, we can transfer the group structure of $W(\mathfrak{A})$ to the image of Δ in such a way that Δ is an isomorphism. If

$$\Phi\Delta = (\pi_1, \tau_1, \sigma_1)$$

and

$$\Psi\Delta = (\pi_2, \tau_2, \sigma_2),$$

then using (5.6)

$$\begin{aligned}(f\cdot x_\gamma)(\Phi\circ\Psi) &= \big((f\pi_1)\cdot(\gamma\tau_1)\cdot x_{\gamma\sigma_1}\big)\Psi \\ &= \big((f\pi_1)\cdot(\gamma\tau_1)\big)\pi_2\cdot(\gamma\sigma_1)\tau_2\cdot x_{(\gamma\sigma_1)\sigma_2} \\ &= (f\pi_1\circ\pi_2)\cdot\big(\gamma[(\tau_1\circ\pi_2)\cdot(\sigma_1\circ\tau_2)]\big)\cdot x_{\gamma\sigma_1\circ\sigma_2}.\end{aligned}$$

Therefore,

$$(\pi_1, \tau_1, \sigma_1)(\pi_2, \tau_2, \sigma_2) = (\pi_1\circ\pi_2, (\tau_1\circ\pi_2)\cdot(\sigma_1\circ\tau_2), \sigma_1\circ\sigma_2),$$

completing the proof.

The proofs of Lemma 5.1 and Theorem 5.2 use extensions of Wenzel's methods. Since $A(\mathfrak{A})$ will correspond to those of triples (π, τ, σ) for which π is the identity it is no surprise that we obtain Wenzel's result as a corollary:

COROLLARY 5.3. $A(\mathfrak{A})$ is *isomorphic to the wreath product*

$$(F_*, \{\cdot\}) \sim P(Y).$$

We also have

COROLLARY 5.4. $W(\mathfrak{A})/A(\mathfrak{A})$ *is isomorphic to* $\big(A_{\overline{F}}(F_*), \{\cdot\}\big)$.

THEOREM 5.5. *The algebra* $\mathfrak{A} = (G, F)$ *where* $G = (G, \{\cdot\})$ *is a group and* $F \subset L(G)$ *is pseudo-saturated if and only if*

$$A_{\overline{F}}(F_*) = A(F_*).$$

Proof. Let $F' = \{l_c \in L(G) \mid c \in F_*\}$, $\mathfrak{A}' = (G, F')$; and let

$$G = \bigcup_{\gamma\in Y} F_* x_\gamma \tag{5.7}$$

be a fixed decomposition of G into disjoint left cosets of F_* in G. By the proof of Theorem 5.2, $A(\mathfrak{A}) = A(\mathfrak{A}')$.

Now if $A_{\overline{F}}(F_*) \neq A(F_*)$, then there is a $\pi \in A(F_*) - A_{\overline{F}}(F_*)$ and by the proof of theorem above, the mapping $\Phi\colon G \to G$ defined relative to (5.7) by

$$(f\cdot x_\gamma)\Phi = f\pi\cdot x_\gamma$$

is a weak automorphism of $\mathfrak{A}'$ but not of $\mathfrak{A}$. Therefore, $W(\mathfrak{A}) \neq W(\mathfrak{A}')$ so that $\mathfrak{A}$ cannot be pseudo-saturated.

Conversely, suppose $A_{\overline{F}}(F_*) = A(F_*)$. Then $W(\mathfrak{A}') = W(\mathfrak{A})$ by the proof of Theorem 5.2. Thus we only need show $\mathfrak{A}'$ is pseudo-saturated.

Let F'' be a set of unary operations on G such that $F' \subset F''$ and $A(\mathfrak{A}') = A(\mathfrak{A}'')$ where $\mathfrak{A}'' = (G, F'')$. Because $R(G) \subset A(\mathfrak{A}) = A(\mathfrak{A}'')$, we have $F'' \subset L(G)$ (Wenzel). Let $F'' = \{b \in G \mid l_b \in F''\}_*$. Since $F' \subset F''$, we have F_* as a subgroup of F''. Suppose F_* is a proper subgroup of F'', and let $X = \{\delta \in Y \mid F'' \cup F_* x_\sigma$ is non-empty$\}$. Then

$$F'' = \bigcup_{\delta \in X} F_* x_\delta$$

with $|X| \geqslant 2$, where $|X|$ is the cardinality of X.

Pick any $f_0 \in F_*$ different than 1 (if F_* is trivial, then $\mathfrak{A}'$ is clearly pseudo-saturated) and pick any $\delta_0 \in X$. According to the proof of Theorem 5.2, there is an automorphism φ of $\mathfrak{A}'$ such that

$$(5.8) \qquad (fx_{\delta_0})\varphi = f\cdot f_0 \cdot x_{\delta_0} \quad \text{for} \quad f \in F_*$$

and

$$(5.9) \qquad (fx_\delta)\varphi = fx_\delta \quad \text{for} \quad f \in F_* \text{ and } \delta \neq \delta_0.$$

Since φ is also an automorphism of $\mathfrak{A}''$ and maps F'' into F'', by the proof of Theorem 5.2, on the coset F'', φ must be of the form

$$f''\varphi = f''\cdot f_1$$

for all $f'' \in F''$ and some $f_1 \in F''$. But (5.9) implies $f_1 = 1$ which is inconsistent with (5.8) since $f_0 \neq 1$. This contradiction implies $|X| = 1$ so that $F' = F''$. By Proposition 4.9, $\mathfrak{A}'$ is pseudo-saturated, as was to be shown.

COROLLARY 5.6. *If G is abelian, then $\mathfrak{A}$ is pseudo-saturated if and only if $\overline{F} = F_*$.*

COROLLARY 5.7. *If F_* is finite, then $\mathfrak{A}$ is pseudo-saturated.*

SECTION 6

Restricted weak automorphisms

6.1. Let $\mathfrak{A} = (A, F)$ be an algebra. A bijective mapping $h\colon A \to A$ is called a *restricted weak automorphism* of $\mathfrak{A}$ if the mapping $\overline{h}\colon f \to f\overline{h}$ is a bijection from F into F, where as before, if f is an n-ary operation on A,

$$x_1 \ldots x_n (f\overline{h}) = (x_1 h^{-1} \ldots x_n h^{-1} f) h$$

for all $x_1, \ldots, x_n \in A$.

Clearly, the set of restricted weak automorphisms of $\mathfrak{A}$ forms a group under composition; the notation for this group will be $(\boldsymbol{W}|F)(\mathfrak{A}) = \big((W|F)(\mathfrak{A}), \{\circ\}\big)$. Every automorphism of $\mathfrak{A}$ is a restricted weak automorphism of $\mathfrak{A}$ so that $\boldsymbol{A}(\mathfrak{A})$ is a subgroup of $(\boldsymbol{W}|F)(\mathfrak{A})$. Also, if $h \in (W|F)(\mathfrak{A})$, then because $\overline{h}$ preserves compositions, $h \in W(\mathfrak{A})$ so that $(\boldsymbol{W}|F)(\mathfrak{A})$ is a subgroup of $\boldsymbol{W}(\mathfrak{A})$. By Corollary 1.4, $\boldsymbol{A}(\mathfrak{A})$ is a normal subgroup of

$(\boldsymbol{W}|F)(\mathfrak{A})$. Indeed, if we define $(A|F)(\boldsymbol{D}F) = \{\sigma \in A(\boldsymbol{D}F) \mid F\sigma = F\}$ and put $(\boldsymbol{A}|F)(\boldsymbol{D}F) = ((A|F)(\boldsymbol{D}F), \{\circ\})$, we can restate the results of Sections 1.3 and 2.2 in the following form:

THEOREM 6.1. *The mapping* $h \to \bar{h}$ *is a homomorphism of* $(\boldsymbol{W}|F)(\mathfrak{A})$ *into* $(\boldsymbol{A}|F)(\boldsymbol{D}F)$ *with kernel* $\boldsymbol{A}(\mathfrak{A})$.

COROLLARY 6.2. $\boldsymbol{A}(\mathfrak{A})$ *is a normal subgroup of* $(\boldsymbol{W}|F)(\mathfrak{A})$.

COROLLARY 6.3. $(\boldsymbol{W}|F)(\mathfrak{A})/\boldsymbol{A}(\mathfrak{A})$ *is isomorphic to a subgroup of* $(\boldsymbol{A}|F)(\boldsymbol{D}F)$.

THEOREM 6.4. *If the algebra* $\mathfrak{A} = (A, F)$ *has a finite basis of* n *elements then* $(\boldsymbol{W}|F)(\mathfrak{A})$ *is a semidirect product of* $\boldsymbol{A}(\mathfrak{A})$ *by* $(\boldsymbol{A}|F)(\boldsymbol{D}F)$. *That is,* $(\boldsymbol{W}|F)(\mathfrak{A})$ *is isomorphic to the group on* $(A|F)(\boldsymbol{D}F) \times G_n(\mathfrak{A})$ *with multiplication*

$$\left(\sigma, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}\right)\left(\tau, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right) = \left(\sigma\tau, \begin{bmatrix} f_1\tau \\ \vdots \\ f_n\tau \end{bmatrix}\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right).$$

6.2. Our main reason for introducing the notion of restricted weak automorphism is to compare the concept of weak isomorphism as defined in [4] with the conventional notion of isomorphism as defined in [3], page 49 and [5], page 34. The comparison is stated as Theorem 6.5 below.

Let $\mathfrak{A} = (A, F)$ and $\mathfrak{B} = (B, G)$ be two algebras. Common to the usual way of defining an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ is the establishing of a bijective correspondence ϱ from F to G which preserves arities. This is usually accomplished by naming the operations of F and G with a common set of names and then requiring that an operation $f \in F$ correspond to the operation $g \in G$ of the same name. Having done this, an isomorphism is then defined as a bijection $\varphi\colon A \to B$ such that for every $f \in F$,

$$(x_1 \ldots x_n f)\varphi = x_1\varphi \ldots x_n\varphi(f\varrho) \tag{6.1}$$

for all $x_1, \ldots, x_n \in A$, where n is the arity of f.

However, it is easily observed that a mapping φ satisfying (6.1) uniquely determines the naming ϱ. Thus we define a bijection $\varphi\colon A \to B$ to be a *restricted weak isomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ if the induced mapping, $\varrho_\varphi\colon f \to f\varrho_\varphi$, where

$$x_1 \ldots x_n(f\varrho_\varphi) = (x_1\varphi^{-1} \ldots x_n\varphi^{-1} f)\varphi,$$

is a bijection from F onto G. Although the notion of restricted weak isomorphism is more "restrictive" in particular situations than Goetz's notion, it is no less general, for Goetz's definition results by replacing F with $\boldsymbol{D}F$ and G with $\boldsymbol{D}G$. In making a close comparison, we must work only with the given fundamental operations of the algebras in question as this is the case in usual isomorphism.

For the given algebras $\mathfrak{A}, \mathfrak{B}$, let I_ϱ denote the set of restricted weak isomorphisms φ from $\mathfrak{A}$ to $\mathfrak{B}$ such that $\varrho_\varphi = \varrho$ (that is, the set of usual isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ with respect to the naming ϱ). We have:

THEOREM 6.5. *If I_ϱ is non-empty, then* $|I_\varrho| = |A(\mathfrak{A})|$, and

$$|\{\varrho \mid I_\varrho \neq \emptyset\}||A(\mathfrak{A})| = |(W|F)(\mathfrak{A})|$$

Proof. The first assertion is clear since if $\varphi_0 \in I_\varrho$, then for any $\varphi \in I_\varrho$, $\varphi_0\varphi^{-1} \in A(\mathfrak{A})$ and the mapping

$$\varphi \to \varphi_0\varphi^{-1}$$

is a bijection from I_ϱ onto $A(\mathfrak{A})$.

The second assertion follows from this and the fact that the set I of all restricted weak isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is the disjoint union of the sets $I_\varrho \neq \emptyset$ and $|I| = |(W|F)(\mathfrak{A})|$.

6.3. For groups of restricted weak automorphisms of finite algebras, we can easily prove a stronger representation theorem then those of Section 3. Namely:

THEOREM 6.6. *Let G be a subgroup of $P(A)$ where A is finite, and let H be a normal subgroup of G. Then there is an algebra $\mathfrak{A} = (A, F)$ such that $H = A(\mathfrak{A})$ and $G = (W|F)(\mathfrak{A})$.*

Proof. Armbrust and Schmidt [1] have proven the first assertion in the conclusion of the theorem, and the proof of the second assertion depends heavily on their construction. Namely, they have shown that there is a single $|A|$-ary operation f on A such that $A\big((A, \{f\})\big) = H$. Let us set $\mathfrak{A}' = (A, \{f\})$. Because G is assumed to be a subgroup of $N_{P(A)}(H)$, we find by the results of Section 4.2 that for any $g \in G$, $f\bar{g}$ is an operation in the saturation $M\mathfrak{A}'$ of $\mathfrak{A}'$. Thus taking $F = \{f\bar{g} \mid g \in G\}$ and setting $\mathfrak{A} = (A, F)$, we have $A(\mathfrak{A}) = H$ because $f \in F$, and G is a subgroup of $(W|F)(\mathfrak{A})$. Now if φ is a restricted weak automorphism of $\mathfrak{A}$, then $F\bar{\varphi} = F$ so in particular, $f\bar{\varphi} = f\bar{g}$ for some $g \in G$. Then $f = f\overline{(g\varphi^{-1})}$ so that $g\varphi^{-1} \in H \subset G$, whence $\varphi \in G$. Therefore, $(W|F)(\mathfrak{A}) = G$, completing the proof.

6.4. Finally, we note that the notion of restricted weak isomorphism can be defined for relational systems as well. A *relational system* is a pair $\mathfrak{R} = (A, R)$ where A is a non-empty set and R is a set of finitary relations on A, that is each $r \in R$ is a subset of A^n, $r \subset A^n$, for some positive integer n called the arity of r. A *restricted weak isomorphism* from the relational system (A, R) to the relational system (B, S) is a bijective mapping $\varphi\colon A \to B$ such that the mapping $\bar{\varphi}\colon r \to r\bar{\varphi}$ given by

$$r\bar{\varphi} = \{(x_1\varphi, \ldots, x_n\varphi) \mid (x_1, \ldots, x_n) \in r\}$$

is a bijection from R onto S.

Every algebra (A, F) is a relational system if we represent an n-ary operation $f \in F$ as the $n+1$-ary relation

$$\{(x_1, \ldots, x_n, x_1 \ldots x_n f) \mid x_1, \ldots, x_n \in A\}$$

and the definition of restricted weak isomorphism applied to this relational system agrees with the former definition for algebras. We therefore use the same notation as above for automorphisms: for the relational system $\mathfrak{R} = (A, R)$, $(\boldsymbol{W}|R)(\mathfrak{R}) = ((\boldsymbol{W}|R)(\mathfrak{R}), \{\circ\})$ denotes the group of restricted weak isomorphisms of $\mathfrak{R}$ onto $\mathfrak{R}$. As before $A(\mathfrak{R})$ denotes the set of automorphisms of $\mathfrak{R}$ that is, the set of those restricted weak automorphisms φ that $\bar{\varphi} = 1_R$. We put $\boldsymbol{A}(\mathfrak{R}) = (A(\mathfrak{R}), \{\circ\})$.

Here again $\boldsymbol{A}(\mathfrak{R})$ is a normal subgroup of $(\boldsymbol{W}|R)(\mathfrak{R})$ but, in general, one can only say that the factor group is isomorphic to a subgroup of the arity-preserving bijections of R. We also note that Theorem 6.5 above is true for relational systems.

However, in this context we can prove easily a representation theorem for the pair $\boldsymbol{A}(\mathfrak{R})$, $(\boldsymbol{W}|R)(\mathfrak{R})$.

THEOREM 6.7. *Let $\boldsymbol{G} = (G, \{\cdot\})$ be a group with normal subgroup $\boldsymbol{H} = (H, \{\cdot\})$. Then there is a relational system $\mathfrak{R} = (X, R)$ such that $(\boldsymbol{W}|R)(\mathfrak{R})$ is isomorphic to $\boldsymbol{G}$ under an isomorphism mapping $A(\mathfrak{R})$ onto H.*

Proof. Birkhoff has shown in [2] that given $\boldsymbol{G}$, there is a partial ordering P of $X = G \cup G^2$ such that $\boldsymbol{G}$ is isomorphic to $\boldsymbol{A}((X, P))$ under the isomorphism $\beta\colon G \to A((X, P))$ given by $g\beta = a_g$ where

$$g_1 a_g = g_1 \cdot g$$

and

$$(g_1, g_2) a_g = (g_1 \cdot g, g_2).$$

Let $G = \bigcup_{i \in I} H_i$ be a decomposition of G into disjoint cosets of H. Each coset H_i can be considered as a unary relation on X. We take $R = \{P\} \cup \cup \{H_i \mid i \in I\}$ and $\mathfrak{R} = (X, R)$.

Now if $\varphi \in (W|R)(\mathfrak{R})$, $P\bar{\varphi} = P$ because P is the only binary relation in R, so that $\varphi = a_g$ for some $g \in G$. Conversely, if $g \in G$, then $P\bar{a}_g = P$, and $H_i \bar{a}_g = H_i \cdot g \in R$ for each $i \in I$. Finally, given H_j, there is an H_i such that $H_i \bar{a}_g = H_j$; we conclude $a_g \in (W|R)(\mathfrak{R})$. Therefore, the mapping β given above is an isomorphism of $\boldsymbol{G}$ onto $(\boldsymbol{W}|R)(\mathfrak{R})$.

$a_g \in A(\mathfrak{R})$ if and only if $H_i \bar{a}_g = H_i$ for all $i \in I$; in particular, $H\bar{a}_g = H$ or $H \cdot g = H$ whence $g \in H$. Conversely, if $g \in H$, then $H_i \cdot g = H_i$ for all $i \in I$ by the normality of $\boldsymbol{H}$ in $\boldsymbol{G}$. Therefore $A(\mathfrak{R}) = \{a_h \mid h \in H\}$, proving the theorem.

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