ANNALES
POLONICI MATHEMATICI
XLII (1983)

The difference method for non-linear elliptic differential equations with mixed derivatives

by JACEK KACZMARCZYK (Kraków)

Dedicated to the memory of Jacek Szarski

Abstract. In this paper we consider a difference method for elliptic differential equation $F(x, u, u_x, u_{xx}) = 0$ with Dirichlet's boundary conditions in an arbitrary bounded domain within $\boldsymbol{R}^n$. A convergence theorem is proved and the error estimate is given.

1. Introduction. Let us consider the elliptic differential equation

$$F(x, u, u_x, u_{xx}) = 0, \tag{1.1}$$

where $x = (x^1, \ldots, x^n)$ is a point of the open and bounded subset Ω of space $\boldsymbol{R}^n$, u is a function defined in $\bar{\Omega}$, u_x is the gradient of u and u_{xx} denotes the $n \times n$ symmetric matrix of second order derivatives with respect to x.

Along with (1.1) we consider the boundary condition

$$u(x) = \bar{u}(x) \quad \text{for } x \in \partial\Omega, \tag{1.2}$$

where $\bar{u}$ is a function defined on $\partial\Omega$.

Malec in his paper [2] uses the well-known seven-point scheme where $F_{w_{ij}}$ are of constant signs (here w_{ij} is the argument of F replaced in the equation by $u_{x_i x_j}$). The constant signs of $F_{w_{ij}}$ does not allow us to consider e.g. the equation $u_{xx} + xyu_{xy} + u_{yy} = 0$ for $(x, y) \in (0, 1) \times (-1, 1)$.

Fitzke [1] has proposed a nine-nodal point difference scheme (for $n = 2$), however, because of the assumption adopted, the study does not even cover the equation $u_{xx} + u_{xy} + u_{yy} = 0$.

In the papers mentioned above only the cubics n-dimensional and the square nets are allowed.

A difference scheme on an arbitrary rectangular net but without mixed derivatives is considered in [3].

There are no restrictions on the sign of $F_{w_{ij}}$ in the method exposed by Voigt [4]. Yet, the existence of a certain matrix is assumed there that is hard to find (particularly for $n > 2$) in the case of the non-linear equation. In the example cited by Voigt, with a Minkowski matrix, also adequate assumptions are fulfilled, warranting the convergence of the method in my paper.

Let me recall Professor Szarski in this place. I participated in his seminar for many years and it was with his discreet support that I worked in the present paper.

2. Assumptions. Let the function F of arguments $x \in \Omega$, $z \in R$, $q = (q_1, \ldots, q_n) \in \boldsymbol{R}^n$, $w = (w_{ij}) \in \boldsymbol{R}^{n^2}$ be of class C^1 with respect to z, q, w and satisfy the assumptions

$$F_{w_{ij}}(x, z, q, w) = F_{w_{ji}}(x, z, q, w) \quad (i, j = 1, \ldots, n), \tag{2.1}$$

$$F_z(x, z, q, w) \leqslant -L, \quad L > 0 \tag{2.2}$$

for all x, z, q, w. Let us assume further that there exists a bounded symmetric matrix $G(x) = \big(G^{ij}(x)\big)$ for $x \in \Omega$ such that

$$|F_{w_{ij}}(x, z, q, w)| \leqslant G^{ij}(x) \quad (i, j = 1, \ldots, n;\ i \neq j) \tag{2.3}$$

and there is a $\varrho_0 \in (0, 1]$ such that

$$\frac{H}{2}\,|F_{q_i}(x, z, q, w)| \leqslant F_{w_{ii}}(x, z, q, w) - \frac{1}{\varrho_0}\sum_{j\neq i} G^{ij}(x) \quad (i = 1, \ldots, n) \tag{2.4}$$

for all x, z, q, w and $H > 0$ sufficiently small.

It follows from (2.3) and (2.4) that $F_w = (F_{w_{ij}})$ is diagonally dominant, which ensures the ellipticity of (1.1).

3. Examples.

(a) If $F(x, z, q, w) = \sum_{i,j=1}^{n} a_{ij}(x) w_{ij} + b(x, z, q, w_{11}, \ldots, w_{nn})$, where $a = (a_{ij})$ is a symmetric diagonally dominant matrix and $b_{w_{ii}} \geqslant 0$, then we put $G^{ij}(x) = |a_{ij}(x)|$. Since in this case $F_{w_{ij}} = a_{ij}$ $(i \neq j)$, (2.3) is satisfied. Condition (2.4) remains in the form

$$\frac{H}{2}\,|b_{q_i}(x, z, q, w_{11}, \ldots, w_{nn})| \leqslant a_{ii}(x) - \frac{1}{\varrho_0}\sum_{j\neq i} |a_{ij}(x)| \quad (i = 1, \ldots, n)$$

since $F_{w_{ii}} = a_{ii} + b_{w_{ii}} \geqslant a_{ii}$.

(b) If in the general non-linear case there is a symmetric, bounded matrix $A(x) = \big(A_{ij}(x)\big)$ such that

$$F_{w_{ii}}(x, z, q, w) \geqslant A_{ii}(x), \quad |F_{w_{ij}}(x, z, q, w)| \leqslant |A_{ij}(x)| \quad (i, j = 1, \ldots, n;\ i \neq j) \tag{3.1}$$

and

$$(3.2)\qquad 0 < g \leqslant A_{ii}(x) - \frac{1}{\varrho}\sum_{j\neq i}|A_{ij}(x)| \qquad (i = 1, \ldots, n);\ \varrho \in (0, 1],$$

then by putting $G^{ij}(x) = |A_{ij}(x)|$ it is simple to verify that assumptions (2.3) and (2.4) are fulfilled as soon as F_{q_i} are bounded (for $F_{q_i} \equiv 0$, g might equal 0). If there is a Minkowski matrix which satisfies besides (3.1) other additional assumptions (see [4]), the condition (3.2) is satisfied as well and hence (2.3) and (2.4).

4. Discretization. Let $[h_j^i]_{j\in Z}$ $(i = 1, \ldots, n)$ be a sequence of numbers such that

$$(4.1)\qquad h = \max_{1\leqslant i\leqslant n}\{\sup_{j\in Z} h_j^i\} < \infty, \qquad \chi = \min_{1\leqslant i\leqslant n}\{\inf_{j\in Z} h_j^i\} > 0$$

and let us put $x_0^i = 0$, $x_{j+1}^i = x_j^i + h_{j+1}^i$, $\bar{h}_j^i = \frac{1}{2}(h_{j+1}^i + h_j^i)$ for $j \in Z$ $(i = 1, \ldots, n)$ (cf. Fig. 1).

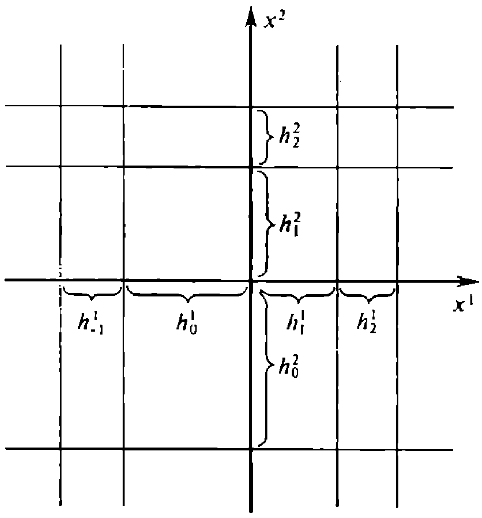

Fig. 1. The space intervals h_j^i (for $n = 2$)

We denote by $\mathscr{M}$ the set of multiindices $m = (m_1, \ldots, m_n)$, where $m_i \in Z$ and we put $x_m = (x_{m_1}^1, \ldots, x_{m_n}^n)$. Let the set of nodal points x_m be denoted by Σ.

For $x_m \in \Sigma$ we put

$$(4.2)\qquad P(x_m) = \bigcup_{i,j=1}^{n}\{x \in \boldsymbol{R}^n\colon\ x_{m_i-1}^i \leqslant x^i \leqslant x_{m_i+1}^i;\ x_{m_j-1}^j \leqslant x^j \leqslant x_{m_j+1}^j;$$
$$x^k = x_{m_k}^k,\ k = 1, \ldots, n,\ k \neq i, j\}.$$

We put $\hat{\Sigma} = \Sigma \cap \bar{\Omega}$ and

$$(4.3)\quad \begin{aligned} H &= \max_{\substack{1\leqslant i\leqslant n\\ j\in Z}}\{h_j^i\colon h_j^i = x_j^i - x_{j-1}^i, \text{ where } x_j^i, x_{j-1}^i \text{ are coordinates of nodal points in } \hat{\Sigma}\},\\ X &= \min_{\substack{1\leqslant i\leqslant n\\ j\in Z}}\{h_j^i\colon h_j^i = x_j^i - x_{j-1}^i, \text{ where } x_j^i, x_{j-1}^i \text{ are coordinates of nodal points in } \hat{\Sigma}\}.\end{aligned}$$

It is obvious that $\chi \leqslant X \leqslant H \leqslant h$.

We denote by ϱ $(0 < \varrho \leqslant 1)$ the number X/H. For $\varrho = 1$ we have the square net.

The nodal point $x_m \in \hat{\Sigma}$ is called: (i) internal nodal point when $P(x_m) \subset \bar{\Omega}$; (ii) type I boundary nodal point when $x_m \in \Omega$ and $P(x_m) \not\subset \bar{\Omega}$ (there exists then an $x' \in P(x_m) \cap \partial\Omega$, such that the segment $[x_m, x']$ is contained in $\bar{\Omega}$); (iii) type II boundary nodal point when $x_m \in \partial\Omega$.

We introduce the sets of multiindices

$$\mathcal{M}_W = \{m \in \mathcal{M}\colon x_m \text{ is an internal point}\},$$
$$\mathcal{M}_{B_1} = \{m \in \mathcal{M}\colon x_m \text{ is a type I boundary point}\},$$
$$\mathcal{M}_{B_2} = \{m \in \mathcal{M}\colon x_m \text{ is a type II boundary point}\}$$

and we put $\mathcal{M}_B = \mathcal{M}_{B_1} \cup \mathcal{M}_{B_2}$ and $\mathcal{M}_{\hat{\Sigma}} = \mathcal{M}_W \cup \mathcal{M}_B$.

Let us denote for $m \in \mathcal{M}$

$$(4.4)\quad \begin{aligned} i(m) &= (m_1, \ldots, m_{i-1}, m_i+1, m_{i+1}, \ldots, m_n)\\ -i(m) &= (m_1, \ldots, m_{i-1}, m_i-1, m_{i+1}, \ldots, m_n)\end{aligned} \qquad (i = 1, \ldots, n).$$

5. Auxiliary lemmas. For $m, \bar{m} \in \mathcal{M}$ let us put

$$d(m, \bar{m}) = \sum_{i=1}^{n} (m_i - \bar{m}_i)^2, \qquad S_m = \{\bar{m} \in \mathcal{M}\colon 0 < d(m, \bar{m}) \leqslant 2\}.$$

If $m \in \mathcal{M}_W$, then $S_m \subset \mathcal{M}_{\hat{\Sigma}}$.

LEMMA 1. *Let us assume*

$$(5.1)\quad \begin{aligned} &\beta\colon \mathcal{M}_W \to R, \quad \eta\colon \mathcal{M}_W \to R, \quad \varepsilon\colon \mathcal{M}_B \to R, \quad y\colon \mathcal{M}_{\hat{\Sigma}} \to R,\\ &a^m\colon S_m \to R \quad \textit{for } m \in \mathcal{M}_W, \quad \|y\| = \max\{|y(m)|\colon m \in \mathcal{M}_{\hat{\Sigma}}\}.\end{aligned}$$

If

$$(5.2)\quad a^m \geqslant 0 \quad \textit{for } m \in \mathcal{M}_W,\ \beta \geqslant \beta_0 > 0,\ \eta \leqslant \eta_0,\ \eta_0 > 0,\ \varepsilon \leqslant \varepsilon_0,$$

where $\beta_0, \eta_0, \varepsilon_0$ *are constants and*

$$(5.3)\quad \sum_{\bar{m}\in S_m} a^m(\bar{m})y(\bar{m}) - \Big(\sum_{\bar{m}\in S_m} a^m(\bar{m}) + \beta(m)\Big)y(m) \geqslant -\eta(m) \quad \textit{for } m \in \mathcal{M}_W,$$

$$(5.4)\quad y(m) \leqslant \varepsilon(m) \quad \textit{for } m \in \mathcal{M}_B,$$

then

$$y(m) \leqslant \max\{\varepsilon_0, \eta_0/\beta_0\} \quad \textit{for } m \in \mathcal{M}_{\hat{\Sigma}}. \tag{5.5}$$

The simple proof of Lemma 1 will be omitted.

LEMMA 2. *Under the assumptions of Lemma* 1 *and*

$$\sum_{\bar{m}\in S_m} a^m(\bar{m})y(\bar{m}) - \Big(\sum_{\bar{m}\in S_m} a^m(\bar{m}) + \beta(m)\Big)y(m) \leqslant \eta(m) \quad \textit{for } m \in \mathcal{M}_W, \tag{5.6}$$

$$y(m) \geqslant -\varepsilon(m) \quad \textit{for } m \in \mathcal{M}_B \tag{5.7}$$

we have

$$y(m) \geqslant \min\{-\varepsilon_0, -\eta_0/\beta_0\} \quad \textit{for } m \in \mathcal{M}_{\hat{\Sigma}}. \tag{5.8}$$

The simple proof of Lemma 2 will be omitted.

From Lemma 1 and Lemma 2, as a simple conclusion, we obtain

LEMMA 3. *Under the assumption of Lemma* 1 *if* $\eta \geqslant 0$, $\varepsilon \geqslant 0$ *and*

$$\Big|\sum_{\bar{m}\in S_m} a^m(\bar{m})y(\bar{m}) - \Big(\sum_{\bar{m}\in S_m} a^m(\bar{m}) + \beta(m)\Big)y(m)\Big| \leqslant \eta(m) \quad \textit{for } m \in \mathcal{M}_W, \tag{5.9}$$

$$|y(m)| \leqslant \varepsilon(m) \quad \textit{for } m \in \mathcal{M}_B, \tag{5.10}$$

then

$$\|y\| \leqslant \max\{\varepsilon_0, \eta_0/\beta_0\}. \tag{5.11}$$

6. Difference expressions. For a function $y\colon \hat{\Sigma} \to R$ we put $y_m = y(x_m)$ and introduce the following difference expressions:

$$(y_m)_i = \frac{1}{2\bar{h}^i_{m_i}}(y_{i(m)} - y_{-i(m)}), \tag{6.1a}$$

$$(y_m)_{ii} = \frac{1}{\bar{h}^i_{m_i}}\left(\frac{y_{i(m)} - y_m}{h^i_{m_i+1}} - \frac{y_m - y_{-i(m)}}{h^i_{m_i}}\right), \tag{6.1b}$$

$$(y_m)^{++}_{ij} = \frac{1}{h^i_{m_i+1}h^j_{m_j+1}}(y_{i(j(m))} - y_{i(m)} - y_{j(m)} + y_m), \tag{6.1c}$$

$$(y_m)^{-+}_{ij} = \frac{1}{h^i_{m_i}h^j_{m_j+1}}(y_{j(m)} - y_{-i(j(m))} - y_m + y_{-i(m)}), \tag{6.1d}$$

$$(y_m)^{+-}_{ij} = \frac{1}{h^i_{m_i+1}h^j_{m_j}}(y_{i(m)} - y_m - y_{i(-j(m))} + y_{-j(m)}), \tag{6.1e}$$

$$(y_m)^{--}_{ij} = \frac{1}{h^i_{m_i}h^j_{m_j}}(y_m - y_{-i(m)} - y_{-j(m)} + y_{-i(-j(m))}), \tag{6.1f}$$

$$(y_m)_{ij} = \tfrac{1}{4}[(y_m)^{++}_{ij} + (y_m)^{-+}_{ij} + (y_m)^{+-}_{ij} + (y_m)^{--}_{ij}] \tag{6.1g}$$

(cf. Fig. 2),

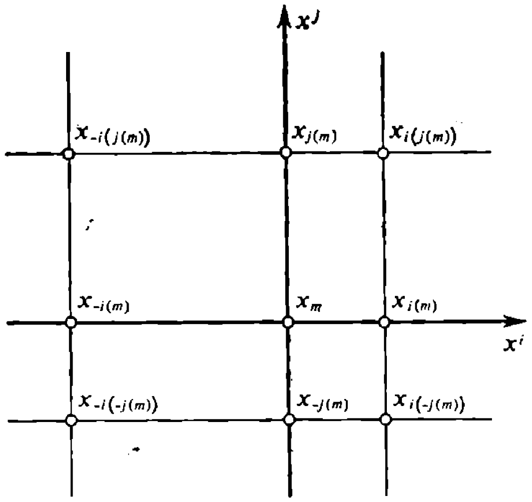

Fig. 2. For simplicity, in Fig. 2 the nodal point x_m is placed at the origin. For example $y_{i(j(m))}$ denotes the value of the function y at the nodal point $x_{i(j(m))}$

$$(6.1\text{h})\qquad (y_m)_\Delta = \sum_{\substack{i,j=1\\ i\neq j}}^{n} G_m^{ij}\Bigg\{\frac{1}{4}\Bigg(\frac{1}{h^i_{m_i+1}h^j_{m_j+1}}\,y_{i(j(m))}+\frac{1}{h^i_{m_i}h^j_{m_j+1}}\,y_{-i(j(m))}+$$
$$+\frac{1}{h^i_{m_i+1}h^j_{m_j}}\,y_{i(-j(m))}+\frac{1}{h^i_{m_i}h^j_{m_j}}\,y_{-i(-j(m))}\Bigg)-$$
$$-\Bigg[\frac{\bar h^j_{m_j}}{h^j_{m_j+1}h^j_{m_j}}\Bigg(\frac{1}{h^i_{m_i+1}}\,y_{i(m)}+\frac{1}{h^i_{m_i}}\,y_{-i(m)}\Bigg)\Bigg]+\frac{\bar h^i_{m_i}\bar h^j_{m_j}}{h^i_{m_i+1}h^i_{m_i}h^j_{m_j+1}h^j_{m_j}}\,y_m\Bigg\}$$

(cf. Fig. 3),

$$(y_m)_{\mathrm{I}} = \big((y_m)_1, \ldots, (y_m)_n\big), \qquad (y_m)_{\mathrm{II}} = \big((y_m)_{ij}\big).$$

Lemma 4. *If u is a function of the class C^2 in $\bar\Omega$, then*

$$(6.2)\qquad \max_{\substack{1\leqslant i\leqslant n\\ m\in\mathscr{M}_W}} |u_{x_i}(x_m)-(u_m)_i| \leqslant \eta_1(H), \qquad \max_{\substack{1\leqslant i,j\leqslant n\\ m\in\mathscr{M}_W}} |u_{x_ix_j}(x_m)-(u_m)_{ij}| \leqslant \eta_2(H),$$

$$(6.3)\qquad \max_{m\in\mathscr{M}_W} |(u_m)_\Delta| \leqslant \eta_3(H),$$

where

$$(6.4)\qquad \lim_{H\to 0}\eta_i(H) = 0 \qquad (i = 1, 2, 3).$$

Proof. We shall prove (6.4) for $i = 3$ only because the proofs for $i = 1, 2$ are easy. For simplicity let us assume that $h^i_j = h$ for $i = 1, \ldots, n$; $j \in Z$ (cf. (4.1)). Then from the symmetry of the matrix G and (6.1h) we have

$$(6.5)\qquad |(u_m)_\Delta| \leqslant \sum_{i<j} G_m^{ij}\left| \frac{1}{2h^2}\left(u_{i(j(m))}+u_{-i(j(m))}+u_{i(-j(m))}+u_{-i(-j(m))}\right)- \right.$$
$$\left. -\frac{1}{h^2}\left(u_{i(m)}+u_{-i(m)}+u_{j(m)}+u_{-j(m)}\right)+\frac{2}{h^2}u_m \right|.$$

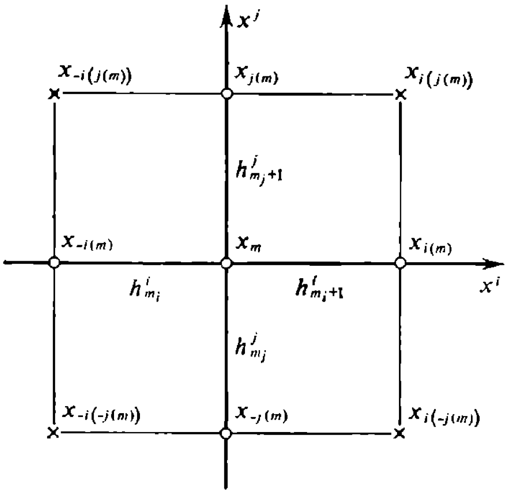

Fig. 3. Let us write h in place of $h^i_{m_i+1}, h^i_{m_i}, h^j_{m_j+1}, h^j_{m_j}$. Then the Laplacian $u_{x_ix_i}+u_{x_jx_j}$ can be approximated with the aid of the difference expression

$$\frac{u_{i(m)}-2u_m+u_{-i(m)}}{(h)^2}+\frac{u_{j(m)}-2u_m+u_{-j(m)}}{(h)^2},$$

as well as with the aid of the quantity $u_{i(j(m))}+u_{-i(j(m))}+u_{i(-j(m))}+u_{-i(-j(m))}-4u_m$ divided by $2h^2$. In a similar way we obtained formula (6.1h)

Using Taylor polynomials we obtain (because u is a function of the class C^2 in $\bar{\Omega}$):

$$\begin{aligned}
u_{i(j(m))} &= u_m+u_{x_i}(x_m)h+u_{x_j}(x_m)h+\tfrac{1}{2}u_{x_ix_i}(x_m)h^2+u_{x_ix_j}(x_m)h^2+\\
&\quad+\tfrac{1}{2}u_{x_jx_j}(x_m)h^2+\delta_{ij}^{++}(x_m)h^2,\\
u_{-i(j(m))} &= u_m-u_{x_i}(x_m)h+u_{x_j}(x_m)h+\tfrac{1}{2}u_{x_ix_i}(x_m)h^2-\\
&\quad-u_{x_ix_j}(x_m)h^2+\tfrac{1}{2}u_{x_jx_j}(x_m)h^2+\delta_{ij}^{-+}(x_m)h^2,\\
(6.6)\qquad u_{i(-j(m))} &= u_m+u_{x_i}(x_m)h-u_{x_j}(x_m)h+\tfrac{1}{2}u_{x_ix_i}(x_m)h^2-u_{x_ix_j}(x_m)h^2+\\
&\quad+\tfrac{1}{2}u_{x_jx_j}(x_m)h^2+\delta_{ij}^{+-}(x_m)h^2,\\
u_{-i(-j(m))} &= u_m-u_{x_i}(x_m)h-u_{x_j}(x_m)h+\tfrac{1}{2}u_{x_ix_i}(x_m)h^2+\\
&\quad+u_{x_ix_j}(x_m)h^2+\tfrac{1}{2}u_{x_jx_j}(x_m)h^2+\delta_{ij}^{--}(x_m)h^2,\\
u_{i(m)} &= u_m+u_{x_i}(x_m)h+\tfrac{1}{2}u_{x_ix_i}(x_m)h^2+\delta_i^{+}(x_m)h^2,\\
u_{-i(m)} &= u_m-u_{x_i}(x_m)h+\tfrac{1}{2}u_{x_ix_i}(x_m)h^2+\delta_i^{-}(x_m)h^2
\end{aligned}$$

(for all $i, j = 1, \ldots, n$; $i \neq j$), where there exists a function $\delta(h)$ such that

$$(6.7)\qquad \begin{array}{lll} |\delta_{ij}^{++}(x_m)| \leqslant \delta(h), & |\delta_{ij}^{-+}(x_m)| \leqslant \delta(h), & |\delta_{ij}^{+-}(x_m)| \leqslant \delta(h), \\ |\delta_{ij}^{--}(x_m)| \leqslant \delta(h), & |\delta_{i}^{+}(x_m)| \leqslant \delta(h), & |\delta_{i}^{-}(x_m)| \leqslant \delta(h) \end{array}$$

(for all $i, j = 1, \ldots, n$; $i \neq j$ and $x \in \mathcal{M}_W$) and

$$(6.8)\qquad \lim_{h\to 0} \delta(h) = 0.$$

Putting (6.6) in (6.5) and reducing the expressions, we get

$$(6.9)\qquad |(u_m)_\Delta| \leqslant \sum_{i<j} G_m^{ij} 6\delta(h).$$

Since the matrix G is bounded, there exists a constant Γ such that

$$(6.10)\qquad G_m^{ij} \leqslant \Gamma \quad \text{for } m \in \mathcal{M}_W \ (i, j = 1, \ldots, n),$$

whence in (6.9) we have

$$(6.11)\qquad |(u_m)_\Delta| \leqslant \sum_{i<j} 6\Gamma\delta(h) = 3n(n-1)\Gamma\delta(h).$$

By (6.8) from (6.11) we obtain (6.3) and (6.4).
This ends the proof of Lemma 4.

7. Difference problem. Let a function $v: \hat{\Sigma} \to R$ be a solution of the difference equations system

$$(7.1)\qquad F\big(x_m, v_m, (v_m)_{\mathrm{I}}, (v_m)_{\mathrm{II}}\big) + (v_m)_\Delta = 0 \quad \text{for } m \in \mathcal{M}_W,$$

$$(7.2)\qquad v_m = \bar{u}(x_m) \quad \text{for } m \in \mathcal{M}_{B_2},$$

$$(7.3)\qquad v_m = \bar{u}(x) \quad \text{for } m \in \mathcal{M}_{B_1},$$

where $x \in \partial\Omega \cap P(x_m)$ (cf. (4.2) and Fig. 4) and $[x_m, x] \subset \bar{\Omega}$.

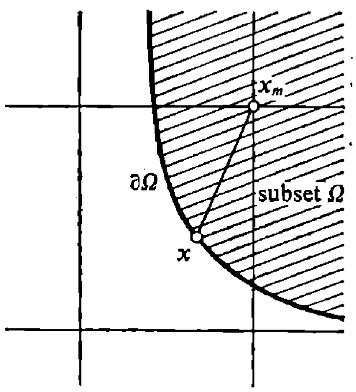

Fig. 4. The point x_m is a type I boundary nodal point and $x \in \partial\Omega$ is a point such that the segment $[x_m, x] \subset \bar{\Omega}$

Remark 1. Normally the difference equations (7.1) associated with the differential equation (1.1) could be written in the form

$$(7.4)\qquad F\big(x_m, v_m, (v_m)_{\mathrm{I}}, (v_m)_{\mathrm{II}}\big) = 0 \quad \text{for } m \in \mathscr{M}_W.$$

The term $(v_m)_\Delta$ in the difference equations (7.1) has appeared because without it the adequate assumptions of Lemmas 1, 2, 3 (cf. (5.2)) which we use in the proof of Theorem 1 are not satisfied (cf. Part 8, Convergence).

LEMMA 5. *If the function F has bounded derivatives $F_{w_{ii}}$ and F_{q_i}, i.e. there are constants g, D such that*

$$(7.5)\qquad F_{w_{ii}}(x, z, q, w) \leqslant D, \qquad |F_{q_i}(x, z, q, w)| \leqslant g \qquad (i = 1, \ldots, n)$$

for all x, z, q, w, and u is a solution of the differential problem (1.1), (1.2) *of the class C^2 in $\bar{\Omega}$, then*

$$(7.6)\qquad F\big(x_m, u_m, (u_m)_{\mathrm{I}}, (u_m)_{\mathrm{II}}\big) + (u_m)_\Delta = \eta_m(H) \quad \textit{for } m \in \mathscr{M}_W,$$

$$(7.7)\qquad u_m = \bar{u}(x) + \varepsilon_m(H) \quad \textit{for } m \in \mathscr{M}_{B_1}$$

where

$$(7.8)\qquad \lim_{H\to 0} \eta(H) = 0, \qquad \lim_{H\to 0} \varepsilon(H) = 0$$

and

$$(7.9)\qquad \eta(H) = \max_{m\in\mathscr{M}_W} |\eta_m(H)|, \qquad \varepsilon(H) = \max_{m\in\mathscr{M}_{B_1}} |\varepsilon_m(H)|.$$

Proof. From (7.6), (1.1) applying the mean value theorem, we have

$$\begin{aligned}(7.10)\quad & |\eta_m(H)| \\ &= \big|F\big(x_m, u_m, (u_m)_{\mathrm{I}}, (u_m)_{\mathrm{II}}\big) + (u_m)_\Delta - F\big(x_m, u_m, u_x(x_m), u_{xx}(x_m)\big)\big| \\ &\leqslant \sum_{i=1}^{n} |F_{q_i}(\sim)|\,|(u_m)_i - u_{x_i}(x_m)| + \sum_{i,j=1}^{n} |F_{w_{ij}}(\sim)|\,|(u_m)_{ij} - u_{x_i x_j}(x_m)| + |(u_m)_\Delta|.\end{aligned}$$

Introducing (7.5), (2.3) and (6.10) in (7.10) and using (6.2), (6.3), we obtain

$$\begin{aligned}(7.11)\qquad |\eta_m(H)| &\leqslant \sum_{i=1}^{n} g\eta_1(H) + \Big[\sum_{\substack{i,j=1\\ i\neq j}}^{n} G^{ij}(x_m) + nD\Big]\eta_2(H) + \eta_3(H) \\ &\leqslant ng\eta_1(H) + [\Gamma n(n-1) + nD]\eta_2(H) + \eta_3(H)\end{aligned}$$

for $m \in \mathscr{M}_W$, whence by (6.4) and (7.9) it follows that

$$(7.12)\qquad \lim_{H\to 0} \eta(H) = 0.$$

Now we shall take into consideration (7.7) and (1.2). We get

$$(7.13)\qquad |\varepsilon_m(H)| = |u_m - \bar{u}(x)| = |u(x_m) - u(x)| \leqslant \sum_{i=1}^{n} |u_{x_i}(\sim)|\,|x^i_{m_i} - x^i| \quad \text{for } m \in \mathscr{M}_{B_1}.$$

Since $x \in P(x_m)$ (cf. (4.2) and Fig. 4), by definition H (cf. (4.3)) we have

$$(7.14)\qquad \sum_{i=1}^{n} |x^i_{m_i} - x^i| \leqslant \sqrt{n}\, H,$$

and since the function u is of the class C^2 in $\bar{\Omega}$, there exists a constant C such that

$$(7.15)\qquad |u_{x_i}(\sim)| \leqslant C \quad (i = 1, \ldots, n).$$

Introducing (7.14) and (7.15) in (7.13), we infer that

$$(7.16)\qquad |\varepsilon_m(H)| \leqslant C\sqrt{n}\, H \quad \text{for } m \in \mathscr{M}_{B_1},$$

whence

$$(7.17)\qquad \varepsilon(H) \leqslant C\sqrt{n}\, H,$$

which with (7.13) gives Lemma 5.

Remark 2. The difference problem (7.1), (7.2), (7.3) and the differential problem (1.1), (1.2) are consistent because $(u_m)_\Delta$ in the difference equations (7.6) satisfy the estimates (6.3), (6.4) and $\eta_m(H)$ in (7.4) satisfy (7.6).

8. Convergence.

THEOREM 1. *We suppose that u is a solution of the class C^2 in $\bar{\Omega}$ of the differential problem* (1.1), (1.2), *v is a solution of the difference equations system* (7.1), (7.2), (7.3), *the function F satisfies the assumption specified in Part* 2, *Assumption, and the derivatives $F_{w_{ii}}$, F_{q_i} are bounded. Then*

1° *the difference method is convergent, i.e.*

$$(8.1)\qquad \lim_{\substack{H\to 0\\ (\varrho \geqslant \varrho_0)}} \|u - v\|_H = 0,$$

where $\|y\|_H = \max\{|y(m)|\colon m \in \mathscr{M}_{\hat{\Sigma}}\}$ *and*

2° *we have the error estimate*

$$(8.2)\qquad \|u - v\|_H \leqslant \max\{\varepsilon(H), \eta(H)/L\} \quad (\varrho \geqslant \varrho_0).$$

Proof. Let us put $r = u - v$. For $m \in \mathscr{M}_B$ from (7.7), (1.2), (7.2) and (7.3) we have

$$u_m - v_m = \begin{cases} \bar{u}(x_m) - \bar{u}(x_m) = 0 & \text{for } m \in \mathscr{M}_{B_2}, \\ \bar{u}(x) + \varepsilon_m(H) - \bar{u}(x) = \varepsilon_m(H) & \text{for } m \in \mathscr{M}_{B_1}, \end{cases}$$

whence

$$|r_m| \leqslant |\varepsilon_m(H)| \leqslant \varepsilon(H) \quad \text{for } m \in \mathscr{M}_B. \tag{8.3}$$

For $m \in \mathscr{M}_W$ from (7.6) and (7.1) and from the mean value theorem, applying the definition of the difference expressions and grouping the terms, we get successively:

$$\eta_m(H) = \left[F\big(x_m, u_m, (u_m)_{\mathrm{I}}, (u_m)_{\mathrm{II}}\big) - F\big(x_m, v_m, (v_m)_{\mathrm{I}}, (v_m)_{\mathrm{II}}\big)\right] + \\ + [(u_m)_\Delta - (v_m)_\Delta], \tag{8.4}$$

$$\eta_m(H) = F_z(\sim) r_m + \sum_{i=1}^{n} F_{q_i}(\sim)(r_m)_i + \sum_{i,j=1}^{n} F_{w_{ij}}(\sim)(r_m)_{ij} + (r_m)_\Delta, \tag{8.5}$$

$$\begin{aligned} \eta_m(H) &= F_z r_m + \sum_{i=1}^{n} F_{q_i} \frac{1}{2\bar{h}^i_{m_i}} (r_{i(m)} - r_{-i(m)}) + \\ &\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} F_{w_{ij}} \tfrac{1}{4}[(r_m)^{++}_{ij} + (r_m)^{-+}_{ij} + (r_m)^{+-}_{ij} + (r_m)^{--}_{ij}] + \\ &\quad + \sum_{i=1}^{n} F_{w_{ii}} \frac{1}{\bar{h}^i_{m_i}} \left(\frac{r_{i(m)} - r_m}{h^i_{m_i+1}} - \frac{r_m - r_{-i(m)}}{h^i_{m_i}}\right) + (r_m)_\Delta \\ &= F_z r_m + \sum_{i=1}^{n} F_{q_i} \frac{1}{2\bar{h}^i_{m_i}} (r_{i(m)} - r_{-i(m)}) + \\ &\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} F_{w_{ij}} \tfrac{1}{4}\Bigg[\frac{1}{h^i_{m_i+1} h^j_{m_j+1}} (r_{i(j(m))} - r_{i(m)} - r_{j(m)} + r_m) + \\ &\quad + \frac{1}{h^i_{m_i} h^j_{m_j+1}} (r_{j(m)} - r_{-i(j(m))} - r_m + r_{-i(m)}) + \\ &\quad + \frac{1}{h^i_{m_i+1} h^j_{m_j}} (r_{i(m)} - r_m - r_{i(-j(m))} + r_{-j(m)}) + \\ &\quad + \frac{1}{h^i_{m_i} h^j_{m_j}} (r_m - r_{-i(m)} - r_{-j(m)} + r_{-i(-j(m))})\Bigg] + \\ &\quad + \sum_{i=1}^{n} F_{w_{ii}} \frac{1}{\bar{h}^i_{m_i}} \left(\frac{r_{i(m)} - r_m}{h^i_{m_i+1}} - \frac{r_m - r_{-i(m)}}{h^i_{m_i}}\right) + \\ &\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} G^{ij}_m \Bigg\{\tfrac{1}{4}\Bigg(\frac{1}{h^i_{m_i+1} h^j_{m_j+1}} r_{i(j(m))} + \frac{1}{h^i_{m_i} h^j_{m_j+1}} r_{-i(j(m))} + \end{aligned} \tag{8.6}$$

$$+\frac{1}{h^i_{m_i+1}h^j_{m_j}}r_{i(-j(m))}+\frac{1}{h^i_{m_i}h^j_{m_j}}r_{-i(-j(m))}\Big)-$$

$$-\left[\frac{\bar{h}^j_{m_j}}{h^j_{m_j+1}h^j_{m_j}}\left(\frac{1}{h^i_{m_i+1}}r_{i(m)}+\frac{1}{h^i_{m_i}}r_{-i(m)}\right)\right]+$$

$$+\frac{\bar{h}^i_{m_i}\bar{h}^j_{m_j}}{h^i_{m_i+1}h^i_{m_i}h^j_{m_j+1}h^j_{m_j}}r_m\Big\},$$

$$(8.7)\qquad \eta_m(H)=\sum_{i\neq j}\Big\{\tfrac{1}{4}(G^{ij}_m+F_{w_{ij}})\frac{1}{h^i_{m_i+1}h^j_{m_j+1}}r_{i(j(m))}+$$

$$+\tfrac{1}{4}(G^{ij}_m-F_{w_{ij}})\frac{1}{h^i_{m_i}h^j_{m_j+1}}r_{-i(j(m))}+$$

$$+\tfrac{1}{4}(G^{ij}_m-F_{w_{ij}})\frac{1}{h^i_{m_i+1}h^j_{m_j}}r_{i(-j(m))}+$$

$$+\tfrac{1}{4}(G^{ij}_m+F_{w_{ij}})\frac{1}{h^i_{m_i}h^j_{m_j}}r_{-i(-j(m))}\Big\}+$$

$$+\sum_{i=1}^{n}\left\{\frac{F_{q_i}}{2\bar{h}^i_{m_i}}+\frac{F_{w_{ii}}}{\bar{h}^i_{m_i}h^i_{m_i+1}}+\sum_{j\neq i}\left[\frac{F_{w_{ij}}}{2h^i_{m_i+1}}\left(\frac{1}{h^j_{m_j}}-\frac{1}{h^j_{m_j+1}}\right)-\right.\right.$$

$$\left.\left.-\frac{G^{ij}_m}{2h^i_{m_i+1}}\left(\frac{1}{h^j_{m_j}}+\frac{1}{h^j_{m_j+1}}\right)\right]\right\}r_{i(m)}+$$

$$+\sum_{i=1}^{n}\left\{\frac{-F_{q_i}}{2\bar{h}^i_{m_i}}+\frac{F_{w_{ii}}}{\bar{h}^i_{m_i}h^i_{m_i}}+\sum_{j\neq i}\left[\frac{F_{w_{ij}}}{2h^i_{m_i}}\left(\frac{1}{h^j_{m_j+1}}-\frac{1}{h^j_{m_j}}\right)-\right.\right.$$

$$\left.\left.-\frac{G^{ij}_m}{2h^i_{m_i}}\left(\frac{1}{h^j_{m_j}}+\frac{1}{\bar{h}^j_{m_j+1}}\right)\right]\right\}r_{-i(m)}-$$

$$-\left\{-F_z+\sum_{i=1}^{n}\frac{F_{w_{ii}}}{\bar{h}^i_{m_i}}\left(\frac{1}{h^i_{m_i+1}}+\frac{1}{h^i_{m_i}}\right)-\right.$$

$$-\tfrac{1}{4}\sum_{i\neq j}F_{w_{ij}}\left(\frac{1}{h^i_{m_i+1}h^j_{m_j+1}}-\frac{1}{h^i_{m_i}h^j_{m_j+1}}-\frac{1}{h^i_{m_i+1}h^j_{m_j}}+\right.$$

$$\left.\left.+\frac{1}{h^i_{m_i}h^j_{m_j}}\right)-\sum_{i\neq j}G^{ij}_m\frac{\bar{h}^i_{m_i}\bar{h}^j_{m_j}}{h^i_{m_i+1}h^i_{m_i}h^j_{m_j+1}h^j_{m_j}}\right\}r_m.$$

From equality (8.7) we shall obtain two inequalities of the form (5.3) and (5.6), the quantity $y(m)$ being replaced by the error r_m, $y(m)=r_m$. It remains to verify the estimates for the coefficients (5.2).

For this purpose let us write for $m \in \mathcal{M}_W$:

(8.8)

$$a^m(\overline{m}) = \begin{cases} \frac{1}{2}(G_m^{ij} + F_{w_{ij}}) \dfrac{1}{h^i_{m_i+1} h^j_{m_j+1}}, & \text{when } \overline{m} = i(j(m)), \\ \frac{1}{2}(G_m^{ij} - F_{w_{ij}}) \dfrac{1}{h^i_{m_i} h^j_{m_j+1}}, & \text{when } \overline{m} = -i(j(m)), \\ \frac{1}{2}(G_m^{ij} - F_{w_{ij}}) \dfrac{1}{h^i_{m_i+1} h^j_{m_j}}, & \text{when } \overline{m} = i(-j(m)), \\ \frac{1}{2}(G_m^{ij} + F_{w_{ij}}) \dfrac{1}{h^i_{m_i} h^j_{m_j}}, & \text{when } \overline{m} = -i(-j(m)), \\ \dfrac{F_{q_i}}{2\overline{h}^i_{m_i}} + \dfrac{F_{w_{ii}}}{\overline{h}^i_{m_i} h^i_{m_i+1}} + \sum_{j \neq i} \Big[\dfrac{F_{w_{ij}}}{2h^i_{m_i+1}} \Big(\dfrac{1}{h^j_{m_j}} - \dfrac{1}{h^j_{m_j+1}} \Big) - & \\ \qquad - \dfrac{G_m^{ij}}{2h^i_{m_i+1}} \Big(\dfrac{1}{h^j_{m_j}} + \dfrac{1}{h^j_{m_j+1}} \Big) \Big], & \text{when } \overline{m} = i(m), \\ \dfrac{-F_{q_i}}{2\overline{h}^i_{m_i}} + \dfrac{F_{w_{ii}}}{\overline{h}^i_{m_i} h^i_{m_i}} + \sum_{j \neq i} \Big[\dfrac{F_{w_{ij}}}{2h^i_{m_i}} \Big(\dfrac{1}{h^j_{m_j+1}} - \dfrac{1}{h^j_{m_j}} \Big) - & \\ \qquad - \dfrac{G_m^{ij}}{2h^i_{m_i+1}} \Big(\dfrac{1}{h^j_{m_j}} + \dfrac{1}{h^j_{m_j+1}} \Big) \Big], & \text{when } \overline{m} = -i(m). \end{cases}$$

We shall prove that

(8.9) $$a^m \geqslant 0.$$

For $\overline{m} = \pm i(\pm j(m))$ by (1.5) we have $|F_{w_{ij}}| \leqslant G^{ij}$, whence $G^{ij} \pm \pm F_{w_{ij}} \geqslant 0$; therefore $a^m(\overline{m}) \geqslant 0$. For $\overline{m} = i(m)$, applying (1.5), (1.6) and the definitions H, X, ϱ, we simply verify that $a^m(\overline{m}) \geqslant 0$. For $\overline{m} = -i(m)$ we reason analogously and hence obtain finally

(8.10) $$a^m(\overline{m}) \geqslant 0 \quad \text{for } \overline{m} \in S_m.$$

This ends the proof of (8.9).

Now we write:

(8.11) $$\beta(m) = -F_s, \quad \beta_0 = L, \quad \eta(m) = \eta_m(H), \quad \eta_0 = \eta(H)$$

and

(8.12) $$\varepsilon(m) = \varepsilon_m(H), \quad \varepsilon_0 = \varepsilon(H) \quad \text{for } m \in \mathcal{M}_B,$$
$$y(m) = r_m \quad \text{for } m \in \mathcal{M}_{\hat{\Sigma}}.$$

From (8.3)

(8.13) $$|y(m)| = |r_m| \leqslant \varepsilon(H) = \varepsilon_0 \quad \text{for } m \in \mathcal{M}_B$$

and from (2.2)

$$(8.14)\qquad \beta(m) = -F_z \geqslant L > 0, \quad L = \beta_0 \quad \text{for } m \in \mathcal{M}_W$$

and according to (7.9)

$$(8.15)\qquad |\eta(m)| = |\eta_m(H)| \leqslant \eta(H) = \eta_0.$$

It is easy to verify that the expression $\sum_{\overline{m}\in S_m} a^m(\overline{m}) + \beta(m)$ (cf. (8.8), (8.11)) is equal to the coefficient of r_m in (8.7) (cf. the last line in formula (8.7)).

From (8.10), (8.13), (8.14), (8.15) it follows that the assumptions of Lemma 3 are satisfied, whence

$$(8.16)\qquad \|r\|_H \leqslant \max\{\varepsilon(H), \eta(H)/L\} \quad (\varrho \geqslant \varrho_0),$$

and hence on the basis of Lemma 5

$$\lim_{\substack{H\to 0\\ (\varrho\geqslant\varrho_0)}} \|u-v\|_H = 0$$

which completes the proof of Theorem 1.

References

[1] A. Fitzke, *Method of difference inequalities for parabolic equations with mixed derivatives*, Ann. Polon. Math. 31 (1975), p. 121–129.

[2] M. Malec, *Schéma des différences finies pour une équation non linéaire partielle du type elliptique aveo dérivées mixtes*, ibidem 34 (1977), p. 119–123.

[3] —, Cz. Mączka, A. Rycerz, *Sur le méthode des différences finies pour les équations paraboliques non linéaires sans dérivées mixtes et avec des conditions aux limities du type de Neumann*, Demonstr. Math. 8 (1975), p. 1–12.

[4] A. Voigt, *The method of lines for nonlinear parabolic differential equations with mixed derivatives*, Numer. Math. 32 (1979), p. 197–207.

Reçu par la Rédaction le 25.02.1981