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Non-negative solutions of some non-linear integral equations

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Abstract. We consider non-negative solutions u of the non-linear convolution equation

$$W \circ u = K * u,$$

where the kernel K is a locally bounded non-negative measurable function vanishing on $(-\infty, 0)$ and W is a convex function $(W(0) = 0)$ satisfying certain additional conditions.

We give theorems concerning the existence and uniqueness of locally bounded measurable solutions vanishing on $(-\infty, 0]$ and positive on $(0, +\infty)$.

1. Introduction. In papers [2], [3] and [4] are considered non-negative solutions u of the equation

$$u^\alpha(x) = \int_0^x K(x-\tau)u(\tau)\,d\tau \qquad (\alpha > 1), \tag{1.1}$$

where K is a non-negative function satisfying certain additional conditions. The solvability of (1.1) is inspected in the function class M_0 defined as the set of all locally bounded measurable functions vanishing on the non-positive half-line and positive on the positive half-line. The existence and uniqueness of solutions u in M_0 is showed in [2], [3] and [4]. The proofs are based on the Banach fixed point theorem.

Here we consider the more general equation

$$W(u(x)) = \int_0^x K(x-\tau)u(\tau)\,d\tau, \tag{1.2}$$

where W is a convex function satisfying certain additional conditions. The kernel K may be of a more general form than in (1.1). Proofs of the existence and uniqueness of solutions in M_0 will be given without use of the Banach fixed point theorem.

We impose the following assumptions on the kernel K:

(K_1) K: $R \to R_+$ is a locally bounded non-negative measurable function ($R_+ = \{x \in R:\ x \geqslant 0\}$),

(K_2) $K(x) = 0$ for $x < 0$,

(K_3) there exist numbers $\varepsilon_1 > 0$, $C_1 > 0$ and $\alpha_1 > 0$ such that $k(x) \geqslant C_1 x^{\alpha_1}$ for $x \in [0, \varepsilon_1]$.

As regards W, we assume:

(W_1) W: $R_+ \to R_+$ is differentiable on R_+,

(W_2) $W(0) = W'(0) = 0$,

(W_3) W' is a strictly increasing continuous function,

(W_4) $\lim_{x \to +\infty} W'(x) = +\infty$,

(W_5) there exist numbers $\varepsilon_2 > 0$, $C_2 > 0$ and $\alpha_2 > 1$ such that $W(x) \leqslant C_2 x^{\alpha_2}$ for $x \in [0, \varepsilon_2]$.

Under these assumptions we will state our results.

2. Some auxiliary lemmas. We introduce certain auxiliary functions and examine their properties.

LEMMA 2.1. *Let*

$$H_1(x) = \begin{cases} W(x)x^{-1} & \textit{for } x > 0, \\ 0 & \textit{for } x = 0. \end{cases} \tag{2.1}$$

Then H_1 is a strictly increasing continuous function from R_+ onto R_+.

Proof. We have

$$\lim_{x \to 0+} W(x)x^{-1} = \lim_{x \to 0+} W'(x) = 0$$

and

$$\lim_{x \to +\infty} W(x)x^{-1} = \lim_{x \to +\infty} W'(x) = +\infty.$$

By (W_3) we get that $W(x)x^{-1}$ is strictly increasing. Hence our lemma is true.

COROLLARY 2.1. *There exists the strictly increasing function F: $R_+ \to R_+$ inverse to H_1. The function F is continuous.*

Remark 2.1. By (W_5) we have

$$F(x) \geqslant [x/C_2]^{1/(\alpha_2 - 1)} \quad \text{for } x \in [0, C_2 \varepsilon_2^{\alpha_2}].$$

LEMMA 2.2. *Let*

$$H_2(x) = \int_0^x W(\tau)\tau^{-1} d\tau. \tag{2.2}$$

Then H_2 is a strictly increasing continuous function from R_+ onto R_+.

The proof of this lemma results from Lemma 2.1.

COROLLARY 2.2. *There exists the strictly increasing continuous function G: $R_+ \to R_+$ inverse to H_2.*

Now we give an a priori estimate of the solution u of (1.2).

LEMMA 2.3. *If $u \in M_0$ is a solution of* (1.2), *then*

$$u(x) \leqslant F\Big(\int_0^x K(\tau)\,d\tau\Big), \tag{2.3}$$

where F denotes the function defined in Corollary 2.1.

Proof. Let $x > 0$. Since

$$W(u(s)) \leqslant [\sup_{s \in I_x} u(s)] \int_0^s K(\tau)\,d\tau \qquad \text{for } s \in I_x,$$

where $I_x = [0, x]$, then

$$W(\sup_{s \in I_x} u(s)) \leqslant \sup_{s \in I_x} u(s) \int_0^x K(\tau)\,d\tau.$$

From the last inequality, by Lemma 2.1, we get

$$\sup_{s \in I_x} u(s) \leqslant F\Big(\int_0^x K(\tau)\,d\tau\Big).$$

Hence (2.3) is true.

Remark 2.2. Let

$$\varphi(x) = \begin{cases} 0 & \text{for } x < 0, \\ F\Big(\int_0^x K(\tau)\,d\tau\Big) & \text{for } x \geqslant 0. \end{cases} \tag{2.4}$$

Then φ is a strictly increasing continuous function belonging to M_0.

Remark 2.3. By (K_3) and Remark 2.1

$$\varphi(x) \geqslant \left[\frac{C_1}{C_2}\right]^{1/(\alpha_2 - 1)} \left[\frac{1}{\alpha_1 + 1}\right]^{1/(\alpha_2 - 1)} x^{(\alpha_1 + 1)/(\alpha_2 - 1)} \tag{2.5}$$

for $x \in J$, where

$$J = \left[0, \left(\frac{C_2}{C_1}(\alpha_1 + 1)\varepsilon_2^{\alpha_2}\right)^{1/(\alpha_1 + 1)}\right]. \tag{2.6}$$

3. Existence of solutions. We define the following operator:

$$T(f) = W^{-1}(K * f) \qquad \text{for } f \in M_0, \tag{3.1}$$

where W^{-1} is inverse to W.

Remark 3.1. For $f_1 \leqslant f_2$ we have $T(f_1) \leqslant T(f_2)$.

Remark 3.2. If $f \in M_0$ is a non-decreasing function, then $T(f)$ is a non-decreasing function belonging to M_0.

We will need certain auxiliary theorems in the proof of the existence of non-negative solutions.

THEOREM 3.1. *The function φ defined by* (2.4) *satisfies*

$$T(\varphi) \leqslant \varphi. \tag{3.2}$$

Proof. For $x > 0$ inequality (3.2) is equivalent to the following one

$$(K * \varphi)(x) \leqslant W\big(\varphi(x)\big). \tag{3.3}$$

We shall prove inequality (3.3). By Remark 2.2 we have

$$(K * \varphi)(x) \leqslant \int_0^x K(\tau)\,d\tau\, F\Big(\int_0^x K(\tau)\,d\tau\Big). \tag{3.4}$$

From (3.4) we get

$$(K * \varphi)(x) \leqslant H_1\Big(F\Big(\int_0^x K(\tau)\,d\tau\Big)\Big) F\Big(\int_0^x K(\tau)\,d\tau\Big). \tag{3.5}$$

By Definition 2.1 we obtain

$$H_1\Big(F\Big(\int_0^x K(\tau)\,d\tau\Big)\Big) F\Big(\int_0^x K(\tau)\,d\tau\Big) = W\Big(F\Big(\int_0^x K(\tau)\,d\tau\Big)\Big). \tag{3.6}$$

From (3.5) and (3.6) we get (3.2).

We first examine equation (1.2) in a particular form. We denote by x_+^α the following function:

$$x_+^\alpha = \begin{cases} x^\alpha & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0 \end{cases} \quad (\alpha > 0). \tag{3.7}$$

This function is in M_0.

THEOREM 3.2. *The equation*

$$u^{\alpha_2}(x) = \left(\frac{C_1}{C_2}\, x_+^{\alpha_1} * u\right)(x) \tag{3.8}$$

has a solution in M_0.

Proof. The following formula is known:

$$\frac{x_+^{\mu_1}}{\Gamma(\mu_1+1)} * \frac{x_+^{\mu_2}}{\Gamma(\mu_2+1)} = \frac{x_+^{\mu_1+\mu_2+1}}{\Gamma(\mu_1+\mu_2+2)} \quad (\mu_1, \mu_2 > 0), \tag{3.9}$$

where Γ is the Euler function (see [1]). Let us note that

$$u_0(x) = \left[\frac{C_1}{C_2}\,\frac{\Gamma(\alpha_1+\alpha_2+2)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\right]^{1/(\alpha_2-1)} x_+^{(\alpha_1+1)/(\alpha_2-1)} \tag{3.10}$$

is a solution of (3.7). The theorem is proved.

We denote by φ_0 the function

$$\varphi_0(x) = \left[\frac{C_1}{C_2(\alpha_1+1)}\right]^{1/(\alpha_2-1)} x_+^{(\alpha_1+1)/(\alpha_2-1)}.$$

Then inequality (2.5) may be written as

$$\varphi(x) \geqslant \varphi_0(x) \quad \text{for } x \in J. \tag{3.11}$$

We now prove the following theorem:

THEOREM 3.3. *The sequence* (f_n) *of functions from* M_0 *defined by*

$$f_0 = \varphi_0, \quad f_{n+1} = T_1(f_n) \quad (n = 0, 1, 2, \ldots), \tag{3.12}$$

where

$$T_1(f) = \left(\frac{C_1}{C_2} x_+^{\alpha_1} * f\right)^{1/\alpha_2} \quad (f \in M_0), \tag{3.13}$$

is convergent to u_0, *where* u_0 *is given by formula* (3.10).

Proof. Let

$$A = \frac{\Gamma(\alpha_1+\alpha_2+2)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}. \tag{3.14}$$

By (3.9) we may write

$$f_n(x) = \left[\frac{C_1}{C_2}\right]^{1/(\alpha_2-1)} \left[\frac{1}{\alpha_1+1}\right]^{1/(\alpha_2-1)\alpha_2^n} A^{\sum_{k=1}^{n} 1/\alpha_2^k} x_+^{(\alpha_1+1)/(\alpha_2-1)}. \tag{3.15}$$

Since $\alpha_2 > 1$, we have

$$\lim_{n\to\infty} f_n(x) = \left[\frac{C_1}{C_2} A\right]^{1/(\alpha_2-1)} x_+^{(\alpha_1+1)/(\alpha_2-1)}. \tag{3.16}$$

The theorem is proved.

Now we give the theorem about the existence of non-trivial non-negative solutions of (1.2).

THEOREM 3.4. *Equation* (1.2) *has a solution in the class* M_0.

Proof. We define the sequence (g_n) of functions from M_0:

$$g_0 = \varphi, \quad g_{n+1} = T(g_n) \quad (n = 0, 1, 2, \ldots). \tag{3.17}$$

Since $\varphi(x) \geqslant \varphi_0(x)$ for $x \in J$, then, by (K_3) and (W_5) we have

$$T(g_n)(x) \geqslant T_1(f_n)(x) \quad \text{for } x \in J \text{ and } n = 0, 1, 2, \ldots, \tag{3.18}$$

where T_1 is defined by (3.13) and f_n by (3.12). By Theorem 3.1 we obtain

$$T^{n+1}(\varphi) \leqslant T^n(\varphi) \quad \text{for } n = 0, 1, 2, \ldots \tag{3.19}$$

Then

$$g_{n+1} \leqslant g_n \quad \text{for } n = 0, 1, 2, \ldots \tag{3.20}$$

Hence there exists $\lim_{n\to\infty} g_n(x)$. We denote this function by $u(x)$. Since (g_n) is a sequence of non-decreasing functions, $u(x)$ is a non-decreasing function. By (3.18) we obtain

$$u(x) \geqslant u_0(x) \quad \text{for } x \in J. \tag{3.21}$$

Since u is non-decreasing, then, by (3.21), we see that $u \in M_0$. The existence of the solution is proved.

Remark 3.1. Every solution $u \in M_0$ of (1.2) is a continuous function.

4. Uniqueness of non-negative solutions. We prepare the following lemma:

LEMMA 4.1. *If $M > 0$ is a number such that $\varphi(x) \leqslant M$ for $x \in I_\delta$, where $I_\delta = [0, \delta]$ and $\delta > 0$, then*

$$T(M)(x) \leqslant M \quad \textit{for } x \in I_\delta. \tag{4.1}$$

Proof. From our assumptions we obtain

$$\int_0^x K(\tau)\,d\tau \leqslant W(M)M^{-1} \quad \text{for } x \in I_\delta. \tag{4.2}$$

By the definition of T we get

$$T(M)(x) = W^{-1}\Big(M \int_0^x K(\tau)\,d\tau\Big). \tag{4.3}$$

Since W^{-1} is an increasing function, then, by (4.2) and (4.3), we have

$$T(M)(x) \leqslant W^{-1}(W(M)). \tag{4.4}$$

From the last inequality we see that the lemma is true.

Now we state the following theorem:

THEOREM 4.1. *If $\varphi(x) \leqslant M$ for $x \in I_\delta$, then the sequence (h_n) of functions defined by*

$$h_0 = M, \quad h_{n+1} = T(h_n), \tag{4.5}$$

is convergent to a solution v of (1.2) *on I_δ such that $v(x) > 0$ for $x \in (0, \delta]$.*

Proof. Since, by assumptions,

$$u(x) \leqslant M \quad \text{on } I_\delta,$$

where $u \in M_0$ is the solution of (1.2) as in Theorem 3.4, we have

$$u(x) \leqslant h_0(x) \quad \text{on } I_\delta. \tag{4.6}$$

From (4.5) and (4.6) we get

$$u(x) \leqslant h_n(x) \quad \text{for } x \in I_\delta \text{ and } n = 0, 1, 2, \ldots \tag{4.7}$$

By Lemma 4.1 we obtain

(4.8) $$T^{n+1}(h_0)(x) \leqslant T^n(h_0)(x) \quad \text{for } x \in I_\delta \text{ and } n = 0, 1, 2, \ldots$$

The last inequality is equivalent to the following one:

(4.9) $$h_{n+1}(x) \leqslant h_n(x).$$

From (4.7) and (4.9) we infer that $\lim_{n\to\infty} h_n(x)$ exists. We denote this function by $v(x)$. The function v satisfies (1.2) on I_δ and, by (4.6), $v(x) > 0$ for $x \in (0, \delta]$.

THEOREM 4.2. *Let $m < M$ and $v(\delta_1) = m$, where $\delta_1 \in (0, \delta)$. Let*

(4.10) $$k_0(x) = \begin{cases} v(x) & \text{for } x \in I_{\delta_1}, \\ m & \text{for } x \in (\delta_1, \delta]. \end{cases}$$

Then the sequence (k_n) of functions defined by $k_{n+1} = T(k_n)$ $(n = 0, 1, 2, \ldots)$ is convergent to v on I_δ.

Proof. Since $T(k_0)(x) = v(x)$ for $x \in I_{\delta_1}$ and $T(k_0)$ is a non-decreasing function, we have

$$T(k_0)(x) \geqslant T(k_0)(\delta_1) = m \quad \text{for } x \in (\delta_1, \delta].$$

Hence

(4.11) $$T(k_0)(x) \geqslant k_0(x) \quad \text{for } x \in I_\delta.$$

From (4.11) we get

(4.12) $$k_n(x) \leqslant k_{n+1}(x).$$

But

(4.13) $$k_0(x) \leqslant M \quad \text{for } x \in I_\delta.$$

We have by (4.13)

(4.14) $$k_n(x) \leqslant h_n(x) \leqslant M \quad \text{for } x \in I_\delta \text{ and } n = 0, 1, 2, \ldots$$

By (4.12) and (4.14) the sequence (k_n) converges to a function υ satisfying equation (1.2) such that

(4.15) $$\upsilon(x) = v(x) \quad \text{for } x \in I_{\delta_1}$$

and

(4.16) $$\upsilon(x) \leqslant v(x) \quad \text{for } x \in (\delta_1, \delta].$$

Let $x \in (\delta_1, \delta]$. Then by the mean value theorem

(4.17) $$v(x) - \upsilon(x)$$
$$= T(v)(x) - T(\upsilon)(x) \leqslant \big(K * (v - \upsilon)\big)(x)\left[W'\big(W^{-1}((K * \upsilon)(x))\big)\right]^{-1}.$$

By (4.15) we have

$$(4.18)\qquad \big(K*(v-\mathbf{v})\big)(x)\Big[W'\Big(W^{-1}\big((K*\mathbf{v})(x)\big)\Big)\Big]^{-1} \leqslant \int_{\delta_1}^{x} K(x-\tau)[v(\tau)-\mathbf{v}(\tau)]\,d\tau\,\big[W'(v(\delta_1))\big]^{-1}$$

for $x\in(\delta_1, \delta]$. Let $M_1 = \sup_{x\in I_\delta} K(x)$ and

$$(4.19)\qquad d(v, \mathbf{v}) = \sup_{\tau\in(\delta_1,\delta]} e^{-\beta\tau}[v(\tau)-\mathbf{v}(\tau)],$$

where $\beta > 0$. We obtain

$$(4.20)\qquad \int_{\delta_1}^{x} K(x-\tau)[v(\tau)-\mathbf{v}(\tau)]\,d\tau \leqslant d(v, \mathbf{v}) \int_{\delta_1}^{x} K(x-\tau)e^{\beta\tau}\,d\tau.$$

From (4.20) we get

$$(4.21)\qquad \int_{\delta_1}^{x} K(x-\tau)[v(\tau)-\mathbf{v}(\tau)]\,d\tau \leqslant d(v, \mathbf{v}) \int_{0}^{x} K(x-\tau)e^{\beta\tau}\,d\tau.$$

By the commutativity of convolution we have

$$(4.22)\qquad \int_{0}^{x} K(x-\tau)e^{\beta\tau}\,d\tau = e^{\beta x}\int_{0}^{x} K(\tau)e^{-\beta\tau}\,d\tau.$$

From (4.20), (4.21) and (4.22) we get

$$(4.23)\qquad \int_{\delta_1}^{x} K(x-\tau)[v(\tau)-\mathbf{v}(\tau)]\,d\tau \leqslant e^{\beta x}M_1\beta^{-1}(1-e^{-\beta x})\,d(v, \mathbf{v}).$$

From (4.23) we obtain

$$(4.24)\qquad \int_{\delta_1}^{x} K(x-\tau)[v(\tau)-\mathbf{v}(\tau)]\,d\tau \leqslant M_1\beta^{-1}e^{\beta x}d(v, \mathbf{v})$$

for $x\in(\delta_1, \delta]$. Now we assume that

$$(4.25)\qquad \beta \geqslant 2M_1\big[W'(v(\delta_1))\big]^{-1}.$$

From (4.17), by (4.18), (4.23) and (4.25), we obtain

$$(4.26)\qquad v(x)-\mathbf{v}(x) \leqslant \tfrac{1}{2}e^{\beta x}d(v, \mathbf{v}) \quad \text{for } x\in(\delta_1, \delta].$$

From the last inequality we infer that

$$d(v, \mathbf{v}) = 0.$$

Hence $v(x) = \mathbf{v}(x)$ for $x\in(\delta_1, \delta]$, too. The theorem is proved.

Remark 4.1. If a sequence of functions $l_n(x)$ is convergent to a solution $l(x)$ of (1.2), then the sequence $l_n(x-\eta)$ is convergent to the function $l(x-\eta)$, which is a solution of (1.2).

THEOREM 4.3. *Equation* (1.2) *has the unique solution in* M_0.

Proof. Let $u_1,\ u_2 \in M_0$ be two solutions of (1.2). Let $\delta > 0$ be any number and $M = \varphi(\delta)$. Let $\eta > 0$ be any number such that $0 < \eta < \delta$ and

$$m = \min\{\min_{[\eta,\delta]} u_1,\ \min_{[\eta,\delta]} u_2\}. \tag{4.27}$$

Let

$$k_0^\eta(x) = v(x) \quad \text{for } x \in I_\delta \tag{4.28}$$

in the case where $v(x) < m$ for $x \in I_{\delta-\eta}$, and let

$$K_0^\eta(x) = \begin{cases} v(x-\eta) & \text{for } x \in [\eta,\ \delta_1], \\ m & \text{for } x \in (\delta_1,\ \delta], \end{cases} \tag{4.29}$$

when $v(\delta_1 - \eta) = m$. Then

$$k_0^\eta(x) \leqslant u_i(x) \leqslant M \quad (i = 1,\ 2). \tag{4.30}$$

Hence

$$k_n^\eta(x) \leqslant u_i(x) \leqslant h_n(x) \quad (i = 1,\ 2), \tag{4.31}$$

where $k_{n+1}^\eta = T(k_n^\eta)$ $(n = 0,\ 1,\ 2, \ldots)$.

Let $n \to \infty$. From (4.31), by Theorems 4.1, 4.2 and Remark 4.1, we obtain

$$v(x-\eta) \leqslant u_i(x) \leqslant v(x) \quad \text{for } i = 1,\ 2 \text{ and } x \in I_\delta. \tag{4.32}$$

Let $\eta \to 0+$. By the continuity of solutions (see Remark 3.1) we get

$$v(x) \leqslant u_i(x) \leqslant v(x) \quad \text{for } i = 1,\ 2.$$

The uniqueness is proved.

5. Final remarks. For the proof of the uniqueness assumption (W_5) is not needed. Assumption (K_3) may be replaced by a more general one, namely:

(K_3') For each $x > 0$ the integral

$$\int_0^x K(\tau)\, d\tau$$

is a positive number.

Under this assumption the uniqueness may be showed.

References

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