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On some theorems on distortion and rotation in the class of k-symmetrical starlike functions of order a

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Introduction. Let $S^*(k)$ $(k = 1, 2, \ldots)$ be the family of all functions of the form

$$w = f(z) = z + \sum_{n=2}^{\infty} c_n^{(k)} z^{(n-1)k+1} \tag{1}$$

regular and univalent in the circle $E = \{z: |z| < 1\}$ which map the circle E onto starlike regions with respect to the point $w = 0$. Thus the functions of this family satisfy the condition

$$\operatorname{re}\frac{zf'(z)}{f(z)} > 0, \qquad z \in E. \tag{2}$$

Consider for a fixed a, $0 \leqslant a < 1$ the family $S_a^*(k)$ of all functions $f(z)$, $|z| < 1$ of form (1) for which

$$\operatorname{re}\frac{zf'(z)}{f(z)} > a, \qquad z \in E. \tag{3}$$

Such functions are called *starlike* of order a, (cf. [3]). We have $S_a^*(k) \subset S^*(k)$, $S_0^*(k) \equiv S^*(k)$.

In this paper we shall make use of the following parametric representation of a function $f(z)$ of the family $S_a^*(k)$:

$$f(z) = z\exp\left\{-\sigma\int_0^{2\pi} \log(1 - z^k \cdot e^{-it})\, d\mu(t)\right\}, \tag{4}$$

where $\sigma = \dfrac{2(1-a)}{k}$ and $\mu(t)$ is a certain non-decreasing function of the variable t in the interval $(-\infty, \infty)$ satisfying the conditions

$$\mu(t+0) = \mu(t), \qquad \mu(t+2\pi) - \mu(t) = 1. \tag{5}$$

If the function $f(z) \in S_a^*(k)$, then there exists a function $\mu(t)$ of the type described above for which formula (4) holds. Conversely, functions defined by formula (4) belong to the family $S_a^*(k)$ (cf. [3]).

In the present paper we prove four theorems of the type of theorems on rotation and distortion in the class $S_a^*(k)$. Here we base on a theorem [4] which we formulate as follows:

THEOREM. *Let*

$$\Phi = \Phi(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N) \tag{6}$$

be a real function of the arguments $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N$ *differentiable in a sufficiently large region* D *and such that* $\operatorname{grad}\Phi \not\equiv 0$. *Let*

$$u_k = \int_0^{2\pi} \varphi_k(t)\,d\mu(t), \qquad k = 1, 2, \ldots, N, \tag{7}$$

where $\varphi_k(t)$, $0 \leqslant t \leqslant 2\pi$ *are given differentiable and periodical functions with the period* 2π *and* $\mu(t) \in G$, G *denoting the family of non-decreasing functions of the variable t defined in the interval* $(-\infty, \infty)$ *and satisfying conditions* (5). *Let for arbitrary numbers* A_k, $\sum_{k=1}^{N} |A_k| \neq 0$ *the function*

$$\sum_{k=1}^{N} A_k \varphi_k'(t) \tag{8}$$

have in the interval $\langle 0, 2\pi\rangle$ *not more than s roots (taking into account their multiplicity). Then the functional*

$$\Phi(\mu) = \Phi\Big(\int_0^{2\pi} \varphi_1(t)\,d\mu(t), \ldots, \int_0^{2\pi} \varphi_N(t)\,d\mu(t)\Big) \tag{9}$$

attains its extremal values in the interval $\langle 0, 2\pi)$ *in the subclass* $G_q \subset G$, *where* G_q *is the family of those functions* $\mu(t)$ *which have not more than* $q = \left[\frac{s+1}{2}\right]$ *jumps in the interval* $0 < t \leqslant 2\pi$.

If the jumps of the extremal function $\mu_0(t)$ *are at the points* t_j, $j = 1, 2, \ldots, q$, *and*

$$u_l = \sum_{j=1}^{q} \lambda_j \varphi_l(t_j), \tag{10}$$

$$\sum_{j=1}^{q} \lambda_j = 1, \qquad \lambda_j \geqslant 0, \tag{11}$$

then t_j *and* λ_j *satisfy the system of equations*

$$\sum_{l=1}^{N} B_l \varphi_l(t_j) - \lambda = 0, \qquad j = 1, 2, \ldots, q, \tag{12}$$

$$\sum_{l=1}^{N} B_l \varphi_l'(t_j) = 0, \qquad j = 1, 2, \ldots, q, \tag{13}$$

where

$$B_l = \Phi'_{u_l}(u_1, u_2, \ldots, u_N) \tag{14}$$

and λ is a real constant.

1.A. Consider, for an arbitrary fixed z, $z \in E$, the functional

$$L(f) = \left| \frac{f(z)^n \cdot f'(z)^m}{z^n} \right|, \tag{1.1}$$

in the family $S^*_a(k)$, where m and n are arbitrary fixed numbers satisfying the condition $m^2 + n^2 \neq 0$. Without any loss of generality we may assume that $z = r$, $0 < r < 1$. Let

$$T = r^k, \quad T_1 = (1-2a)r^k. \tag{1.2}$$

Since

$$[L(f)]^2 = \left[\frac{f(r)}{r} \cdot \frac{\overline{f(r)}}{r}\right]^n \cdot [f'(r) \cdot \overline{f'(r)}]^m,$$

we have

$$2\log L(f) = 2\log\left| \frac{f(r)^n \cdot f'(r)^m}{r^n} \right| \tag{1.3}$$

$$= n\log\frac{f(r)}{r} + n\log\frac{\overline{f(r)}}{r} + m\log f'(r) + m\log\overline{f'(r)}.$$

Differentiating function (4) we obtain for $z = r$

$$f'(r) = \int_0^{2\pi} \frac{1+T_1 e^{-it}}{1-Te^{-it}}\, d\mu(t) \cdot \exp\left\{-\sigma \int_0^{2\pi} \log(1-Te^{-it})\, d\mu(t)\right\}. \tag{1.4}$$

By (4), (1.2) and (1.4) formula (1.3) becomes

$$2\log L(f) = (m+n)\sigma u_1 + m\log u_2 + m\log u_3, \tag{1.5}$$

where

$$u_l = \int_0^{2\pi} \varphi_l(t)\, d\mu(t), \quad 1 = 1, 2, 3, \tag{1.6}$$

and

$$\varphi_1(t) = \log\frac{1}{(1-Te^{-it})(1-Te^{it})}, \tag{1.7}$$

$$\varphi_2(t) = \frac{1+T_1 e^{-it}}{1-Te^{-it}}, \quad \varphi_3(t) = \overline{\varphi_2(t)}.$$

Finding the derivatives of functions (1.7) we get

(1.8)
$$\varphi_1'(t) = \frac{iT(e^{it}-e^{-it})}{(1-Te^{-it})(1-Te^{it})}, \quad \varphi_2'(t) = \frac{-ie^{-it}(T_1+T)}{(1-Te^{-it})^2}, \quad \varphi_3'(t) = \overline{\varphi_2'(t)}.$$

Let A_l $(l = 1, 2, 3)$ be arbitrary numbers not vanishing simultaneously. Then function (8) has the form

(1.9)
$$\sum_{l=1}^{3} A_l\varphi_l'(t) = A_1\frac{iT(x-1/x)}{(1-Tx)(1-T/x)} + A_2\frac{-i(1/x)(T_1+T)}{(1-T/x)^2} + A_3\frac{ix(T_1+T)}{(1-Tx)^2},$$

where $x = e^{it}$, or the form

$$\sum_{l=1}^{3} A_l\varphi_l'(t) = \frac{P(x)}{(1-Tx)^2(x-T)^2}, \tag{1.10}$$

where $P(x)$ is a polynomial of degree $s \leqslant 4$.

Consider the function

$$\Phi(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (m+n)\sigma\tilde{u}_1 + m\log\tilde{u}_2 + m\log\tilde{u}_3 \tag{1.11}$$

of three variables $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ defined in the cartesian product $D_1 \times D_2 \times D_3$, where D_1 denotes the real axis, D_2 and D_3 are open planes, $\tilde{u}_1 \epsilon D_1$, $\tilde{u}_2 \epsilon D_2$, $\tilde{u}_3 \epsilon D_3$, $\tilde{u}_3 = \bar{\tilde{u}}_2$. We see that function (1.11) and the functional $\Phi(u_1, u_2, u_3)$, where u_l $(l = 1, 2, 3)$, are defined by formulae (1.6) and (1.7) satisfy the assumptions of the theorem given in the introduction and by (1.10) the function $\mu = \mu_0(t)$ for which the functional $\Phi(u_1, u_2, u_3)$ attains its extremal value has not more than

$$q = \left[\frac{s+1}{2}\right] = 2 \tag{1.12}$$

jumps. Now we shall write explicitely the system of equations (12)-(13).

By (1.12) it takes the form

$$\begin{aligned} B_1\varphi_1(t_1)+B_2\varphi_2(t_1)+B_3\varphi_3(t_1)-\lambda &= 0, \\ B_1\varphi_1(t_2)+B_2\varphi_2(t_2)+B_3\varphi_3(t_2)-\lambda &= 0, \\ B_1\varphi_1'(t_1)+B_2\varphi_2'(t_1)+B_3\varphi_3'(t_1) &= 0, \\ B_1\varphi_1'(t_2)+B_2\varphi_2'(t_2)+B_3\varphi_3'(t_2) &= 0, \end{aligned} \tag{1.13}$$

where

$$B_1 = \frac{\partial\Phi}{\partial u_1} = (m+n)\sigma, \quad B_2 = \frac{\partial\Phi}{\partial u_2} = \frac{m}{u_2}, \quad B_3 = \frac{\partial\Phi}{\partial u_3} = \frac{m}{u_3}. \tag{1.14}$$

By (1.12) and (10) we have

$$
\begin{aligned}
u_1 &= \lambda_1\varphi_1(t_1)+\lambda_2\varphi_1(t_2),\\
u_2 &= \lambda_1\varphi_2(t_1)+\lambda_2\varphi_2(t_2),\\
u_3 &= \lambda_1\varphi_3(t_1)+\lambda_2\varphi_3(t_2),
\end{aligned}
\tag{1.15}
$$

with

$$\lambda_1+\lambda_2 = 1. \tag{1.16}$$

It can easily be proved that the function $\mu_0(t)$ has exactly one jump-point. Denote this point by t_1. Let $f^*(z)$ be the extremal function in relation to functional (1.1), i.e. such a function that

$$L(f^*) \geqslant L(f), \quad \text{or} \quad L(f^*) \leqslant L(f)$$

for each $f(z)\in S_a^*(k)$.

Since $\mu_0(t)$ has one jump-point t_1, the system of equations (1.13) can be reduced to a system of two equations, where t_1 is the root of the equation

$$B_1\varphi_1'(t)+B_2\varphi_2'(t)+B_3\varphi_3'(t) = 0 \tag{1.17}$$

with the unknown t to which the greatest value of the functional $L(f)$ corresponds.

Simultaneously we have

$$u_1 = \varphi_1(t_1), \quad u_2 = \varphi_2(t_1), \quad u_3 = \varphi_3(t_1). \tag{1.18}$$

By (1.14) and (1.18) equation (1.17) becomes

$$(m+n)\sigma\varphi_1'(t)+\frac{m\varphi_2'(t)}{\varphi_2(t)}+\frac{m\varphi_3'(t)}{\varphi_3(t)} = 0. \tag{1.19}$$

Substituting $\varphi_j(t)$ and $\varphi_j'(t)$ $(j = 1,2,3)$ from (1.7) and (1.8) to equation (1.19) we obtain after some transformations the equation

$$
\begin{aligned}
\sin t\cdot[2\sigma T_1T(m+n)\cos t+\sigma T(m+n)(1+T_1^2)+\\
+m(T_1+T)(1+T_1T)] = 0.
\end{aligned}
\tag{1.20}
$$

By (1.5), (1.18) and (1.7) we get

$$L^* = L(f^*) = \frac{|1+T_1e^{it_1}|^m}{|1-Te^{it_1}|^{(m+n)\delta+m}}. \tag{1.21}$$

In our further study we shall distinguish two cases: 1° $m+n = 0$, 2° $m+n \neq 0$.

In case 1° functional (1.1) takes the form

$$L(f) = \left|\frac{zf'(z)}{f(z)}\right|^m, \tag{1.22}$$

in case 2° it becomes

$$(1.23)\qquad L(f) = \left|\frac{f(z)^{1-\mu}\cdot f'(z)^{\mu}}{z^{1-\mu}}\right|^{m+n}, \quad \mu = \frac{m}{m+n}.$$

In what follows without any loss of generality we may restrict ourselves to the study of functionals

$$(1.24)\qquad E(f) = \left|\frac{zf'(z)}{f(z)}\right|$$

and

$$(1.25)\qquad H(f) = \left|\frac{f(z)^{1-\mu}\cdot f'(z)^{\mu}}{z^{1-\mu}}\right|$$

and to take into account in the final results their relation to functional (1.1): $L(f) = [E(f)]^m$ in the case $m+n = 0$ and $L(f) = [H(f)]^{m+n}$ if $m+n \neq 0$.

Now we shall find the extremal values of functional (1.24). In this case equation (1.20) takes the form

$$(1.26)\qquad (T_1+T)(1+T_1T)\cdot\sin t = 0$$

and the extremal values of functional (1.24) according to formula (1.21) are given by the equalities

$$(1.27)\qquad E_s(f^*) = \left|\frac{1+T_1e^{it_s}}{1-Te^{it_s}}\right|$$

where t_s, $t_s \epsilon (0, 2\pi\rangle$ are the roots of equation (1.26). From (1.26) we have $\sin t = 0$, so

$$(1.28)\qquad e^{it_1} = 1 \quad\text{or}\quad e^{it_2} = -1, \quad t_1 \text{ and } t_2 \epsilon (0, 2\pi\rangle.$$

Thus we ultimately obtain the following extremal values of functional (1.24):

$$(1.29)\qquad E_1^* = \frac{1+T_1}{1-T} \quad\text{and}\quad E_2^* = \frac{1-T_1}{1+T}.$$

Now we proceed to functional (1.25) for which equation (1.20) takes the form

$$(1.30)\qquad \sin t\cdot[2\sigma T_1T\cos t + \sigma T(1+T_1^2) + \mu(T_1+T)(1+T_1T)] = 0.$$

To each root t_s, $0 < t_s \leqslant 2\pi$ of equation (1.30) there corresponds an extremal value of functional (1.25) given by the formula

$$(1.31)\qquad H_s^* = \frac{|1+T_1e^{it_s}|^{\mu}}{|1-Te^{it_s}|^{\sigma+\mu}}.$$

Analysing equation (1.30) we find its solutions and the extremal values of the corresponding functional (1.25). Thus we arrive at conclusions formulated in the theorem we formulate next.

Let, according to formula (1.31),

$$H_s^* = \frac{|1+T_1 e^{i\arccos x_s}|^{\mu}}{|1-Te^{i\arccos x_s}|^{\sigma+\mu}}, \quad s = 1, 2, 3, \tag{1.32}$$

where $x_1 = 1$, $x_2 = -1$ and x_3 is a root of the equation

$$g(x) = 2\sigma T_1 Tx + \sigma T(1+T_1^2) + \mu(T_1+T)(1+T_1 T) = 0, \tag{1.33}$$

where $T = r^k$, $T_1 = (1-2a)T$, $\sigma = 2(1-a)/k$, $r = |z|$.

Let, moreover,

$$a_j = a_j(r, k, a) = \frac{[1-(-1)^j \cdot T_1]^2}{k(1+T_1 T)}, \quad j = 1, 2. \tag{1.34}$$

For fixed r, k, a we denote by I_l and J_l $(l = 1, 2, 3, 4, 5)$, respectively, the intervals of values of the parameter μ:

I_l: $(-\infty, -\sigma\rangle$, $(-\sigma, -a_1\rangle$, $(-a_1, -a_2)$, $\langle -a_2, 0)$, $\langle 0, \infty)$ for $0 \leqslant a < \frac{1}{2}$ and

J_l: $(-\infty, -a_2\rangle$, $(-a_2, -a_1)$, $\langle -a_1, -\sigma)$, $\langle -\sigma, 0)$, $\langle 0, \infty)$ for $\frac{1}{2} \leqslant a < 1$.

THEOREM 1. *The following sharp estimations of the functionals $E(f)$ and $H(f)$ hold in the family $S_a^*(k)$:*

$$E_2^* \leqslant E(f) \leqslant E_1^* \tag{1.35}$$

and

$$\begin{aligned} &H_2^* \leqslant H(f) \leqslant H_1^* && \text{for } \mu \in I_5 \cup J_4 \cup J_5, \\ &H_1^* \leqslant H(f) \leqslant H_2^* && \text{for } \mu \in I_1 \cup J_1 \cup J_3, \\ &\hat{H}_1^* \leqslant H(f) \leqslant \hat{H}_2^* && \text{for } \mu \in I_2 \cup I_4, \\ &\tilde{H}_1^* \leqslant H(f) \leqslant \tilde{H}_2^* && \text{for } \mu \in I_3 \cup J_2, \end{aligned} \tag{1.36}$$

where

$$\hat{H}_1^* = \min[H_1^*, H_2^*], \qquad \hat{H}_2^* = \max[H_1^*, H_2^*],$$
$$\tilde{H}_1^* = \min[H_1, H_2, H_3], \qquad \tilde{H}_2^* = \max[H_1^*, H_2^*, H_3^*].$$

The equality signs in estimations (1.35) *and* (1.36) *hold on the circumference $|z| = r$ for the functions*

$$f_s^*(z) = \frac{z}{(1-z^k e^{i\arccos x_s})^{\sigma}}, \quad s = 1, 2, 3,$$

respectively.

COROLLARY. *If* $f(z)\in S_a^*(k)$, *then*

$$\frac{r}{(1+r^k)^\sigma}\leqslant |f(z)|\leqslant\frac{r}{(1-r^k)^\sigma}, \tag{1.37}$$

$$\frac{1-(\sigma k-1)r^k}{(1+r^k)^{1+\sigma}}\leqslant |f'(z)|\leqslant\frac{1+(\sigma k-1)r^k}{(1-r^k)^{1+\sigma}}, \tag{1.38}$$

estimations (1.37) *and* (1.38) *being sharp.*

In particular, hence follow the known estimations of this function in the class S^* and more general in the class S.

B. Denote by $\Sigma(k)$ the class of functions of the form

$$u = F(\zeta) = \zeta+\sum_{n=1}^{\infty}\frac{a_{nk-1}}{z^{nk-1}} \tag{1.39}$$

regular and univalent in the circle $|\zeta|>1$ for $\zeta\neq\infty$ with a single pole at the point $\zeta=\infty$ and satisfying the condition $F(\zeta)\neq 0$, $z\in G$, $G=\{\zeta: \infty>|\zeta|>1\}$. Let $\Sigma^*(k)$ denote the subclass of starlike functions of the family $\Sigma(k)$ i.e. the subclass of functions mapping the region G onto a region whose complement to the plane is a starlike region with respect to the point $w=0$. A function $F(\zeta)$ belongs to the family $\Sigma^*(k)$ if and only if

$$F(\zeta)=\frac{1}{f(1/\zeta)} \tag{1.40}$$

for some function $f(z)$ of the family $S^*(k)$.

Functions $F(\zeta)$ of the family $\Sigma^*(k)$ are called *starlike of order* a in the circle G, if

$$\operatorname{re}\frac{\zeta F'(\zeta)}{F(\zeta)}>a \quad \text{for every } \zeta\in G. \tag{1.41}$$

Denote by $\Sigma_a^*(k)$ the family of functions $F(\zeta)$ of form (1.39) satisfying condition (1.41). Obviously $F(\zeta)\in\Sigma_a^*(k)$ if and only if $F(\zeta)=1/f(1/\zeta)$, $f(z)\in S_a^k(k)$.

From Theorem 1 we obtain immediately a theorem on estimation of the functionals

$$\theta(F)=\left|\frac{\zeta F'(\zeta)}{F(\zeta)}\right| \tag{1.42}$$

and

$$\Delta(F)=\left|\frac{\zeta^{\mu+1}\cdot F'(\zeta)^\mu}{F(\zeta)^{\mu+1}}\right| \tag{1.43}$$

in the class $\Sigma_a^*(k)$.

In fact, let

$$\theta_j^* = \frac{\varrho^k - (-1)^j \beta}{\varrho^k + (-1)^j}, \quad j = 1, 2 \tag{1.44}$$

and

$$\Delta_s^* = \frac{|\varrho^k + \beta e^{i \arccos x_s}|^{\mu} \cdot \varrho^{\beta+1}}{|\varrho^k - e^{i \arccos x_s}|^{\sigma+\mu}}, \quad s = 1, 2, 3, \tag{1.45}$$

where $x_1 = 1$, $x_2 = -1$ and x_3 is a root of the equation

$$h(x) = 2\sigma\beta\varrho^k x + \sigma(\varrho^{2k} + \beta^2) + \mu(1+\beta)(\varrho^{2k} + \beta) = 0, \tag{1.46}$$

with $\beta = 1 - 2a$, $\varrho = |\zeta| > 1$, $T = 1/\varrho^k$. $T_1 = \beta T$.

Let I_l and J_l $(l = 1, 2, 3, 4, 5)$ denote the same intervals of values of the parameter μ as in Theorem 1.

THEOREM 2. *The following sharp estimations hold in the family* $\Sigma_a^*(k)$:

$$\theta_2^* \leqslant \theta(F) \leqslant \theta_1^*, \tag{1.47}$$

and

$$\begin{aligned} \Delta_2^* \leqslant \Delta(F) \leqslant \Delta_1^* \quad &\textit{for } \mu \in I_5 \cup J_4 \cup J_5, \\ \Delta_1^* \leqslant \Delta(F) \leqslant \Delta_2^* \quad &\textit{for } \mu \in I_1 \cup J_1 \cup J_3, \\ \hat{\Delta}_1^* \leqslant \Delta(F) \leqslant \hat{\Delta}_2^* \quad &\textit{for } \mu \in I_2 \cup I_4, \\ \tilde{\Delta}_1^* \leqslant \Delta(F) \leqslant \tilde{\Delta}_2^* \quad &\textit{for } \mu \in I_3 \cup J_2, \end{aligned} \tag{1.48}$$

$$\hat{\Delta}_1^* = \min[\Delta_1^*, \Delta_2^*], \qquad \hat{\Delta}_2^* = \max[\Delta_1^*, \Delta_2^*],$$
$$\tilde{\Delta}_1^* = \min[\Delta_1^*, \Delta_2^*, \Delta_3^*], \qquad \tilde{\Delta}_2^* = \max[\Delta_1^*, \Delta_2^*, \Delta_3^*].$$

The equalities in estimations (1.47) *and* (1.48) *occur for the circumference* $|\zeta| = \varrho > 1$ *for functions*

$$F_s^* = \frac{(\zeta^k - e^{i \arccos x_s})^{\sigma}}{\zeta^{\beta}}, \quad s = 1, 2, 3,$$

respectively.

COROLLARY. *If the function* $F(\zeta) \in \Sigma_a^*(k)$, *then*

$$\frac{(\varrho^k - 1)^{\sigma}}{\varrho^{\beta}} \leqslant |F(\zeta)| \leqslant \frac{(\varrho^k + 1)^{\sigma}}{\varrho^{\beta}} \tag{1.49}$$

and

$$\frac{1}{\tilde{\Delta}_2^*} \leqslant |F'(\zeta)| \leqslant \frac{1}{\tilde{\Delta}_1^*} \tag{1.50}$$

for $0 \leqslant a < 1$, $k = 1$, $T \in (0, 1)$ *and for* $\frac{1}{2} < a < 1$, $k \geqslant 2$, $T > T_2$,

$$\frac{\varrho^k - \beta}{\varrho^{\beta+1}(\varrho^k + 1)^{1-\sigma}} \leqslant |F'(\zeta)| \leqslant \frac{\varrho^k + \beta}{\varrho^{\beta+1}(\varrho^k - 1)^{1-\sigma}} \tag{1.50'}$$

for $0 \leqslant a \leqslant \frac{1}{2}$, $k \geqslant 2$, $T \in (0,1)$ *and for* $\frac{1}{2} < a < 1$, $k \geqslant 2$ *and* $T \leqslant T_2$ *with*

$$\tilde{\Delta}_1^* = \min\left[\frac{\varrho^{\beta+1}}{(\varrho^k+\beta)(\varrho^k-1)^{\sigma-1}}, \frac{\varrho^{\beta+1}}{(\varrho^k-\beta)(\varrho^k+1)^{\sigma-1}}, \frac{\varrho^{\beta+1}}{|\varrho^k+\beta e^{i\arccos x_3}|\cdot|\varrho^k-e^{i\arccos x_3}|^{\sigma-1}}\right],$$

$$\tilde{\Delta}_2^* = \max\left[\frac{\varrho^{\beta+1}}{(\varrho^k+\beta)(\varrho^k-1)^{\sigma-1}}, \frac{\varrho^{\beta+1}}{(\varrho^k-\beta)(\varrho^k+1)^{\sigma-1}}, \frac{\varrho^{\beta+1}}{|\varrho^k+\beta e^{i\arccos x_3}|\cdot|\varrho^k-e^{i\arccos x_3}|^{\sigma-l}}\right],$$

x_3 *is a root of the equation*

$$2\beta\varrho^k x+[(\varrho^{2k}+\beta^2)-k(\varrho^{2k}+\beta)] = 0,$$

moreover,

$$T_2 = \frac{k}{k-\beta}\left[\sqrt{\frac{1-\sigma}{-\beta}}-\frac{1}{k}\right].$$

Estimations (1.49) *and* (1.50) *are sharp.*

Hence follow in particular the estimations of these functions in the class $\Sigma^*(k)$ and in the class Σ^*:

(1.51) $$\frac{(\varrho-1)^2}{\varrho} \leqslant |F(\zeta)| \leqslant \frac{(\varrho+1)^2}{\varrho}$$

and

(1.52) $$\frac{\varrho^2-1}{\varrho^2} \leqslant |F'(\zeta)| \leqslant \frac{\varrho^2+1}{\varrho^2}.$$

Equalities in estimations (1.51) and (1.52) occur on the circumference $|\zeta| = \varrho$ for functions

$$F_s^*(\zeta) = \frac{(\zeta-e^{i\arccos x_s})^2}{\zeta}, \quad s = 1, 2, 3,$$

respectively, with $x_1 = 1$, $x_2 = -1$, $x_3 = 0$.

2.A. Now we shall deal with functional

(2.1) $$K(f) = \arg\frac{f(z)^n\cdot f'(z)^m}{z^n}, \quad f(z)\in S_a^*(k),$$

where m and n are arbitrary fixed real numbers satisfying the condition $m^2+n^2 \neq 0$ and $\arg\frac{f(z)^n\cdot f'(z)^m}{z^n}$ is a singlevalued branch $U(z)$ of the multi-valued function $w(z) = \arg\frac{f(z)^n\cdot f'(z)^m}{z^n}$, such that $U(0) = 0$.

In the same manner as in the preceding section we assume without any loss of generality that $z = r > 0$. Since

$$(2.2)\qquad 2i\arg\frac{f(r)^n\cdot f'(r)^m}{r^n} = \log\frac{f(r)^n\cdot f'(t)^m}{r^n} - \log\overline{\frac{f(r)^n\cdot f'(r)^m}{r^n}}$$

$$= m\log f'(r) - m\log\overline{f'(r)} + n\log\frac{f(r)}{r} - n\log\overline{\frac{f(r)}{r}};$$

then by (4), (1.4) and (1.2) in (2.2) we obtain

$$(2.3)\ 2i\arg\frac{f(r)^n\cdot f'(r)^m}{r^n} = (m+n)\sigma\cdot u_1 + m\log u_2 - m\log u_3 = i\Psi(u_1, u_2, u_3),$$

where

$$(2.4)\ u_1 = \int_0^{2\pi}\frac{1-Te^{it}}{1-Te^{-it}}\,d\mu(t),\qquad u_2 = \int_0^{2\pi}\frac{1+T_1e^{-it}}{1-Te^{-it}}\,d\mu(t),\qquad u_3 = \bar{u}_2.$$

By (7) we obtain from (2.4)

$$(2.5)\qquad \varphi_1(t) = \log\frac{1-Te^{it}}{1-Te^{-it}},\qquad \varphi_2(t) = \frac{1+T_1e^{-it}}{1-Te^{-it}},\qquad \varphi_3(t) = \overline{\varphi_2(t)}.$$

Thus

$$(2.6)\ \varphi_1'(t) = \frac{-iT(e^{it}+e^{-it}-2T)}{(1-Te^{it})(1-Te^{-it})},\qquad \varphi_2'(t) = \frac{-ie^{-it}(T_1+T)}{(1-Te^{-it})^2},\qquad \varphi_3'(f) = \overline{\varphi_2'(t)}.$$

Then similarly as in the first section we prove that the extremal function $\mu_0(t)$ belongs to the family G_1. It has one jump-point; thus $u_l = \varphi_l(t_1)$ $(l = 1, 2, 3)$, t_1 being the abscissa of the jump-point. This abscissa is a root of the equation

$$(2.7)\qquad B_1\varphi_1'(t) + B_2\varphi_2'(t) + B_3\varphi_3'(t) = 0$$

in which the coefficients B_l $(l = 1, 2, 3)$ according to formula (1.4) for functional (2.3) take the form

$$(2.8)\qquad B_1 = \frac{1}{i}(m+n)\sigma,\qquad B_2 = \frac{1}{i}\frac{m}{u_2},\qquad B_3 = -\frac{1}{i}\frac{m}{u_3}.$$

By equalities $u_l = \varphi_l(t_1)$ $(l = 1, 2, 3)$ and formulae (2.5), (2.6) and (2.8) equation (2.7) becomes

$$(2.9)\quad g(x) = 2(m+n)\sigma T_1Tx^2 + [m(T_1+T)(1-T_1T) + \sigma T(m+n)(1+$$

$$+T_1^2 - 2T_1T)]x + [m(T_1^2-T^2) - (m+n)\sigma T^2(1+T_1^2)] = 0,$$

where $x = \cos t$.

To every root x_0, $-1 \leqslant x_0 \leqslant 1$ of equation (2.9) there correspond two extremal values K^* and $-K^*$, of functional (2.1), where

$$(2.10)\quad K^* = [(m+n)\sigma+m]\arctan\frac{T\sin(\arccos x_0)}{1-T\cos(\arccos x_0)}+ \\ +m\arctan\frac{T_1\sin(\arccos x_0)}{1+T_1\cos(\arccos x_0)}$$

and $0 \leqslant \arccos x_0 \leqslant \pi$.

In fact by (2.3) and $u_1 = \varphi_1(t_1)$ we obtain successively

$$K^* = \frac{(m+n)\sigma}{2}\arg\frac{1-Te^{it_1}}{1-Te^{-it_1}}+\frac{m}{2}\arg\frac{1+T_1e^{-it_1}}{1+T_1e^{it_1}}+\frac{m}{2}\arg\frac{1-Te^{it_1}}{1-Te^{-it_1}}$$

and

$$K^* = [(m+n)\sigma+m]\arg(1-Te^{it_1})+m\arg(1+T_1e^{-it_1}).$$

Hence easily (2.10) follows.

Similarly as in the first section we distinguish in our further study two cases:

$$1^\circ\ m+n = 0 \text{ and } 2^\circ\ m+n \neq 0.$$

In case 1° functional (2.1) takes the form

$$(2.11)\qquad K(f) = m\left[\arg\frac{zf'(z)}{f(z)}\right],$$

and in case 2° it becomes

$$(2.12)\qquad K(f) = (m+n)\left[\arg\frac{f(z)^{1-\mu}\cdot f'(z)^{\mu}}{z^{1-\mu}}\right],\qquad \mu = \frac{m}{m+n}.$$

Thus without any loss of generality in the sequel it sufficient to study, the functionals

$$(2.13)\qquad M(f) = \arg\frac{zf'(z)}{f(z)}$$

and

$$(2.14)\qquad N(f) = \arg\frac{f(z)^{1-\mu}\cdot f'(z)^{\mu}}{z^{1-\mu}}$$

and to taking into account relations (2.1) and (2.11)-(2.12) in the final results.

Consider now functional (2.13). In this case equation (2.9) takes the form

$$(2.15)\qquad g_1(x) = (T_1+T)(1-T_1T)x+T_1^2-T^2 = 0,\qquad x = \cos t$$

and has always a solution

$$(2.16) \qquad x_0 = \frac{T-T_1}{1-T_1T}$$

in the interval $\langle -1, 1\rangle$. Thus the extremal values of functional (2.13) are equal to M_0^* and $-M_0^*$, where

$$(2.17) \quad M_0^* = \arctan\frac{T\sin(\arccos x_0)}{1-T\cos(\arccos x_0)} + \arctan\frac{T_1\sin(\arccos x_0)}{1+T_1\cos(\arccos x_0)}.$$

Proceed now to functional (2.14). Equation (2.9) takes now the form

$$(2.18) \quad g_2(x) = 2\sigma T_1Tx^2 + [\mu(T_1+T)(1-T_1T)+\sigma T(1+T_1^2-2T_1T)]x + \\ + [\mu(T_1^2-T^2)-\sigma T^2(1+T_1^2)] = 0.$$

We have

$$(2.19) \quad \begin{aligned} g_2(0) &= 4a(1-a)T^2\cdot[-\mu-b_1], \\ g_2(1) &= 2(1-a)T(1-T)(1+T_1)[\mu+b_2], \\ g_2(-1) &= 2(1-a)T(1+T)(1-T_1)[-\mu-b_3], \end{aligned}$$

where

$$(2.20) \quad b_1(r,k,a) = \frac{1+T_1^2}{2ak}, \quad b_j(r,k,a) = \frac{1+(-1)^j\cdot T_1}{k}, \quad j = 2, 3.$$

Analysing equation (2.18) we find its solutions and the corresponding extremal values of functional (2.14) and arrive eventually at the conclusions formulated in Theorem 3.

For fixed values of r, k, a denote respectively by A_s, B_s, C_s ($s = 1, 2, 3, 4, 5$) the intervals of values of the parameter μ:

A_s: $(-\infty, -\sigma\rangle$, $(-\sigma, -b_2\rangle$, $(-b_2, -b_3)$, $\langle -b_3, 0)$, $\langle 0, \infty)$, with $0 \leqslant a \leqslant \frac{1}{2}$ and $0 < T < 1$,

B_s: $(-\infty, -b_3\rangle$, $(-b_3, -b_2)$, $\langle -b_2, -\sigma)$, $\langle -\sigma, 0)$, $\langle 0, \infty)$, with $\frac{1}{2} < a < 1$ and $0 < T \leqslant \sqrt{2}-1$ and with $\frac{1}{2} < a < a_2 = \dfrac{1+T^2}{2T(1+T)}$ and $\sqrt{2}-1 < T < 1$,

C_s: $(-\infty, -b_3\rangle$, $(-b_3, -b_2)$, $\langle -b_2, -\sigma)$, $\langle -\sigma, 0)$, $\langle 0, \infty)$ with $a_2 \leqslant a < 1$ and $\sqrt{2}-1 < T < 1$.

THEOREM 3. *The following sharp estimations of the functionals $M(f)$ and $N(f)$ hold in the family $S_a^*(k)$:*

$$(2.21) \qquad |M(f)| \leqslant M_0^*$$

and

$$(2.22)\quad \begin{aligned} &|N(f)| \leqslant N_1^* && \text{for } \mu \in A_5 \cup B_4 \cup B_5 \cup C_4 \cup C_5, \\ &|N(f)| \leqslant -N_1^* && \text{for } \mu \in A_1, \\ &|N(f)| \leqslant \hat{N}_1^* && \text{for } \mu \in A_2 \cup A_4 \cup B_1 \cup B_3 \cup C_1 \cup C_3, \\ &|N(f)| \leqslant \tilde{N}^* && \text{for } \mu \in A_3 \cup B_2 \cup C_2, \end{aligned}$$

with

$$\hat{N}_1^* = \max[N_1^*, -N_1^*], \qquad \tilde{N}^* = \max[N_1^*, N_2^*, -N_1^*, -N_2^*],$$

where

$$N_s^* = (\sigma+\mu)\arctan\frac{T\sin(\arccos x_s)}{1-T\cos(\arccos x_s)} + \mu\arctan\frac{T_1\sin(\arccos x_s)}{1+T_1\cos(\arccos x_s)}, \qquad s = 1, 2,$$

$x_s, x_s \in \langle -1, 1\rangle$ $(s = 1, 2)$ *are the roots of the equation*

$$(2.23)\quad g_2(x) = 2\sigma T_1 T x^2 + [\mu(T_1+T)(1-T_1T) + \sigma T(1+T_1^2-2T_1T)]x + \\ + [\mu(T_1^2-T^2) - \sigma T^2(1+T_1^2)] = 0.$$

If $\mu \in A_3 \cup B_2 \cup C_2$ equation (2.23) has two different roots $x_s \in \langle -1, 1\rangle$ $(s = 1, 2)$; in the remaining intervals it has one root x_1.

In estimations (2.21) and (2.22) equalities hold on the circumference $|z| = r$ for the functions

$$f^*(z) = \frac{z}{\left(1 - z^k e^{i\arccos\frac{T-T_1}{1-T_1T}}\right)^\sigma} \quad \text{and} \quad f_s^*(z) = \frac{z}{(1-z^k e^{i\arccos x_s})^\sigma}, \quad s = 1, 2.$$

Taking $\mu = 1 \in A_5$ and then $\mu = 0 \in A_5$ we obtain from Theorem 3 the following

COROLLARY. *If* $f(z) \in S_a^*(k)$, *then*

$$(2.24)\quad |\arg f'(z)| \leqslant (1+\sigma)\arctan\frac{T\sin(\arccos x_0)}{1-T\cos(\arccos x_0)} + \\ + \arctan\frac{T_1\sin(\arccos x_0)}{1+T_1\cos(\arccos x_0)}$$

(cf. [4]) *and*

$$(2.25)\quad \left|\arg\frac{f(z)}{z}\right| \leqslant \sigma\arctan\frac{T\sin(\arccos x_0)}{1-T\cos(\arccos x_0)},$$

estimations (2.24) *and* (2.25) *being sharp.*

In particular from estimations (2.21) and (2.25) we obtain for $a = 0$ the known estimations in the family $S^*(k)$:

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqslant 2\arctan T = \arcsin\frac{2T}{1+T^2}$$

(cf. [1]) and

$$\left|\arg\frac{f(z)}{z}\right| \leqslant \frac{2}{k}\arctan\frac{T\sin(\arccos T)}{1-T\cos(\arccos T)} = \frac{2}{k}\arcsin T,$$

(cf. [2]).

B. Making use of the correspondence between the classes $S_a^*(k)$ and $\Sigma_a^*(k)$ we obtain immediately from Theorem 3 the theorem on the estimation of functionals

$$\Psi(F) = \arg\frac{\zeta F'(\zeta)}{F(\zeta)} \tag{2.26}$$

and

$$\chi(F) = \arg\frac{\zeta^{\mu+1}\cdot F'(\zeta)^\mu}{F(\zeta)^{\mu+1}} \tag{2.27}$$

in the class of functions $\Sigma_a^*(k)$. Let A_s, B_s and C_s ($s = 1, 2, 3, 4, 5$) denote the same intervals of the parameter μ, as in Theorem 3.

THEOREM 4. *In the family of functions $\Sigma_a^*(k)$ the following sharp estimations*

$$|\Psi(F)| \leqslant \Psi_0^* \tag{2.28}$$

hold, where

$$\Psi_0^* = \arctan\frac{\sin(\arccos x_0)}{\varrho^k - \cos(\arccos x_0)} + \arctan\frac{\beta\sin(\arccos x_0)}{\varrho^k + \beta\cos(\arccos x_0)},$$

x_0 is the root of the equation

$$\hat{g}_1(x) = (\beta+1)(\varrho^{2k}-\beta)x+\beta^2-1 = 0$$

and

$$\begin{aligned} &|\chi(F)| \leqslant \chi_1^* && \text{for } \mu\in A_5 \cup B_4 \cup B_5 \cup C_4 \cup C_5,\\ &|\chi(F)| \leqslant -\chi_1^* && \text{for } \mu\in A_1,\\ &|\chi(F)| \leqslant \hat{\chi}_1^* && \text{for } \mu\in A_2 \cup A_4 \cup B_1 \cup B_3 \cup C_1 \cup C_3,\\ &|\chi(F)| \leqslant \tilde{\chi}^* && \text{for } \mu\in A_3 \cup B_2 \cup C_2, \end{aligned} \tag{2.29}$$

with

$$\hat{\chi}_1^* = \max[\chi_1^* - \chi_1^*], \quad \tilde{\chi}^* = \max[\chi_1^*, \chi_2^*, -\chi_1^*, -\chi_2^*],$$

where

$$\chi_s^* = (\sigma+\mu)\arctan\frac{\sin(\arccos x_s)}{\varrho^k-\cos(\arccos x_s)}+\mu\arctan\frac{\beta\sin(\arccos x_s)}{\varrho^k+\beta\cos(\arccos x_s)},$$

$\beta = 1-2a$, $\varrho = |\zeta| > 1$, $T = 1/\varrho^k$, $T_1 = \beta T$ *and* $x_s, x_s \in \langle -1, 1\rangle$ $(s = 1, 2)$ *are the roots of the equation*

$$(2.30)\quad \hat{g}_2(x) = 2\sigma\beta\varrho^{2k}x^2+[\mu\varrho^k(\beta+1)(\varrho^{2k}-\beta)+\sigma\varrho^k(\varrho^{2k}+\beta^2-2\beta)]x+ \\ +[\mu\varrho^{2k}(\beta^2-1)-\sigma(\varrho^{2k}+\beta^2)] = 0.$$

If $\mu\in A_3 \cup B_2 \cup C_2$, then equation (2.30) has two different roots $x_s \in\langle -1, 1\rangle$ $(s = 1, 2)$. In the remaining intervals it has one root x_1.

Equalities occur in estimations (2.28) and (2.29) on the circumference $|\zeta| = \varrho$ for the functions

$$F_s^*(\zeta) = \frac{(\zeta^k-e^{i\arccos x_s})^\sigma}{\zeta^\beta},\quad s = 0, 1, 2.$$

From Theorem 4 we obtain for $\mu = 0$ and $\mu = -1$ the following

COROLLARY. *If* $F(\zeta)\in\Sigma_a^*(k)$, *then*

$$(2.31)\qquad \left|\arg\frac{F(\zeta)}{\zeta}\right| \leqslant \sigma\arctan\frac{\sin(\arccos x_0)}{\varrho^k-\cos(\arccos x_0)},$$

where $x_0, x_0\in\langle -1, 1\rangle$ *is a root of the equation*

$$2\beta\varrho^{2k}x^2+\varrho^k(\varrho^{2k}+\beta^2-2\beta)x-(\varrho^{2k}+\beta^2) = 0,\quad x = \cos t$$

and

$$(2.32)\qquad \begin{aligned} &|\arg F'(\zeta)| \leqslant \tilde{\chi}^* && \text{for } 0\leqslant a<1 \text{ and } k = 1,\\ &|\arg F'(\zeta)| \leqslant -\chi_1^* && \text{for } 0\leqslant a<\tfrac{1}{2} \text{ and } k\geqslant 2,\\ &|\arg F'(\zeta)| \leqslant \hat{\chi}_1^* && \text{for } \tfrac{1}{2}\leqslant a<1 \text{ and } k\geqslant 2,\end{aligned}$$

where

$$\hat{\chi}_1^* = \max[\chi_1^*, -\chi_1^*],\qquad \tilde{\chi}^* = \max[\chi_1^*, \chi_2^*, -\chi_1^*, -\chi_2^*],$$

$$\chi_s^* = -(1-\sigma)\arctan\frac{\sin(\arccos x_s)}{\varrho^k-\cos(\arccos x_s)}-\arctan\frac{\beta\sin(\arccos x_s)}{\varrho^k+\beta\cos(\arccos x_s)}$$

and $x_s, x_s\in\langle -1, 1\rangle$ $(s = 1, 2)$ *are the roots of the equation*

$$2\sigma\beta\varrho^{2k}x^2-[\varrho^k(\beta+1)(\varrho^{2k}-\beta)-\sigma\varrho^k(\varrho^{2k}+\beta^2-2\beta)]x-[\varrho^{2k}(\beta^2-1)+ \\ +\sigma(\varrho^{2k}+\beta^2)] = 0,$$

estimations (2.31) *and* (2.32) *being sharp.*

In particular for $a = 0$ we obtain the sharp estimations in the family $\Sigma^*(k)$:

$$\left|\arg\frac{F(\zeta)}{\zeta}\right| \leqslant \frac{2}{k}\arcsin\frac{1}{\varrho^k},\qquad \left|\arg\frac{\zeta F'(\zeta)}{F(\zeta)}\right| \leqslant 2\arctan\frac{1}{\varrho^k}.$$

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