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THE ORIGINS OF THE CONCEPT OF DIMENSION

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The paper tells the story of how the problem of dimension came about and what vicissitudes have marked a half a century pursue to grasp the intuition of dimension and to shape it mathematically.

The origins of the concept of dimension can be traced back as far as the "Elements" of Euclid (365?-300? B. C.). Its book I starts with the following definitions:

1. *A point is what has no part.*
2. *A line is what has length but not width.*
3. *The extremities of a line are points.* (...)
5. *A surface is what has length and width only.*
6. *The extremities of a surface are lines.*

And book XI of "Elements" adds the following two:

1. *A solid is what has length, width, and depth.*
2. *An extremity of a solid is a surface.*

The definitions seem to set up sound intuitive basis to our understanding of dimension as measured by the number of parameters needed for a description of points belonging to an object: point needs 0 parameters, line is that which needs 1 parameter, surface — 2, solid — 3. Since geometrical objects considered until the middle of XIX century easily fell into one of the four classes: points, lines, surfaces, and solids, the intuitive attitude has been quite satisfactory for more than two thousand years, there was no need for an extra precision and, quite understandably, nobody bothered about it. It was only in the second half of XIX century, together with the appearance of new ideas in algebra and differential geometry, that the situation changed.

First came into being objects whose points are described by systems of n real numbers, where n is any positive integer. They made their first appearance in "The Linear Calculus of Extension" of 1844 and "The Calculus of Extension" of 1862 (cf. [16]) by Hermann Günther Grassmann

(1809-1877), and in the qualifying lecture made by Georg Bernhardt Riemann (1826-1866) for the title of Privatdozent in 1854. The latter has been published posthumously in 1868 under the title "On the Hypotheses Which Lie at the Foundation of Geometry" [36] and the objects are already called there n-dimensional manifolds, the name used to this day. The dimensional qualification seemed quite natural from the intuitive point of view, but the trend towards precision (remember that it was a period of instillation of rigor into analysis) and a prevailing attitude have opened the question: what does the word "dimension" actually mean and whether is it used here properly?

The first who consciously tackled that question was Georg Cantor (1845-1918). In 1874 he has started his revolutionary work on the theory of infinite sets and found out, to his astonishment, that the naive understanding of infinity, as of the one and only one, cannot be maintained any longer: the set of algebraic numbers is countable whereas the set of reals is not. The discovery turned to be prolific (e.g., it yields immediately that there are uncountably, hence infinitely many transcendental numbers) and the idea suggested itself that the infinite cardinal numbers can be ordered in the way which corresponds to the passage from 1-dimensional to more-dimensional objects. On the way of Cantor's thinking there is an invaluable testimony: a correspondence between him and Richard Dedekind (1831-1916) which covered all his active life and parts of which have been published in his collected works [13] and separately in 1937 by E. Noether and J. Cavaillès [1]. In 1874 Cantor wrote to Dedekind as follows:

"Concerning questions which make me recently busy, also the following seems to be worth of presenting out of this way of thinking: Can a surface (e.g., a square including boundary) cover a line (e.g., a straight-line segment including ends) in a one-valued manner so that each point of the surface corresponds to one point of the line and, conversely, to each point of the line belongs one point of the surface?" ([1], p. 20).

Cantor adds that the answer must afford serious difficulties, although "no" is so suggestive that one may consider the proof almost superfluous.

And three years later he proved that the answer is "yes" by showing that there exists a one-to-one correspondence between points of the unit segment and points of the unit n-dimensional cube, where n is any positive integer or even countable infinite (see [1], p. 29-34; a complete proof of a more general theorem is given in [11] = [14], III, 2). He was so startled ("je le vois, mais je ne le crois pas") that it raised his doubts about the validity of the whole question of dimension. He wrote to Dedekind ([1], p. 34):

..."Now it seems to me that all philosophical or mathematical reasonings, which make use of that faulty assumption, are invalid. And even more, the distinction between objects of different dimensions must be looked for in quite different areas than that of the characteristic number of independent coordinates."

In an answer, Dedekind tried to defend the previous intuition of dimension for manifolds ([1], p. 37-38):

..."Against it I raise (despite your theorem or rather as a result of cogitations initiated by your theorem) my conviction or my belief (...) that the dimensional number of a continuous manifold is, as before, its first and the most important invariant, and so I must stand up for all the hitherto writers in that matter. But I agree willingly with you that the constancy of the dimensional number requires a proof and as long as such a proof is not provided one should doubt it. I do not, however, doubt in that constant, although your theorem apparently destroys it. All writers make namely the tacit and quite natural assumption that for any new description of points of a continuous manifold by new coordinates the latter should also (in general) be continuous functions of the old ones (...). Thus for the time being I believe in the following theorem: «Let there be a mutually one-valued and complete correspondence between points of one continuous manifold A of a dimensions on one side and of another continuous manifold B of b dimensions on the other. Then the correspondence, if a and b are not equal, is necessarily everywhere discontinuous.»"...

Cantor agreed and in the next few years tried to prove the Dedekind's conjecture. For some time he believed to be in a possession of a proof that there does not exist a continuous and even many-to-one mapping of one manifold onto another (of a different dimension), but there were some gaps and so eventually, in 1879, he gave it up ([1], p. 39-49). Luckily for him, for in 1890 Peano will show that such mappings do exist. Then he turned to attempts made in the meantime by some other people, rightly criticized them for incorrectness and presented his own proof, in turn rightly criticized by some of them somewhat later (see [12] = [14], III, 3; cf. Anmerkung in [14], p. 138). After 1879 Cantor apparently lost his interest and ceased to be engaged in the work around the question of dimension (but later he will yet provide a definition of a plane curve, i.e. of a 1-dimensional object lying in the plane).

Thus the question remained open, but the matter has been to some extent clarified: if dimension was to be really "the first and most important" geometrical invariant, it had to be preserved by one-to-one and continuous mappings of one manifold onto another. And since the simplest manifold is a euclidean space itself, the problem can be put in the following form: does there exist, for two euclidean spaces of different dimensions, a one-to-one and continuous mapping from one space onto the other? The expected answer was "no", but the next thirty years have seen only incorrect or partial solutions, e.g. Eugen Netto (1846-1919) [32] and Jacob Lüroth (1844-1910) [23].

Manifolds were not the only geometrical objects investigated in that time. Another large class was that of lines (curves), the abundance of which has appeared in analytic and differential geometry. What a line is, was understood intuitively, but the time has come to define it precisely. The first who attempted such a definition was Camille Jordan (1838-1922). In his famous "Cours d'Analyse" [18] he has bound the concept of a line with that of continuity:

..."A line, being defined as the place of successive positions of a moving point, will be represented, in the case of a plane move, by a system of two equations

$$x = f(t), \quad y = g(t),$$

where f and g are functions of an independent variable t which can be viewed upon as a time. If those functions are continuous, the curve will be called continuous." (see [18], I, p. 90).

Thus a line has been defined as a continuous image of an interval and, in praise of Jordan's contribution, each such an image has been later called a Jordan line. One is tempted to think that the intuition has been seized rightly, since each figure classified so far as a line was a Jordan line and nobody doubted that also conversely, each Jordan line was a figure that we want to be called a line.

However, in 1890 Giuseppe Peano (1858-1932) has constructed a continuous mapping on the unit interval, the image of which was the full unit square [38]. In this way and in a sharp contrast to an intuition the full square has turned up to be a Jordan line.

This was a great surprise and the discovery has become the subject of vivid investigations carried by men like Ernesto Cesàro (1859-1906), Hans Hahn (1879-1924), David Hilbert (1862-1943), Eliakim Hastings Moore (1862-1932), Arthur M. Schoenflies (1853-1928), Wacław Sierpiński (1882-1969), and several others (see survey article [22]). They have thrown much light on the matter but the very fact remained and the confusion brought about by it can be well seen in a remark of Felix Klein (1849-1925) who once said that nothing is more obscure than the concept of a curve [19].

Nearly thirty years later Hans Hahn and Stefan Mazurkiewicz (1888-1945) have characterized, in general topological terms, the class of compact metric Jordan lines, i.e. of those compact metric spaces which are continuous images of a segment. They proved (cf. [17], [25]) that it coincides with the class of locally connected continua or, as many call it, Peano continua. This is a vast class of continua and contains many objects which are rather far from what we want to be called a line, e.g. cubes of all dimensions, including infinite-dimensional Hilbert cube.

On the other hand, there are obvious "lines" which are not locally connected, e.g. graph of the function $y = \sin 1/x$, $0 < x \leqslant 1$, together with the segment $[-1, 1]$ of y-axis.

Thus the Jordan's attempt failed.

Discovery of Peano was also a blow to the hope of relating the general concept of dimension to the number of continuous parameters needed for a description. Moreover, if one could combine Cantor's construction of a one-to-one mapping from a segment onto a square with that of Peano, thus asserting a one-to-one and continuous (consequently, homeomorphic) correspondence between the points of the segment and of the square — the dimension would be of no geometric value whatsoever. Whether such a combination be possible, could be decided by a solution of the basic problem: are two euclidean spaces of distinct dimensions homeomorphic to each other?

But now the expected negative solution of that problem could not suffice. In the series of six papers "Über unendliche lineare Punktmannigfaltigkeiten" (Mathematische Annalen, 1879-1884 = [14], III, 4) Georg Cantor defined open, closed, dense, perfect, connected etc. subsets of euclidean spaces and Italian mathematicians Giulio Ascoli (1843-1896), Vita Volterra (1860-1940), Cesare Arzelà (1847-1912) transferred these notions to sets the elements of which are functions. This flux of ideas was the beginning of general topology with its multitude of general topological spaces. The range of considered objects raised tremendously and revealed ones of complexity never before dreamt of. And in spite of unsuccessful attempts at the definition that has been hitherto made, the old belief that the dimension is "the first and most important" invariant of a space was so strong that the whole question has been transplanted to the much wider area of topology. But we should also notice the difference: the intuitive basis of dimension, as expressed by Euclid and then commonly accepted, has been now broken to pieces by the discovery of Peano and his followers, the whole concept seemed dubious (remember that the problem of dimension of euclidean spaces was still open in its entirety), and there was no idea what it should be with respect to general topological spaces and how to define it. It is then no wonder that there came a period fo multidirectional searches, unsuccessful attempts and wanderings. Rather wonderful was the fact that the work went on incessantly without waiting for a solution of the basic problem for euclidean spaces (soon solved anyway) and that within the life of one generation the whole work of analysis and construction has been done.

But first came unsuccessful attempts. Such was Jordan's and now we mention another two.

Cantor defined a line in the plane as a continuum with the following property [13]: for each point of the continnum and any neighbourhood of it (in the plane) there is a point in that neighbourhood which does not belong to the continuum. In other words, a line in the Cantor sense is any nowhere dense subcontinuum of the plane. The definition excludes pathologies of Peano type and in the considered range of plane subcontinua agrees well with the intuition accepting those and only those continua as lines which should be lines. Nevertheless, it is unacceptable: the definition is essentially restricted to subcontinua of the plane and it is "external" by referring to the complement of the figure.

Maurice Fréchet (1878-1956) has patterned his definition of a dimensional type [15] after the concept of cardinality (later P. Mahlo has coined the term homoie for it). If two figures A and B are such that each is homeomorphic with the subset of the other, we say that both have the same dimensional type and write $dA = dB$, e.g. interval and the whole line. If A is homeomorphic with the subset of B, but B is not homeomorphic with any subset of A, then the dimensional type of A is lesser than that of B and we write $dA < dB$. And if neither $dA = dB$ nor $dA < dB$ nor $dA > dB$, the dimensional types of A and B are incomparable, e.g. a triod and a circle. There are many problems on dimensional types and, although they may appear interesting in themselves, from the viewpoint of dimension it is also a drive into corner: there are too many dimensional types and the interrelations are too ramified. No wonder then that the concept was soon abandoned and well forgotten.

The last years before the outbreak of World War I have seen an abundance of intriguely interesting papers which came close to the concept of dimension and proved its validity but fell just short of reaching it. Three names deserve particular mention here: Poincaré, Lebesgue, and Brouwer.

Henri Poincaré (1854-1912) was the man who thought the whole problem over and came to fine conclusions. However, he had no time to put them in a clear mathematical form and all we know about his way of thinking lies hidden in two articles: one published 1912 in a philosophical journal [34] and another found after his death and published [35] only in 1920. In the first he writes:

..."Of all the theorems of Analysis Situs (the former name of topology — R. D.), the most important is that which we express by saying that space has three dimensions. It is this proposition that we are about to consider, and we shall put the question in these terms: when we say that space has three dimensions, what do we mean?"...

As the following excerpts will show, he tried to find an answer by returning to the roots but considering them from a distinct point of view. Looking back at Euclid, one can see that the stress is now put not on the number of parameters but on the shape of extremities. According to Poincaré it suffices to know what cuts are necessary to divide geometrical objects (for some reason he calls them continua) into pieces, because then one can proceed inductively and evaluate dimension of those objects by dimensions of their cuts. Poincaré writes:

..."If to divide a continuum C, cuts which form one or several continua of one dimension suffice, we shall say that C is a continuum of two dimensions; if cuts which form one or several continua of at most two dimensions suffice, we shall say that C is a continuum of three dimensions; and so on.

To justify this definition it is necessary to see whether it is in this way that the geometers introduce the notion of three dimensions at the beginning of their work. Now, what we see? Usually they begin by defining surfaces as the boundaries of solids or pieces of space, lines as the boundaries of surfaces, points as the boundaries of lines, and they state that the same procedure cannot be carried further.

This is just the idea given above: to divide space, cuts that are called surfaces are necessary; to divide surfaces, cuts that are called lines are necessary; to divide lines, cuts that are called points are necessary; we can go no further and a point cannot be divided, a point not being a continuum. Then lines, which can be divided by cuts which are not continua, will be continua of one dimension; surfaces, which can be divided by continuous cuts of one dimension, will be continua of two dimensions; and finally space, which can be divided by continuous cuts of two dimensions, will be a continuum of three dimensions."

Stressing the inductive nature of the geometric meaning of dimension and the possibility of disconnecting a space by subsets of lower dimension, Poincaré penetrated the problem deeply. He repeats:

..."I want to base the establishing of the number of dimensions upon the concept of a cut. (...) A continuum has n dimensions if it can be divided into pieces with the help of one or more cuts which are continua of $n-1$ dimensions."

Another approach, only later estimated at its proper value, has been proposed by Henri Lebesgue (1875-1941). In a paper of 1911 he writes [20]:

..."If each point of a region D of n dimensions belongs to at least one of finitely many closed sets $E_1, \ldots, E_p$ and if these sets are sufficiently small, then at least $n+1$ of them have common points."

By a *region* Lebesgue means here an open and bounded (i.e., of a finite diameter) subset of an n-dimensional euclidean space. The proposition stems out from the observation of the pattern on the surface of a wall or of honey comb: a square can be covered by arbitrarily small "bricks" in such a way that no point of it belongs to more than three of them, and if "bricks" are sufficiently small, then some three must have a point in common. In a similar way, a cube in n-dimensional euclidean space can be decomposed into arbitrarily small "bricks" such that not more than $n+1$ meet. Lebesgue's proposition is equivalent to say that this number cannot be reduced further, i.e. for any decomposition into a finite number of sufficiently small "bricks" there must be a point common to at least $n+1$ of them.

Trivial as it may seem, the real meaning of the observation consists in the fact that it discloses a simple topological property discerning euclidean spaces of distinct dimensions. In particular, it implies that there does not exist a homeomorphism between euclidean spaces of distinct dimensions.

The observation has been fully proved later by Brouwer [4] and in 1921 by Lebesgue himself [21]. And when the topological terminology evolved sufficiently, it has become the definition of a so-called covering dimension: covering dimension of a topological space X is the least integer n, denoted $\dim X$, such that for each finite open cover of X there exists a finite open refinement of rank $n+1$ (i.e., the intersection of any $n+2$ members of the refinement is empty). Defined in topological terms, the covering dimension is a topological invariant and the observation of Lebesgue (now a theorem) states that it agrees well with the intuition yielding proper values for euclidean spaces.

Relying upon some ideas of Lebesgue [20] and upon his own analysis of the phenomenon discovered by Peano [33], Stefan Mazurkiewicz has come to the following concept [24]: dimension of a compact metric space X is the least positive integer n with the property that there exists a mapping f from a closed subset of the Cantor set onto X such that each counter-image $f^{-1}(x)$, where $x \in X$, consists of at most $n+1$ points. Writing during the war, in the language of a nation which has not existed in political maps of that time, and in a journal of a limited propagation — the concept has been doomed to oblivion from the very beginning. Later it turned to be equivalent with the concept of Menger and Urysohn, but never exercised any influence.

In the same year 1911, when Lebesgue made his observation, Leitzen Egbert Jan Brouwer (1882-1965) has started his series of papers concerning dimension. In [2] he proved that euclidean spaces of distinct dimensions are not homeomorphic to each other, thus being the first to confirm the common conviction (although stated simultaneously, Lebesgue's observation has been proved much later). However, he has not made use of any clearly defined dimensional invariant (a closer examination reveals the invariant: for a sufficiently small $\varepsilon > 0$ there does not exist a continuous mapping from the n-dimensional cube into $(n-1)$-dimensional polyhedron which moves each point for at most ε), and so the paper did not imply the definition of dimension. But it proved that the dimension of manifolds, which by definition are locally euclidean, is well defined. Thus in the range of manifolds the intuition as exposed by Euclid remains valid. In [3] Brouwer generalized the previous result on the invariance of dimension of euclidean spaces by proving the well-known theorem on the invariance of domain: in an n-dimensional manifold a one-to-one and continuous image of an open subset remains open. Then he turned to the ideas of Poincaré. Of particular interest is here the paper [4]. Brouwer starts with a critic of Poincaré's view, emphasizing its susceptibility of various interpretations, e.g. the meaning of a continuum (not defined there), and ambiguity of words "one or more" (actually how many?). Then he writes:

..."All these faults vanish if we change the recurrent definition of Poincaré as follows:

Let X be any normal set, and A, B, C three subsets of X which are closed in X and have no common points. Sets A and B are called separated in X by C, if each connected and closed subset of X, which has points in common both with A and B, contains at least one point of C. The expression X *has dimensional degree* n, where n is any rational integer, means now that for each choice of A and B there is a set C, separating A and B, which has dimensional degree at most $n-1$. Further, the expression X *has dimensional degree* 0 or *dimensional degree infinity* means that X does not contain any subset which is a continuum or that one cannot attribute to X either 0 or any integer as its degree of dimension."

The crucial point in the conception of Poincaré is the notion of a cut. Once we know what a cut is, it suffices to establish the family of sets of dimension 0 and proceed inductively: sets of dimension 1 are those which have sufficiently many cuts of dimension 0, sets of dimension 2 are those which have sufficiently many cuts of dimensions 0 and 1, etc. And that is what Brouwer did. He defined cuts as above and established sets of dimension (dimensional degree) 0 as those which (in the modern terminology) do not contain any non-degenerate connected subset. The value of his definition has been shown immediately by the theorem that each n-dimensional manifold has dimensional degree equal exactly to n.

One year after the appearance of the last Brouwer's paper related to the problem of dimension, World War I had broken up and most of the hitherto achievements were to be forgotten. After war the theory will revive, but on a much simpler basis.

Of the growing necessity of the concept of dimension may testify the paper of Sierpiński [37] in which he considered the following property of a subset X of a euclidean space: for each point p of X and each $\varepsilon > 0$ the set X is the union of two disjoint and closed (in X) subsets A and B such that $p \in A$ and A lies within a ball of diameter ε. With the appearance of dimension theory the property will become equivalent to say that X is 0-dimensional.

The modern concept of dimension has been proposed by Karl Menger and P. S. Urysohn (1898-1924).

Menger claims that he had sent first a note containing his definition to the journal Monatshefte für Mathematik und Physik as early as autumn 1921 (cf. [28], p. 83-86), but the definition appeared in print only in 1923 (see [27]). The definition runs as follows:

..."A non-empty set M in the space R_m (i.e., m-dimensional euclidean space — R. D.) is called n-dimensional if 1. for each point p and each open neighbourhood $U_1(p)$ there exists an open neighbourhood $U_2(p) \subset U_1(p)$ the boundary of which has at most $(n-1)$-dimensional common part with M, and if 2. M contains at least one point q for which there exists an open neighbourhood U such that M meets the boundary of any open neighbourhood $U_1(q) \subset U$ in a set which contains $(n-1)$-dimensional part. (-1)-dimensional is empty set."

And in a shorter form (cf. [28], p. 80):

..."A space is called n-dimensional, if n is the least number such that each point of the space is contained in arbitrarily small neighbourhoods with at most $(n-1)$-dimensional boundaries."

The definition follows the line proposed by Poincaré and is much simpler than the dimensional degree of Brouwer. Laying closer comparison of the two for a while aside, let us follow a mental experiment of Menger ([28], p. 78 ff.) which is perhaps the best way to make clear the underlying intuition of Menger's concept:

"... A simple experiment with an individual object, for the fulfillment of which we may think of a solid as made of a wood, of a surface as made of a thin metal sheet, and of a curve as made of a fine wire, gives characteristic result for its dimension. The experiment consists in extracting a point together with the points of a vicinity from the object. If we want to extract a point together with the vicinity from a wooden solid, we must pull out a saw and cut across some surfaces. If we want to extract a point together with the vicinity from a metal surface, we use scissors and cut the surface along some curves. If we want to extract from a curve a point together with the all points of its vicinity, then pincers, no matter how the curve is ramified or tangled, suffice to cut the curve in discrete points through. And finally, when we consider a sand form and from it we also want to extract a point together with all points in its vicinity, then we see that no tool is necessary, because in discrete objects nothing is to separate.

Now there are also objects which bear themselves with respect to their dimension differently in different points. Consider a wooden solid to which a metal surface is added and, moreover, in some points a wire. We call the object 3-dimensional in its entirety, but at the same time we say that in some points it behaves like a surface and in some points like a curve. (...)

To make these results precise for general spaces, let us notice that the vicinity employed in the experiment corresponds to a neighbourhood of that point. And that what in the experiment must be passed to extract such a neighbourhood, what connects the neighbourhood with the rest of the space — corresponds obviously to the boundary of that neighbourhood. ..."

As we have noticed, two men have rendered services to the cause of dimension. The second was Paul S. Urysohn who found in 1924 his untimely death in the Atlantic when the incoming storm has brought the swimmer upon rocks near Le Batz (France). Three years earlier his professor D. F. Yegorov (1869-1931) has put to him the problem to find a definition of a line such that in the plane were equivalent to that of Cantor but differed essentially in the manner of formulation: new definition had to be "internal", i.e. without appealing to external space.

The friend of P. S. Urysohn, P. S. Aleksandrov, recalls ([40], Introduction):

"... Paul begun to think immediately. (...) And very soon the object of his considerations became a general concept of dimension. Making incessant trials, he kept working all the summer 1921. Constructing examples to show why this or that effort is invalid, he has been passing from one possibility to another. Two months of unusually deep meditations elapsed, but finally, some morning at the end of August, Paul roused from sleep with the ready and now known to everybody definition of an inductive dimension. This happened in the village Burkovo, near Bolševo, on the coast of Klasma river, where a group of young mathematicians from Moscow spent their holidays. And that very morning, while bathing in Klasma, he told me his definition and in a talk which lasted several hours afterwards he sketched the plan of dimension theory with a long series of theorems which were then only hypotheses and nobody knew how to treat them but which were in the next few months proved one by one. The sketched plan has been completely fulfilled during winter 1921/22 and in spring of 1922 the theory has been done."

In September 1922 there has appeared a small note [38] (complete exposure of the theory has been published posthumously in Fundamenta Mathematicae [39]) containing the following definition:

"Let C be a set, and p its given point. We say that a subset B of C ε-separates point p if the difference $C-B$ can be decomposed into union of two sets A and D such that

1. sets A and D are separated, i.e. they do not meet and neither contains any point of convergence of the other,
2. set A contains the point p,
3. set A is contained in a ball of radius ε.

Sets and points of dimension n will be defined inductively. For that purpose assume that sets and points of dimension $< n$ have been already defined.

If a point p is not of a dimension $< n$ with respect to C, but for each ε it can be ε-separated by a set of dimension $< n$, then we say that p is of dimension n with respect to C. And if each point of C is of dimension $\leqslant n$ with respect to C and there are points, with respect to which C is of dimension n, then C itself is called a set of dimension n."

The definition of Urysohn is valid, literally speaking, for metric spaces only, but if one replaces metric balls of arbitrarily small radius around p by (in modern terminology) arbitrarily small neighbourhoods of p — one gets a definition which has a meaning for general topological spaces. And moreover, since the set B from the Urysohn's definition either is a boundary of a neighbourhood of p or contains such a boundary and, conversely, since each boundary of a neighbourhood of p ε-separates point for some ε — both definitions, of Menger and of Urysohn alike, express the same intuition. And since both start at the same level -1, which is the dimension of empty and only empty set, both are equivalent.

We have already noted that the definition of Menger appeared in print later than that of Urysohn, but Menger claims to have it earlier. This has stirred up a harsh controversy on priority (cf. an introduction of P. S. Aleksandrov to collected papers of P. S. Urysohn [40]), but it is now rather commonly accepted that both definitions have been obtained independently.

Another bitter controversy concerning priority has aroused between Brouwer [6]-[8] and Menger [29]-[31]; cf. commentaries of H. Freudenthal to [10].

Brouwer claimed that his dimensional degree had to be equivalent to Menger's dimension but a misprint spoiled it and, moreover, that even keeping it one can construct another but equally good theory of dimension. Neither argument seems to be convincing. Brouwer used his definition as it was to prove such basic results as that concerning dimension of euclidean spaces. And making a stronger appeal in his definition of a cut to connectedness which is quite difficult to deal with, he made his definition of dimensional degree much more complicated and apparently much more distant from the intuition than Menger's definition of dimension. The difference between the two definitions starts already at the level 0: a set has Brouwer's dimensional degree 0 if it does not contain any connected subset, and a set has dimension 0 in the sense of Menger-Urysohn if for each point of it there exist arbitrarily small neighbourhoods with empty boundary. Thus each set of dimension 0 has dimensional degree 0, but it turned out soon that not conversely: there exist sets of each positive dimension, including infinity, which do not contain any connected subset consisting of more than one point and thus which have dimensional degree 0 [26]. Thus the class of 0-dimensional spaces is contained in and is not equal to the class of spaces with dimensional degree 0, and this leads (an easy inductive proof) to the inequality

$$\text{dimensional degree of } X \leqslant \text{dimension of } X$$

for each topological space X, where on the left-hand side may appear 0 and (for the same X) on the right-hand side even infinity.

It follows that the two definitions, that of Menger-Urysohn and that of Brouwer, lead to distinct stratifications of general topological spaces with respect to their dimension or dimensional degree and that stratification with respect to dimension is more subtle.

On the other hand, the two definitions are based upon the same geometric intuition, both are inductive and, in fact, for a vast class of spaces, including locally connected metric continua, both are equivalent. As Brouwer himself has admitted [7], if one replaces the Brouwer definition of a cut by that of Urysohn and starts induction at the level not 0 but -1 (dimension of empty set), the two definitions become fully equivalent. Thus the two definitions are in a sense close to each other but not identical. And since the theory based upon the definition of Menger-Urysohn has gained imminent success, while the possible theory based upon the definition of Brouwer offered serious difficulties due to the more complicated concept of a cut – nobody, including Brouwer himself, undertook the building of the latter. The hard, good work on dimensional degree has sunk into oblivion (but the work which aroused out of it, e.g. the important theorem on invariance of domain, has maintained its great value).

The appearance of the definition of Menger-Urysohn closes the period of about half a century search after a good mathematical expression of a concept of dimension and opens the period of evolving dimension theory based upon it, which lasts to the present days. In the period of search two trends could be observed. The first, one may call it geometric, consisted in the analysis of a relation, suggested by the tradition and the nature of the then considered objects (lines, surfaces, euclidean spaces, manifolds), between the number of parameters needed for the description of an object and the dimension of the object, its "first and most important" geometric invariant. The relation has been soon clarified in the form of the basic problem "Can two euclidean spaces of distinct dimensions be homeomorphic to each other?" and the negative solution of that problem some thirty years later has proved the geometric validity of the notion of dimension, at least for manifolds. The milestones of the geometric trend are the names Cantor, Dedekind, Peano and Brouwer. The unexpected discovery of Peano has shown the limits of applicability of that relation and it was the beginning of the second trend which most rightly can be called topological. It was a search after a new, deeper, intuitive basis of dimension and its mathematical shaping, which had to be appliable to general topological spaces and be compatible with the already well established dimension of euclidean spaces. The milestones here are works of Peano, Poincaré, Brouwer, Menger and Urysohn. Somewhat aside lies the original and important by its consequences observation made by Lebesgue.

The story told above shows clearly that a construction of a working mathematical concept is by no means a trivial task. In the case of dimension it took about half a century of hard work of many of the brightest men of the time. Much of that work has been made in vain, but in a sense each contributed to the final solution. Such was the price for a progress. And in the early twenties there are at hand two good definitions of dimension: inductive of Menger-Urysohn and covering of Lebesgue, and the problem of dimension is well installed within that new branch of mathematics which justifies its appearance and in turn is justified by its validity — topology. Gaining momentum, the problem of dimension will in a few years expand to the still growing dimension theory.

Added in proof. After the article had been written, I learned of early Bernard Bolzano's (1741-1848) analysis of the problem of dimension (cf. Dale M. Johnson, *Prelude to dimension theory: the geometrical investigations of Bernard Bolzano*, Archive for the History of Exact Sciences 17 (1977), p. 262-295). Bolzano's interest, spanning virtually his entire lifetime, has led him to ingenious definitions of a line, surface and solid in 1817 and to new ones in 1830's and 40's, based upon some set-theoretical and topological methods of his own. However, all this was far ahead of his time and he had no followers to take the task up where he had left it. His contribution has thus remained totally unknown to the people who some thirty years after his death raised the problem anew as well as to those who eventually solved it.

REFERENCES

[1] *Briefwechsel Cantor-Dedekind*, Paris 1937.

[2] L. E. J. Brouwer, *Beweis der Invarianz der Dimensionenzahl*, Mathematische Annalen 70 (1911), p. 161-165.

[3] — *Beweis der Invarianz des n-dimensionalen Gebiets*, ibidem 71 (1912), p. 305-313.

[4] — *Über den natürlichen Dimensionsbegriff*, Journal für die reine und angewandte Mathematik 142 (1913), p. 146-152.

[5] — *Berichtigung zur Abhandlung „Über den natürlichen Dimensionsbegriff"*, ibidem 153 (1923), p. 253.

[6] — *Über den natürlichen Dimensionsbegriff*, Proceedings Akademie Amsterdam 26 (1923), p. 795-800.

[7] — *Bemerkungen zum natürlichen Dimensionsbegriff*, ibibem 27 (1924), p. 635-638.

[8] — *Zum natürlichen Dimensionsbegriff*, Mathematische Zeitschrift 21 (1924), p. 312-314.

[9] — *Zur Geschichtschreibung der Dimensionstheorie*, Proceedings Akademie Amsterdam 31 (1928), p. 953-957.

[10] — Collected papers.

[11] G. Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik 84 (1978), p. 242-258.

[12] — *Über einen Satz aus der Theorie der stetigen Mannigfaltigkeiten*, Göttinger Nachrichten (1879), p. 127-135.

[13] — *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, Leipzig 1883.

[14] — *Gesammelte Abhandlungen*, Berlin 1932.

[15] M. Fréchet, *Les dimensions d'un ensemble abstrait*, Mathematische Annalen 68 (1910), p. 145-168.

[16] H. G. Grassmann, *Gesammelte mathematische und physikalische Werke*, 3 volumes, Leipzig 1894-1911. (Volume I contains both *Die lineare Ausdehnungslehre* of 1844 and *Die Ausdehnungslehre* of 1862.)

[17] H. Hahn, *Über die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist*, Jahresbericht der Deutschen Mathematiker-Vereinigung 23 (1914), p. 318-322.

[18] C. Jordan, *Cours d'analyse de l'Ecole Polytechnique*, 2 volumes, Paris 1893-1894.

[19] F. Klein, *Gutachten, betreffend den dritten Band der Theorie der Transformationsgruppen von S. Lie anläßlich der ersten Verteilung des Lobatschewsky-Preises*, Mathematische Annalen 50 (1898), p. 583-600.

[20] H. Lebesgue, *Sur la non-applicabilité de deux domaines appartenant respectivement à n et n+p dimensions*, ibidem 70 (1911), p. 162-168.

[21] — *Sur la correspondance entre les points de deux espaces*, Fundamenta Mathematicae 2 (1921), p. 257-285.

[22] A. Lelek, *O funkcjach Peany* [*On functions of Peano*], Prace Matematyczne 7 (1962), p. 127-140. [in Polish]

[23] J. Lüroth, *Über Abbildung von Mannigfaltigkeiten*, Mathematische Annalen 63 (1907), p. 222-238.

[24] S. Mazurkiewicz, *O punktach wielokrotnych krzywych wypelniających obszar plaski* [*On multiple points of curves filling a plane region*], Prace Matematyczno-Fizyczne 26 (1915), p. 113-120. [in Polish]

[25] — *Sur les lignes de Jordan*, Fundamenta Mathematicae 1 (1920), p. 166-209.

[26] — *Sur les problèmes κ et λ de Urysohn*, ibidem 10 (1929), p. 311-319.

[27] K. Menger, *Über die Dimension von Punktmengen*, Monatshefte für Mathematik und Physik 33 (1923), p. 148-160; ibidem 34 (1924), p. 137-161.

[28] — *Dimensionstheorie*, Leipzig - Berlin 1928.

[29] — *Zur Dimensions- und Kurventheorie, Unveröffentlichte Aufsätze aus den Jahren 1921-1923*, Monatshefte für Mathematik und Physik 36 (1929), p. 411-432.

[30] — *Antwort auf eine Note von Brouwer*, ibidem 37 (1930), p. 175-182.

[31] — *Eine Zuschrift von K. Menger an H. Hahn, betreffend Mengers Antwort auf eine Note von Brouwer in Bd. 37 der Monatshefte*, ibidem 40 (1933), p. 233.

[32] E. Netto, *Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik 86 (1879), p. 263-268.

[33] G. Peano, *Sur une courbe qui remplit toute une aire plane*, Mathematische Annalen 36 (1890), p. 157-180.

[34] H. Poincaré, *Pourquoi l'espace a trois dimensions?*, Revue de Metaphysique et de Morale 20 (1912), p. 483-504.

[35] — *Dernières pensées*, Paris 1920.

[36] G. B. Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, Berlin 1923 (several editions).

[37] W. Sierpiński, *Sur les ensembles connexes et non connexes*, Fundamenta Mathematicae 2 (1921), p. 81-95.

[38] P. Urysohn, *Les multiplicités cantoriennes*, Comptes Rendus de l'Académie de Paris 175 (1922), p. 440-442.

[39] — *Mémoire sur les multiplicités cantoriennes*, Fundamenta Mathematicae 7 (1925), p. 30-137; ibidem 8 (1926), p. 225-351.

[40] П. С. Урысон, *Труды по топологии и другим областям математики*, 2 volumes, Москва-Ленинград 1951.

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