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ATTRACTORS OF MAPS OF THE INTERVAL

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§ 1. Statement of results

Let Λ be a class of mappings $f\colon [0, 1] \hookleftarrow$ satisfying the following conditions:

(1) $f \in C^3$,
(2) f *has finitely many critical points,*
(3) f has negative Schwarzian derivative

$$Sf = f'''/f' - \frac{3}{2}(f''/f')^2 < 0 \quad \text{outside critical points.}$$

In the present paper, we study the typical (with respect to Lebesgue measure λ) behaviour of orbits $\{f^n(x)\}_{n=0}^{\infty}$ of a map $f \in \Lambda$. The majority of papers on this subject are devoted to *unimodal maps*, i.e. maps having one critical point (see [1], [2] containing further references).

Let $\omega(x)$ denote the limit set of an orbit $\{f^n(x)\}_{n=0}^{\infty}$. Following Milnor [3], by an *attractor* we will mean a compact set $A \subset [0, 1]$ such that the set $\varrho(A) = \{x\colon \omega(x) \subset A\}$ has positive Lebesgue measure λ and, moreover, $\lambda(\varrho(A) \setminus \varrho(A')) > 0$ for any proper closed subset $A' \subset A$. An attractor is called *indecomposable* if it is not the union of strictly smaller attractors, it is *minimal* if it does not contain strictly smaller attractors.

An interval $I \subset [0, 1]$ is *periodic* if $f^p(I) \subset I$; in this case $\bigcup_{k=0}^{p-1} f^k(I)$ is called a *cycle of intervals*. If additionally the map $f^p\colon I \hookleftarrow$ is topologically transitive (i.e. has a dense orbit), then I is called a *transitive* (*periodic*) interval.

A set $X \subset [0, 1]$ is called *wandering* if $f^n(X) \cap X = \emptyset$ ($n = 1, 2, \ldots$). A map f is *conservative* if it has no wandering sets of positive measure, and *ergodic* if there is no decomposition $[0, 1] = X_1 \cup X_2$, where X_i are disjoint invariant sets of positive measure.

Finally, a point x is called *recurrent* if $x \in \omega(x)$. Now we may formulate the main results of the present paper.

THEOREM 1. *Let $f \in \Lambda$. Then for a.e. $x \in [0, 1]$ one of the following statements holds*:

(a) *$\omega(x)$ is a periodic orbit,*

(b) *$\omega(x)$ is the cycle of a transitive interval I of period say p. The map f^p: $I \hookleftarrow$ is conservative and ergodic,*

(c) *$\omega(x) = \omega(c)$ for some recurrent critical point c.*

Thus, there are attractors of three kinds. By Singer's theorem (see [1]) each attractor of kind (a) attracts some critical point or endpoint of $[0, 1]$. Each attractor of kind (b) contains some critical point and does not contain strictly smaller attractors of kind (c) (the last follows from the proof, see § 2). So we obtain the following result which provides the answer to some Milnor's questions [3]:

COROLLARY 1. (a) *A map $f \in \Lambda$ with d critical points has at most $d+2$ indecomposable attractors.*

(b) *Let $f \in \Lambda$ be a unimodal map with a critical point c and f: $c \to 1 \to 0$. Then f has a unique attractor A and $\omega(x) = A$ for a.e. x.*

An interval $J \subset [0, 1]$ is called a *homterval* if

(1) the iterates f^n have no critical points inside J ($n \in \mathbf{N} \equiv \{0, 1, 2, \ldots\}$),

(2) $\omega(x)$ is infinite for $x \in J$.

COROLLARY 2. *Suppose that each recurrent critical point of $f \in \Lambda$ is periodic. Then*

(a) *f has no homtervals,*

(b) *each minimal attractor of f is a cycle of periodic points or a transitive interval,*

(c) *$\omega(x)$ coincides with some minimal attractor for a.e. $x \in [0, 1]$.*

THEOREM 2. *Let $f \in \Lambda$ be unimodal with critical point c and let $f^{(n)}(c) \neq 0$ for some $n > 0$. If f is topologically transitive, then f is ergodic.*

COROLLARY 3. *Let $f \in \Lambda$ be unimodal with critical point c and let $f^{(n)}(c) \neq 0$ for some $n > 0$. Then f has at most one absolutely continuous invariant measure. If f has such a measure μ, then the dynamical system (f, μ) is ergodic.*

We make use of the following properties of functions with negative Schwarzian derivative:

THE FIRST DISTORTION LEMMA. *Let* φ: $I \to J$ *be a monotone function,* $S\varphi < 0$. *Let* Y_1, Y_2 *be measurable sets in* J *such that* $\sup Y_1 \leqslant y \leqslant \inf Y_2$. *Then*

$$|(\varphi^{-1})'(y)| \leqslant \max_{i=1,2} \lambda(\varphi^{-1}(Y_i))/\lambda(Y_i).$$

THE SECOND DISTORTION LEMMA. *Let* φ: $I \to J$ *be a monotone function,* $S\varphi < 0$ E, *be a measurable set in* J, $0 < \eta < 1$. *Divide* J *into two intervals* J^- *and* J^+ *by a point* $y \in \operatorname{int}(J)$. *Then there exists an interval* $K_\eta = [y, z_\eta]$ *which is contained in* J^γ *for some* $\gamma \in \{\pm 1\}$ *such that*

$$\frac{\lambda(\varphi^{-1}(E \cap K_\eta))}{\lambda(\varphi^{-1}(K_\eta))} \geqslant \eta \frac{\lambda(E \cap J^\gamma)}{\lambda(J^\gamma)}. \tag{1}$$

The First Distortion Lemma easily follows from the properties of functions with negative Schwarzian derivative. The Second Distortion Lemma will be proved in § 4.

In § 2 a preliminary version of Theorem 1 is proved (Lemmas 2,3). The proof is completed in § 3 (Lemmas 4, 8), except ergodicity in the case (b). The last property and Theorem 2 are proved in § 4. The main tool there is the Second Distortion Lemma. We conclude the paper with some remarks concerning the general smooth case (§ 5).

§ 2. λ-wandering points and conservativity

By an interval we mean an open, closed or semi-open interval; (α, β) denotes an (open) interval with endpoints α, β without the assumption that $\alpha \leqslant \beta$.

We say that f is *monotone* on an interval I if $f'(x) \neq 0$ inside I. Let C be the set of critical points of f; $H_n(x)$ be the maximal closed interval containing x on which f^n is monotone. Put $M_n(x) = f^n(H_n(x))$. The endpoints of $M_n(x)$ belong to the set $\bigcup_{k=1}^{n} f^k(C \cup \{0, 1\})$. Let $r_n(x)$ be the distance from $f^n(x)$ to the nearest end-point of the interval $M_n(x)$. This function will play a crucial role below.

A point x is called *wandering* if it has a wandering neighbourhood. The set of nonwandering points is denoted by Ω. The structure of Ω for an arbitrary continuous map of the interval is described in [4]. This description implies

LEMMA 1. *Let* $f \in A$. *Then*

(a) $\operatorname{int}(\Omega)$ *is the union of all minimal transitive intervals* I_k,

(b) $\operatorname{int}(I_k) \cap \operatorname{int}(I_j) = \emptyset$ $(k \neq j)$ *and each* $\operatorname{orb}(I_k)$ *contains a critical point,*

(c) *for any interval* $J \subset \operatorname{orb}(I_k)$ *there exists* N *such that* $f^N(J) = I_k$.

The last property means that the map $f^p\colon I_k \hookleftarrow$ is mixing (where p is the period of the interval I_k).

A point x will be called *λ-wandering* if any neighbourhood of x contains a wandering set of positive measure. Let W_λ denote the set of λ-wandering points and Ω_λ be its complement. It is clear that W_λ is closed and $x \in W_\lambda \setminus f(C \cup \{0, 1\})$ implies $f^{-1}(x) \in W_\lambda$.

LEMMA 2. (a) *Ω_λ is the union of the interiors (in the intrinsic topology of* $[0, 1]$) *of some transitive intervals,*

(b) *the map $f\colon \Omega_\lambda \hookleftarrow$ is conservative,*

(c) *the limit set $\omega(x)$ for a.e. $x \in \Omega_\lambda$ coincides with the cycle of the transitive interval containing x.*

Proof. It is clear that $\Omega_\lambda \subset \operatorname{int}(\Omega)$. Let I be a transitive interval of period p, $L = \bigcup_{k=0}^{p-1} f^k(I)$. Suppose that L contains a wandering set X of positive measure. Let x be a density point of X which does not belong to $\bigcup_{n=1}^{\infty} f^n(C \cup \{0, 1\})$. Then x and all its preimages are λ-wandering. Since the set of preimages is dense in L (Lemma 1), we obtain $L \subset W_\lambda$ and (a), (b) follow.

Let $L = \bigcup_{k=0}^{p-1} f^k(I)$ be the cycle of transitive intervals such that $\operatorname{int}(I) \subset \Omega_\lambda$. Consider an open interval $J \subset L$ and the invariant compact set $K = L \setminus \bigcup_{n=0}^{\infty} f^{-n}(J)$. By Lemma 1 the set K is nowhere dense. Suppose that $\lambda(K) > 0$. Consider any interval $R \subset L \setminus K$. Lemma 1 implies that $\lambda(f^n(R) \cap K) > 0$ for some n. Choose minimal such n and put $S = R \cap f^{-n}(K)$. It is easy to see that the set $T = S \setminus \bigcup_{i=1}^{n-1} f^{-i}(S)$ wanders and has positive measure. This contradicts the conservativity of $f\colon L \hookleftarrow$. Hence $\lambda(K) = 0$. To finish the proof consider a countable base of intervals J_s and the corresponding sets K_s. We have $\lambda(\bigcup K_s) = 0$ and $\omega(x) = L$ for $x \in [0, 1] \setminus (\bigcup K_s)$.

LEMMA 3 (cf. [5]). *For a.e. $x \in [0, 1]$ one of the following statements holds:*

(a) $r_n(x) \to 0$ $(n \to \infty)$;

(b) *$f^N(x) \in \Omega_\lambda$ for some N.*

Proof. Fix $\varepsilon > 0$ and $0 < \varrho < \varepsilon$. By $V(\alpha)$ denote the complement to the α-neighbourhood of Ω_λ. Divide $V(\varepsilon - \varrho)$ into intervals J_k such that $\lambda(J_k) < \varrho/2$. Each J_k contains a wandering set S_k of positive measure. Let $\delta = \min_k \lambda(S_k)$.

Now consider the set

$$X_n = X_n(\varepsilon, \varrho) = \{x: f^n(x) \in V(\varepsilon), r_n(x) \geqslant \varrho\}.$$

Connected components $X_{n,j}$ of X_n are intervals of the form $X_n \cap H_{n,j}$, where $H_{n,j}$ are maximal intervals on which f^n is monotone. The interval $f^n(X_{n,j})$ is contained in $M_{n,j} = f^n(H_{n,j})$ and the length of each component $I^\gamma_{n,j}$ ($\gamma = \pm 1$) of $M_{n,j} \backslash f^n(X_{n,j})$ is at least ϱ. Hence each $I^\gamma_{n,j}$ contains some interval $J_k \equiv J_{n,j}$ and the corresponding set $S_k \equiv S^\gamma_{n,j}$, where $k = k(n, j, \gamma)$. Let f_j^{-n}: $M_{n,j} \to H_{n,j}$ be the branch of the inverse function. The First Distortion Lemma implies

$$\lambda(X_{n,j}) < \max_{\gamma = \pm 1} \frac{\lambda(f_j^{-n}(S^\gamma_{n,j}))}{\delta}.$$

Summation over j and n gives

$$\sum_{n=1}^{\infty} \lambda(X_n) \leqslant \frac{1}{\delta} \sum_k \sum_{n=1}^{\infty} \lambda(f^{-n}(S_k)).$$

Since S_k wanders, the last sum us finite. By Borel–Cantelli lemma,

$$\lambda\{x \mid \exists\, n_k \to \infty: x \in X_{n_k}\} = 0.$$

Hence for a.e. x the following property is valid:

Property P. *If* $n_k \to \infty$ *and* $\inf \operatorname{dist}(f^{n_k}(x), \Omega_\lambda) > 0$, *then* $r_{n_k}(x) \to 0$ $(k \to \infty)$.

It remains to check that if x satisfies Property P and $\operatorname{dist}(f^{l_k}(x), \Omega_\lambda) \to 0$ for some sequence $l_k \to \infty$ but $f^n(x) \notin \bar{\Omega}_\lambda$ $(n \in N)$, then $r_{l_k}(x) \to 0$ $(k \to \infty)$.

The set $\partial(\Omega_\lambda) \cap \omega(x) = \bar{\Omega}_\lambda \cap \omega(x)$ is finite and invariant. Hence, the problem reduces to the case when $f^{l_k}(x)$ tends to a periodic point $\alpha \in \partial(\Omega_\lambda)$. To complete the proof it remains to make a simple analysis of different cases (α is attracting, neutral, repelling). It will be omitted.

§ 3. Asymptotic behavior of orbits for which $r_n(x) \to 0$ $(n \to \infty)$

Results of this section preceding Lemma 8 concern an arbitrary smooth piecewise monotone map f: $[0, 1] \circlearrowleft$. Now we introduce notation which we use up to the end of the section. For a critical point $c \in C$ let $[c^-, c]$, $[c, c^+]$ denote maximal intervals (maybe degenerate) on which all iterates f^n are monotone. By c^γ for $\gamma \in \{\pm 1\}$ we denote one of the points c^+, c^-. Note that $\omega(c) = \omega(c^+) = \omega(c^-)$.

Let $C(x)$ be the set of $c \in C$ with the following property: for any $\varepsilon > 0$ there exist $n, t \in N$ $(t \leqslant n)$ such that c is an endpoint of the interval $f^{n-t}(H_n(x))$ and $|f^n(x) - f^t(c)| < \varepsilon$. If, moreover, c is the right-hand endpoint of $f^{n-t}(H_n(x))$, then we set $c^- \in C^-(x)$. Similarly we define the set $C^+(x)$.

Let $\beta > 0$. Then $p(\beta)$ denotes the least p for which f^p has a critical point $d^\gamma(c)$ in each interval $(c^\gamma, c^\gamma+\gamma\beta)$; $\delta = \delta(\beta)$ is some number for which the interval $I^\gamma(c, \beta, \delta) = (c^\gamma+\gamma\delta, c^\gamma+\gamma(\beta-\delta))$ also contains $d^\gamma(c)$. Finally,

$$\varepsilon(\beta) = \min\{\lambda(f^k(I^\gamma(c, \beta, \delta))): k = 0, \ldots, p(\beta)-1, c^\gamma \in C^+(x) \cup C^-(x)\}.$$

LEMMA 4. *Let $r_n(x) \to 0$ $(n \to \infty)$. Then one of the following statements holds:*

(a) $x \in \{0, 1\} \cup \bigcup_{n=0}^{\infty} f^{-n}(C)$,

(b) *x belongs to a homterval with endpoints from the set* $\{0, 1\} \cup \bigcup_{n=0}^{\infty} f^{-n}(C)$,

(c) *the orbit of x tends to an attracting or neutral cycle,*

(d) *each point $c^\gamma \in C^\gamma(x)$ belongs to $\omega(x)$; in this case if $c \neq c^\gamma$, then $[c, c^\gamma]$ is a homterval.*

Proof. Let $\beta > 0$, $c^\gamma \in C^\gamma(x)$ and assume for definiteness $\gamma = -1$. Find $n, t \in N$ $(t \leqslant n)$ such that c is the right-hand endpoint of $f^{n-t}(H_n(x))$ and $|f^n(x)-f^t(c)| < \varepsilon(\beta)$. If $f^{n-t}(x) \geqslant c^-$, one of the cases (a)–(c) holds.

Assume that $f^{n-t}(x) < c^-$ and show that

$$f^{n-t}(x) > c^- - \beta. \tag{2}$$

If $t < p(\beta)$, then the inequality $|f^t(c)-f^t(c^- - \beta)| \geqslant \varepsilon(\beta)$ implies (2). If $t \geqslant p(\beta)$ the function f^t is not monotone on the interval $(c^- - \beta, c)$. Since it is monotone on the interval $[f^{n-t}(x), c]$, we obtain (2) again. Since $\beta > 0$ is arbitrary, $c^- \in \omega(x)$.

It remains to note that $[c^-, c]$ is not a homterval only in the case when c^- is a neutral periodic point and $\{f^n(x)\}$ tends to its cycle. Then the case (c) takes place.

Let X be the set of x such that $r_n(x) \to 0$ $(n \to \infty)$ but x does not satisfy conditions (a)–(c) of Lemma 4. For a finite set $S \subset [0, 1]$ denote by S_* the set of maximal elements of S with respect to the following quasiordering: $x \prec y$ if $\omega(x) \subset \omega(y)$. Put $\omega(S) = \bigcup_{x \in S} \omega(x)$.

LEMMA 5. *If $x \in X$, then* $\omega(x) = \omega(C_*(x)) = \omega(C_*^+(x) \cup C_*^-(x))$.

Proof. It is easy to see that the definitions of X and $C(x)$ and the finiteness of the set of critical points of f imply that $\omega(x) \subset \omega(C(x) \cup \{0, 1\})$. If we cannot remove some $a \in \{0, 1\}$ from the right-hand side of the inclusion, then a is the endpoint of $H_n(x)$ for all $n \in N$. Then either $x = a$ or the orbit of x tends to a cycle, or $[a, x]$ is a homterval. All the possibilities are excluded for $x \in X$. Thus $\omega(x) \subset \omega(C(x)) = \omega(C_*(x))$.

On the other hand, by Lemma 4 $\omega(C_*^+(x) \cup C_*^-(x)) \subset \omega(x)$. It remains to remark that $\omega(C_*(x)) = \omega(C_*^+(x) \cup C_*^-(x))$ since $\omega(c) = \omega(c^+) = \omega(c^-)$ for any $c \in C$.

LEMMA 6. *Let $x \in X$. Then any point $c^\gamma \in C_*^\gamma(x)$ is recurrent.*

Proof. Lemmas 4–5 imply that $c^\gamma \in \omega(b)$ for some $b \in C(x)$. Hence $\omega(c) = \omega(c^\gamma) \subset \omega(b)$. Since c is a maximal element of the ordered set $C(x)$, we have $\omega(c^\gamma) = \omega(b)$. Consequently $c^\gamma \in \omega(c^\gamma)$.

There exist a number $\varrho > 0$ and a function $\sigma(x) > 0$ $(x \in X)$ with the following properties:

Property A. *If $b, c \in C$ and $0 < |f^n(c) - b^\gamma| < \varrho$ then $b^\gamma \in \omega(c)$.*

Property B. *Let $x \in X$, $c \in C$, $t \leqslant n$. Let c be the right-hand endpoint of the interval $f^{n-t}(H_n(x))$ and $|f^n(x) - f^t(c)| < \sigma(x)$. Then $c^- \in C^-(x)$ (correspondingly $c^+ \in C^+(x)$).*

LEMMA 7. *Let $x \in X$. Then $\omega(x) = \omega(c)$ for each $c \in C_*(x)$.*

Proof. Fix $c^\gamma \in C_*^\gamma(x)$. To be definite, let $\gamma = +1$. Let $0 < \alpha < \delta(\varrho) \equiv \delta$, $\alpha(\|f'\| + 1) < \varrho$, where $\|\cdot\|$ denotes the supnorm. There exists $k \in N$ such that $f^k(x) \in (c^+, c^+ + \alpha)$ and $r_n(x) < \min(\alpha, \varepsilon(\varrho), \sigma(x))$ for $n \geqslant k$. Let us show that $\operatorname{dist}(f^{k+j}(x), \omega(c)) < \alpha$ for $j \in N$ (where $\operatorname{dist}(y, Y) = \inf\{|y - z|: z \in Y\}$). Suppose that this is not the case and let j be the least natural number for which the reverse inequality holds. Note that $j > 0$ since $\operatorname{dist}(f^k(x), \omega(c)) \leqslant f^k(x) - c^+ < \alpha$.

For appropriate $t \leqslant k+j$ one endpoint b of the interval $f^{k+j-t}(H_{k+j}(x))$ is a critical point and $|f^{k+j}(x) - f^t(b)| = r_{k+j}(x)$. Let $\operatorname{sgn}(f^{k+j-t}(x) - b) = \gamma$. Then $b^\gamma \notin \omega(c)$. Indeed, otherwise

$$\operatorname{dist}(f^{k+j}(x), \omega(c)) \leqslant |f^{k+j}(x) - f^t(b^\gamma)| \leqslant r_{k+j}(x) < \alpha.$$

This contradicts the choice of j.

Further, since $|f^{k+j}(x) - f^t(b)| = r_{k+j}(x) < \sigma(x)$, we have $b^\gamma \in C^\gamma(x)$ (Property B). Let us show that $c^\pm \notin \omega(b)$. Indeed, otherwise $\omega(c) \subset \omega(b)$. Since $c \in C_*(x)$ we obtain $\omega(c) = \omega(b)$ and $b^\gamma \in C_*^\gamma(x)$. By Lemma 6 the point b^γ is recurrent and hence $b^\gamma \in \omega(c)$. This is a contradiction.

Now consider 3 cases.

(1) Let $t = 0$. Then

$$\begin{aligned}\operatorname{dist}(b, \omega(c)) &\leqslant |b - f^{k+j}(x)| + \operatorname{dist}(f^{k+j}(x), \omega(c)) \\ &\leqslant r_{k+j}(x) + \|f'\| \operatorname{dist}(f^{k+j-1}(x), \omega(c)) \leqslant \alpha + \|f'\|\alpha < \varrho,\end{aligned}$$

and hence $b^\gamma \in \omega(c)$ (Property A). We obtain a contradiction.

(2) Let $0 < t < j$. We have $\operatorname{dist}(f^{k+j-t}(x), \omega(c)) < \alpha < \delta$ and $\operatorname{dist}(b^\gamma, \omega(c)) \geqslant \varrho$ (by Property A). Consequently, $K \equiv [f^{k+j-t}(x), b] \supset$

$[b^\gamma + \gamma(\varrho - \delta),\, b]$. Since f^t is monotone on K, we obtain $t < p(\delta)$ and hence $\lambda(f^t(K)) \geqslant \varepsilon(\varrho)$. This contradicts the fact that $\lambda(f^t(K)) = r_{k+j}(x) < \varepsilon(\varrho)$.

(3) Let $t \geqslant j$. Set $\varkappa = t - j$ and consider the interval $J = f^\varkappa(K) = [f^k(x), f^\varkappa(b)]$. Since f^j is monotone on J and $j > 0$, we have $f^\varkappa(b) \geqslant c$. If $f^\varkappa(b) \in [c, c^+] \equiv L$, then $\omega(b) = \omega(L) \ni c^+$ which contradicts what has been proved above. Consequently, $f^\varkappa(b) \geqslant c^+ + \varrho$. Besides $f^k(x) < c^+ + \delta$, and hence $J \supset [c^+ + \delta, c^+ + \varrho]$. Using the monotonicity of $f^j | J$ once more, we obtain that $j < p(\varrho)$. Hence, $r_{k+j}(x) = \lambda(f^j(J)) \geqslant \varepsilon(\varrho)$ and we obtain a contradiction again.

LEMMA 8. *Let $f \in A$, $x \in X$. Then $\omega(x) = \omega(c)$ for some recurrent point $c \in C$.*

Proof. Let $b^\gamma \in C_*^\gamma(x)$. We may assume that $\gamma = -1$. By Lemmas 6–7 $\omega(x) = \omega(b^-) \ni b^-$. If $b^- = b$ then b is the required critical point. In what follows we assume that $b^- < b$. Let Γ be the family of homtervals J such that one of the endpoints of each J is a critical point and $\omega(J) = \omega(x)$. We have $[b^-, b] \in \Gamma$.

There exists a family Δ of maximal homtervals such that

(1) endpoints of a homterval $K \in \Delta$ do not belong to $\{0, 1\} \cup \bigcup_{n=0}^{\infty} f^{-n}(C)$,
(2) if $f^n(K) = f^m(J)$ for some $K, J \in \Delta$, $n, m \in N$, then $K = J$,
(3) for each $J \in \Gamma$ there exist $K \in \Delta$ and $s(J)$ such that $f^{s(J)}(J) \subset K$,
(4) each homterval $K \in \Delta$ contains some homterval $f^{s(J)}(J)$, where $J \in \Gamma$.

Set $s = \max\{s(J)\colon J \in \Gamma\}$, $\mu = \min\{\lambda(f^i(J))\colon J \in \Gamma, i = 0, 1, \ldots, s-1\}$. For any homterval J, the symbols $H_n(J)$, $M_n(J)$, $\sigma(J)$, $C(J)$, etc. are meaningful. Let $\sigma = \min\{\sigma(K)\colon K \in \Delta\}$. We have $\sigma > 0$. Denote by $M_n^+(J)$, $M_n^-(J)$ the right-hand and the left-hand components of $M_n(J) \setminus f^n(J)$ and let $H_n^+(J)$, $H_n^-(J)$ be the corresponding components of $H_n(J) \setminus J$ (i.e. $f^n(H_n^+(J)) = M_n^+(J)$).

Define the set $S \subset N$ as follows: $n \in S$ if for some $K_n \in \Delta$ the homterval $[u_n, v_n] = f^n(K_n)$ is nearer to b^- than all homtervals $f^i(K)$ ($K \in \Delta$, $i = 0, \ldots, n-1$). The homterval K_n is uniquely determined in view of the Property (2) of the family Δ. Since $\omega[b^-, b] \ni b^-$, homtervals $f^n(K_n)$ accumulate to b^-. Besides they are contained in the left-hand semi-neighbourhood of b^- (in view of property (1) of Δ). Let $\alpha = \frac{1}{2}\min(\varrho, \delta, \mu, b - b^-)$ and homtervals $f^n(K_n)$ are contained in the left-hand α-semi-neighbourhood of b^-.

Let $M_n(K_n) = [f^{\tau_n}(a_n), f^{t_n}(c_n)]$, where $a_n, c_n \in C$, $t_n, \tau_n \leqslant n$. Consider two cases.

(1) $f^{t_n}(c_n) < b^- + \alpha$ for some $n \geqslant N$. Then $|f^{t_n}(c_n) - b^-| < \alpha < \varrho/2$ and

hence $\omega(c_n) \supset \omega(b^-) = \omega(x)$ (Property A). Besides $\lambda(M_n^+(K_n)) = f^{t_n}(c_n) - v_n < 2\alpha < \sigma$ and hence $c_n^\gamma \in C^\gamma(K_n)$ for some γ (Property B). By Lemma 4 $c_n^\gamma \in \omega(K_n) = \omega(x)$. Thus, $c_n^\gamma \in \omega(c_n) = \omega(x)$. If $c_n^\gamma = c_n$, then c_n is the required critical point.

Let $c_n^\gamma \neq c_n$. Then $J_n = [c_n^\gamma, c_n] \in \Gamma$. If $t_n < s$ then $f^{t_n}(c_n) \geqslant v_n + \lambda(f^{t_n}(J_n)) \geqslant b^- - \alpha + \mu > b^- + \alpha$ which contradicts the assumption. Hence $t_n \geqslant s$. Then $f^{t_n}(J_n) = f^{l_n}(L_n)$, where $L_n \in \Delta$, $l_n = t_n - s(J_n) < n$. The homterval $f^{l_n}(L_n)$ lies to the right of $f^n(K_n)$, to the left of b (since $f^{t_n}(c_n) < b^- + \alpha < b$, and does not intersect $[b^-, b]$ (Property 1 of the family Δ). Hence $f^{l_n}(L_n)$ is nearer to b^- than $f^n(K_n)$. This is a contradiction.

(2) $f^{t_n}(c_n) \geqslant b^- + \alpha$ for all $n \in S$, $n \geqslant N$. Then $\lambda(M_n^+(K_n)) \geqslant \alpha$. Besides $\lambda(H_n^\pm(K_n)) \to 0$, $\lambda(f^n(K_n)) \to 0$ and $\min \lambda(K_n) > 0$. So

$$\lambda(M_n^+(K_n))/\lambda(f^n(K_n)) > \lambda(H_n^+(K_n))/\lambda(K_n)$$

for large n. It follows from the First Distortion Lemma that

$$\lambda(M_n^-(K_n))/\lambda(f^n(K_n)) \leqslant \lambda(H_n^-(K_n))/\lambda(K_n) \to 0 \quad (n \to \infty). \tag{3}$$

Hence

$$u_n - f^{\tau_n}(a_n) = \lambda(M_n^-(K_n)) < \alpha < \delta$$

for large n. Besides $u_n > b^- - \alpha$. Hence

$$b^- > f^{\tau_n}(a_n) > b^- - 2\alpha \geqslant b^- - \varrho.$$

These properties imply (as in the case 1) that

$$a_n^\gamma \in \omega(a_n) = \omega(x) \quad \text{for some } \gamma.$$

If $a_n^\gamma = a_n$ for some n, then a_n is the required critical point.

Suppose that $a_n^\gamma \neq a_n$ for all $n \in S$, $n \geqslant N$. Then $I_n = [a_n, a_n^\gamma] \in \Gamma$. The set of homtervals I_n is finite and $f^{\tau_n}(I_n) \subset M_n^-(K_n)$. Hence

$$\lambda(f^{\tau_n}(I_n)) \leqslant r_n(K_n) \to 0$$

and we have $\tau_n \to \infty$.

Let $U_n = f^{s(I_n)}(I_n)$, $V_n \in \Delta$ be the maximal homterval containing U_n, $m_n = \tau_n - s(I_n)$. Intervals U_n, V_n have a common endpoint $f^{s(I_n)}(a_n^\gamma)$ since this point belongs to $\omega(x)$. Hence $V_n \subset U_n \cup H_{m_n}^-(U_n)$. We have $M_{m_n}^+(U_n) \supset M_n^+(K_n)$ and therefore $\lambda(M_{m_n}^+(U_n)) > \alpha$. Besides, $\lambda(H_{m_n}^+(U_n)) \to 0$ since $m_n \geqslant \tau_n - s \to \infty$. So

$$\lambda(M_{m_n}^+(U_n))/\lambda(f^{m_n} U_n) \geqslant \lambda(H_{m_n}^+(U_n))/\lambda(U_n).$$

The First Distortion Lemma implies

$$\lambda(M^{-}_{m_n}(U_n))/\lambda(f^{m_n}(U_n)) \leqslant \lambda(H^{-}_{m_n}(U_n))/\lambda(U_n) \leqslant D.$$

Since $f^{m_n}(V_n) \subset M^{-}_{m_n}(U_n) \cup U_n$, we obtain

$$\lambda(f^{m_n}(V_n)) \leqslant (D+1)\cdot\lambda(f^{m_n}(U_n)) \leqslant (D+1)\cdot\lambda(M^{-}_n(K_n)).$$

The last inequality and (3) imply

$$\lambda(f^{m_n}(V_n))/\lambda(f^n(K_n)) \to 0 \quad (n\to\infty). \tag{4}$$

Let l be the element of the set S preceding n. Let us show that

$$\lambda(f^l(K_l)) < \lambda(f^n(K_n)). \tag{5}$$

Indeed, if $l = m_n$ and $K_l = \overset{\circ}{V}_n$ then (4) implies (5). Otherwise the homterval $f^l(K_l)$ is placed between the homtervals $f^{m_n}(V_n)$ and $f^n(K_n)$. Hence, $f^l(K_l) \subset M^{-}_n(K_n)$ and (3) implies (5).

Thus, the sequence $\{\lambda(f^n(K_n))\}_{n\in S}$ increases, which is impossible. The Lemma is proved.

§ 4. Ergodicity

The main technical tool in this Section is the Second Distortion Lemma. Now we prove it.

LEMMA 9. *Consider two mass systems m_k and M_k concentrated in points y_k and Y_k correspondingly ($k = 1, 2, \ldots, n$). Let c be the centre of gravity of the first system, C_l be the centre of gravity of the first l masses of the second system. Assume that $Y_k \geqslant y_k$ and $M_{k+1}/M_k \geqslant m_{k+1}/m_k$. Then there exists l such that $C_l \geqslant c$.*

Proof. Suppose that the sequence $\{y_k\}_{k=0}^{n}$ is decreasing. Then $y_1 > c$ and we may set $l = 1$. If this is not the case, there exists k such that $y_{k+1} \geqslant y_k$. Replace the pair of masses m_k, m_{k+1} by the mass $m = m_k + m_{k+1}$ concentrated in the centre of gravity $y = (m_k y_k + m_{k+1} y_{k+1})/(m_k + m_{k+1})$. Similarly proceed with M_k, M_{k+1}. We obtain two systems of $(n-1)$ masses satisfying the conditions of the lemma. Induction completes the proof.

Proof of the Second Distortion Lemma. From the well-known properties of functions with negative Schwarzian derivative it follows that the function $|(\varphi^{-1})'|$ is monotone on some interval J^{γ}. Observe now that inequality (1) is invariant under affine transformations of I, J. Therefore we may assume that $y = 0$, $J^{\lambda} = \varphi^{-1}(J^{\lambda}) = [0, 1]$ and E is contained in $[0, 1]$. Let $q = \lambda(E)$, $\psi = \varphi^{-1}$: $[0, 1] \hookleftarrow$. Then (1) takes the form

$$\frac{\lambda(\psi(E\cap K))}{\lambda(\psi(K))} \geqslant \eta q. \tag{6}$$

Divide $[0, 1]$ into intervals $J_1, \ldots, J_n$ such that $|\psi'(x)|/|\psi'(y)| \geqslant \eta$ for $x, y \in J_k$. Let $E_k = E \cap J_k$. There exists $\xi_k \in J_k$ for which

$$\lambda(\psi(J_k)) = |\psi'(\xi_k)| \cdot \lambda(J_k).$$

Let $|\psi'(\xi_k)| = M_k$, $\lambda(J_k) = \lambda_k$. Then we have

$$\lambda\Big(\psi\big(\bigcup_{k=1}^{l} J_k\big)\Big) = \sum_{k=1}^{l} M_k \lambda_k. \tag{7}$$

But $\lambda(\psi(E_k)) = \int_{E_k} |\psi'(z)|\, dz \geqslant \eta M_k \lambda(E_k)$. Putting $\lambda(E_k) = \mu_k \cdot \lambda(E) = \mu_k q$ we obtain

$$\lambda\Big(\psi\big(\bigcup_{k=1}^{l} E_k\big)\Big) \geqslant \eta q \sum_{k=1}^{l} M_k \mu_k. \tag{8}$$

Observe that

$$\sum_{k=1}^{n} \lambda_k = 1 = \sum_{k=1}^{n} \mu_k. \tag{9}$$

Now make use of Lemma 9 for $m_k = 1$ and M_k concentrated in $y_k = Y_k = \mu_k - \lambda_k$ $(k = 1, \ldots, n)$. The conditions of Lemma 9 are valid since the sequence $\{M_k\}_{k=1}^{n}$ increases. By (9) we have $c = 0$. Hence there exists l for which

$$\sum_{k=1}^{l} M_k(\mu_k - \lambda_k) \geqslant 0. \tag{10}$$

It follows from (7), (8), (10) that

$$\lambda\Big(\psi\big(\bigcup_{k=1}^{l} E_k\big)\Big) \geqslant \eta q \lambda\Big(\psi\big(\bigcup_{k=1}^{l} J_k\big)\Big).$$

The interval $K_\eta = \bigcup_{k=1}^{l} J_k$ is the required.

LEMMA 10. *Let L be the cycle of a transitive interval. Suppose that a map $f\colon L \hookleftarrow$ is not ergodic. Then $r_n(x) \to 0$ $(n \to \infty)$ for a.e. $x \in L$.*

Proof. Let $L = X_1 \cup X_2$ be a partition of L into two invariant sets of positive measure. Then $\lambda(X_1 \cap J) > 0$ for any interval J. Indeed, otherwise $\lambda(X_1 \cap f^n(J)) = 0$ for $n \in N$ and by Lemma 1 $\lambda(X_1) = 0$. Consequently, $\lambda_1(\varepsilon) \equiv \inf \lambda(X_1 \cap J) > 0$, where the infimum is taken over all intervals $J \subset [0, 1]$ of length ε.

Consider the set $Y = \{x\colon \lim r_n(x) > 0\}$. Let $x \in Y$. Then $r_{n_k} > \varepsilon$ for appropriate $\varepsilon > 0$, $n_k \to \infty$. Apply to $f^{n_k}\colon H_{n_k}(x) \to M_{n_k}(x)$ the Second

Distortion Lemma. We obtain that for some interval $K_j \subset H_{n_j}(x)$ for which x is one of the endpoints, the following estimate holds:

$$\frac{\lambda(K_j \cap X_1)}{\lambda(K_j)} \geqslant \frac{1}{2}\frac{\lambda_1(\varepsilon)}{\varepsilon}.$$

Since a transitive interval does not contain a homterval,

$$\lambda(K_j) \leqslant \lambda(H_{n_j}(x)) \to 0 \quad (j \to \infty).$$

Hence, the upper density of X_1 at x

$$\left(\text{i.e.} \varlimsup_{\varepsilon \to 0} \frac{\lambda(X_1 \cap [x-\varepsilon, x+\varepsilon])}{2\varepsilon}\right) \text{ is positive.}$$

Consequently, x is not a density point of X_2. By the Lebesgue Theorem $\lambda(Y \cap X_2) = 0$.

Interchanging the roles of X_1 and X_2 we obtain $\lambda(Y \cap X_1) = 0$. Hence $\lambda(Y) = 0$, and the lemma is proved.

Now the proof of Theorem 1 is completed and we pass to the proof of Theorem 2.

From here on we assume that $f \in A$ is a unimodal map with a maximum point $c, f^{(n)}(c) \neq 0$ for some $n > 0$. Without loss of generality we may assume that f: $c \to 1 \to 0$ (see [1]). Denote this class of maps by $\mathfrak{D}$.

There exists a point $\gamma \in (c, 1]$ such that $f(\gamma) = f(0)$. Define an involution τ: $[0, \gamma] \hookleftarrow$ as follows: $\tau(x) = x'$, where $f(x) = f(x')$. The property $f^{(n)}(c) \neq 0$ implies that τ is smooth. For $b \in [0, \gamma]$ put $U_b = (b, b')$.

LEMMA 11. *Let $f \in \mathfrak{D}$ be mixing. Suppose that the orbit of a point x intersects a neighbourhood U_b. Let n be the first moment for which $f^n(x) \in U_b$. Then $M_n(x) \supset U_b$.*

Proof. Let x divide $H_n(x)$ into intervals H_n^+ and H_n^-, c divide U_b into U_b^+ and U_b^-. Then for every $\mu = \pm 1$ and appropriate $\gamma \in \{\pm 1\}$, $l < n$ we have

$$f^l(H_n^\mu) = [c, f^l(x)] \supset U_b^\gamma.$$

If $f^{n-l}(c) \in U_b$, then $f^{n-l}([c, f^l(x)]) \subset U_b$ and hence $f^{n-l}(U_b) = f^{n-l}(U_b^\gamma) \subset U_b$. But this is impossible for a mixing map.

A set will be called *symmetric* if it is τ-invariant. Theorem 2 is the immediate consequence of the following:

LEMMA 12. *Let a map f: $[0, 1] \hookleftarrow$ be transitive. Let a set $X \subset [0, 1]$ of positive measure λ be f-invariant and symmetric in a neighbourhood of the critical point c. Then c is a density point of X.*

Proof. By Lemma 1 we may assume that f is mixing. Using the results of [6], or [7], or [8], or theorem 1 of the present paper we obtain that $\omega(x) \ni c$ for a.e. $x \in [0, 1]$. Fix some density point $x \in X$ of X for which $\omega(x) \ni c$. Put $Y = [0, 1] \setminus X$ and for an interval $J = (a, b)$ denote by

$$\varrho(a, b) = \varrho(J) = \lambda(Y \cap J)/\lambda(J)$$

the density of Y in J. Consider two cases.

(1) The lower density of Y at c is positive:

$$\underline{\lim}_{b \to c} \varrho(c-b, c+b) > 0.$$

Since τ is smooth and X is symmetric, we have $\varrho(U_b) \geqslant \varepsilon > 0$ for all $b \neq c$.

Let $f^{n_0}(x) \in (0, \gamma)$, n_{k+1} be the first moment in which the orbit $\{f^n(x)\}_{n=n_k+1}^{\infty}$ intersects the neighbourhood $U_k \equiv U_{f^{n_k}(x)}$; let x_k be that one of points $f^{n_k}(x)$, $\tau(f^{n_k}(x))$ which lies to the left of c.

Since $U_k = \bigcup_{i=k}^{\infty} (U_i \setminus U_{i+1})$, the inequality $\varrho(U_{k_j} \setminus U_{k_{j+1}}) \geqslant \varepsilon > 0$ holds for some sequence $k_j \to \infty$. Since τ is smooth and X is symmetric, we obtain

$$\varrho(x_{k_j}, x_{k_{j+1}}) \geqslant L^{-1} \varrho(U_{k_j} \setminus U_{k_{j+1}}) \geqslant \varepsilon_1 > 0,$$

where L is the Lipschitz constant for τ, $\varepsilon_1 = \varepsilon L^{-1}$. Besides, the estimate

$$\varrho(x_k', x_{k+1}) \geqslant \min\left(\varrho(x_k', c), \varrho(c, x_{k+1})\right) \geqslant \varepsilon_1 > 0$$

is valid for all $k \in N$. Thus, in both intervals into which $x_{k_{j+1}}$ divides U_{k_j}, the density of Y is at least ε_1.

Put $V_k = f^{-n_k}(U_k) \cap H_{n_k}(x)$. By Lemma 11, f^{n_k} monotonically maps V_k onto U_k. Using the Second Distortion Lemma we obtain that in some semi-neighbourhood of x which is contained in V_k, the density of Y is at least $\varepsilon_1/2$. Since f has no homtervals, $\lambda(V_k) \to 0$. Thus, the set $Y = [0, 1] \setminus X$ has positive upper density at x. We have a contradiction.

(2) The lower density of Y at c is equal to zero:

$$\underline{\lim}_{b \to c} \varrho(c-b, c+b) = 0.$$

If the conclusion of lemma is not true then

$$\overline{\lim}_{b \to c} \varrho(c-b, c+b) > 0.$$

Using the symmetry of X and smoothness of τ we find sequences $\{a_k\}$, $\{b_k\}$ satisfying

$$b_k \in (a_k, c), \quad a_k \to c, \tag{11}$$

$$\varrho(a_k, c) \geqslant \varepsilon, \quad \varrho(a'_k, c) \geqslant \varepsilon > 0, \tag{12}$$

$$\varrho(b_k, c) < \delta_k, \varrho(b'_k, c) < \delta_k, \quad \text{where } \delta_k \to 0, \tag{13}$$

$$\varrho(U_d) \leqslant \varrho(U_{a_k}) \quad \text{for } d \in U_{a_k} \setminus U_{b_k}. \tag{14}$$

Let $n = n_k$ be the first moment in which $f^n(x) \in U_{a_k}$. Put $x_k = f^{n_k}(x)$. Replacing if necessary a_k, b_k by a'_k, b'_k, we may assume that $x_k \in (a_k, c)$. We have

$$\varrho(x_k, a'_k) \geqslant \frac{\lambda(Y \cap (c, a'_k))}{|a_k - a'_k|} \geqslant \frac{1}{1+L}\varrho(U_{a_k}) \geqslant \frac{\varepsilon}{1+L} > 0.$$

To estimate the density of Y in (a_k, x_k) consider two cases.

(a) $x_k \in (a_k, b_k)$. By (14) $\varrho(U_{x_k}) \leqslant \varrho(U_{a_k})$ and hence

$$\varrho((x_k, a_k) \cup \tau(x_k, a_k)) = \varrho(U_{a_k} - U_{x_k}) \geqslant \varrho(U_{a_k}) \geqslant \varepsilon.$$

Using the smoothness of τ and symmetry of X once more, we obtain $\varrho(x_k, a_k) \geqslant L^{-1}\varepsilon$.

(b) $x_k \in (b_k, c)$. Then

$$\lambda(Y \cap (a_k, c)) = \lambda(Y \cap (a_k, b_k)) + \lambda(Y \cap (b_k, c)) \leqslant |b_k - a_k| + \delta_k |c - b_k|.$$

Therefore

$$\varepsilon \leqslant \varrho(a_k, c) \leqslant \frac{|b_k - a_k|}{|a_k - c|} + \delta_k$$

and hence

$$|b_k - a_k| \geqslant \frac{\varepsilon}{2}|c - a_k|$$

for sufficiently large k.

Finally, we obtain

$$\varrho(a_k, x_k) \geqslant \frac{\varepsilon}{2}\frac{\lambda((a_k, b_k) \cap Y)}{|a_k - b_k|} = \frac{\varepsilon}{2}\varrho(a_k, b_k) \geqslant \frac{\varepsilon^2}{2}$$

(the last estimate follows from (14)).

Thus, in both cases (a), (b) densities $\varrho(a_k, x_k)$ and $\varrho(x_k, a'_k)$ are separated from zero. Now using the Second Distortion Lemma and Lemma 11 as in the case 1, we obtain that upper density of Y at x is positive. We have a contradiction again. The proof is now completed.

Proof of Corollary 3. Using results of [4], absence of homtervals [6] and Theorem 1, we see that for the unique attractor A of f one of the following statements holds:

(a) A is a cycle,

(b) A is contained in a cycle of transitive intervals;

(c) A is a solenoid (in the sense of [4], i.e. a Cantor set on which f has a specific dynamics and, as a corollary of it, is topologically conjugate to a transitive shift on a group).

It is clear that the support of any absolutely continuous invariant measure is contained in A. Hence, in case (a) such measure does not exist. If it exists in case (c), then it coincides with the unique measure for f/A. Finally, in case (b), the required follows from Theorem 2.

§ 5. Concluding remarks

It seems plausible that a complete description of attractors of $f \in \Lambda$ can be achieved (cf. [3], [4]).

MAIN CONJECTURE. Each attractor of $f \in \Lambda$ is either a periodic orbit, or a cycle of transitive interval, or a solenoid.

To establish the Main Conjecture it is sufficient to solve the following problems.

PROBLEM 1. $f \in \Lambda$ has no homtervals.

PROBLEM 2. If $f \in \Lambda$ is topologically transitive, then it is conservative.

In conclusion we formulate one result concerning smooth maps.

THEOREM 3. *Let* f: $[0, 1] \hookleftarrow$ *be a piecewise monotone* C^∞*-map. Assume that for any critical point* c *there exists* $n > 0$ *such that* $f^{(n)}(c) \neq 0$. *Then for generic* $x \in [0, 1]$ *the limit set* $\omega(x)$ *is either a periodic orbit, or a cycle of transitive intervals, or* $\omega(c)$, *where* c *is a recurrent critical point.*

The proof is similar to that of Theorem 1, but one uses results of [4] and estimates from [9] instead of the First Distortion Lemma.

Added in proof. (1) Problem 1 was solved by one of the authors for maps $f \in \Lambda$ with non-degenerate critical points (M. Yu. Lyubich, 1987). Using the estimates from [9], the authors extended this result onto the smooth case. The proof is to appear in "Ergodic Theory and Dynamical Systems".

(2) Theorem 2 can be extended onto the poly-modal case in the following form: for any indecomposable attractor A the restriction $f|\{x: \omega(x) = A\}$ is ergodic. The proof is to appear in "Algebra and Analysis", 1989, N1.

(3) We have proved also that $\lambda(A) > 0$ for every non-periodic and non-solenoidal attractor A and there exists a unique σ-finite invariant measure μ on A absolutely continuous with respect to the Lebesgue measure. Moreover, if μ is finite then $h_\mu(f) > 0$.

(4) Finally, if $A = \omega(c)$ is not a cycle of intervals then $\omega(x) = A$ for all $x \in A$ and $h(f|A) = 0$.

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