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ON MAPPINGS BETWEEN QUASI-ALGEBRAS

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A theory of mappings between quasi-algebras is a generalization of that of homomorphisms. A *homomorphism* between quasi-algebras of the type F is any mapping φ which fulfils the following basic mapping formulas:

$$\varphi\big(f(x, \xi < n(f))\big) = f(\varphi(x_\xi), \xi < n(f)), \tag{1}$$

where $f \in F$ is an operator symbol. The mappings φ which satisfy the basic mapping-formulas of the form

$$\varphi\big(f(x_\xi, \xi < n(f))\big) = \tau(\varphi(x_\xi), \xi < n(f)), \tag{2}$$

where $f \in F$ and τ is a G-term belonging to $P(f)$, are *P-homomorphisms* between quasi-algebras of the type F and of the type G. For a theory of P-homomorphisms see my paper [4].

The general basic-mapping-formula has the following form:

$$\varphi_\mu\big(f(x_\xi, \xi < n(f))\big) = \tau(\varphi_\sigma(x_\xi), \sigma < a, \xi < n(f)), \tag{3}$$

where $\mu < a$, f is an operator symbol in F, and τ is a G-mapping-term.

The systems φ_σ, $\sigma < a$, of mappings of quasi-algebras of the type F into quasi-algebras of the type G, which satisfy a family P of general basic mapping-formulas, are called ***systems of P-mappings***. A theory of systems of P-mappings between algebras with finitary operations is given by Fujiwara [1]. In this paper we give a generalization of this theory not only for arbitrary algebras but also for quasi-algebras with infinitary partial operations. Moreover, we obtain the existence theorems on $\mathfrak{B}$-free systems of $\mathfrak{B}$-P-mappings and $\mathfrak{B}$-P-direct sum of quasi-algebras for arbitrary quasi-primitive class $\mathfrak{B}$ of quasi-algebras, and for any family P of general basic mapping-formulas. We also consider the notion of independence with respect to any family P of general basic mapping-formulas, i. e. *P-independence*, and we obtain some results similar to those of Marczewski [2] and Schmidt [3].

§ 1. Quasi-algebras. Let k be any ordinal number and let A be any set. By a *k-ary partial operation in A* we understand any mapping f of a subset of the set A^k (of all sequences of the type k in A) into A. The value of k-ary partial operation f in A for a sequence $(a_\xi < k)$ — if it exists — will be denoted by $f(a_\xi, \xi < k)$. The partial operations defined on the whole set A^k are called *k-ary operations in A*. Let $G = \{g, \ldots\}$ be any set of *operator symbols*. By $n(g)$, where $g \in G$, will be denoted the *rank* of the operator symbol g, i. e. the ordinal number n for which g is n-ary. Any system

$$\boldsymbol{A} = \langle A, (g_{\boldsymbol{A}}, g \in G)\rangle,$$

where A is a set and $g_{\boldsymbol{A}}$ is an $n(g)$-ary partial operation in A for all $g \in G$, is called a *quasi-algebra of the type G*. If, moreover, in the system $\boldsymbol{A}$ a $g_{\boldsymbol{A}}$ is an operation in A for $g \in G$, then $\boldsymbol{A}$ is said to be an *algebra of the type G*. In the sequel the quasi-algebras will be denoted by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ and their sets by $A, B, C, \ldots$

A subset B of a set A is called *closed with respect to a k-ary partial operation f in A* provided that, for all sequences $(b_\xi, \xi < k)$ belonging to B^k, if f is defined for $(b_\xi, \xi < k)$ then the value $f(b_\xi, \xi < k)$ belongs to B. Let $\boldsymbol{A} = \langle A, (g_{\boldsymbol{A}}, g \in G)\rangle$ be any quasi-algebra of the type G and let B be a subset of A closed with respect to $g_{\boldsymbol{A}}$ for all $g \in G$. Then the subset B and also the system $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g \in G)\rangle$ where $g_{\boldsymbol{B}} = g_{\boldsymbol{A}}|B$, is called a *subquasi-algebra of* $\boldsymbol{A}$. Any intersection of subquasi-algebras of $\boldsymbol{A}$ is also a subquasi-algebra of $\boldsymbol{A}$. Thus for any subset M of A there exists the least subquasi-algebra $\overline{\boldsymbol{M}}$ of $\boldsymbol{A}$ containing M called *generated* by M. If $\overline{\boldsymbol{M}} = \boldsymbol{A}$, then M is said to be a *set of generators for* $\boldsymbol{A}$. Let $\boldsymbol{A} = \langle A, (g_{\boldsymbol{A}}, g \in G)\rangle$ and $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g \in G)\rangle$ be two quasi-algebras of the type G. A mapping h of A into B is said to be a *homomorphism of* $\boldsymbol{A}$ *into* $\boldsymbol{B}$ provided that for all $g \in G$ and all sequences $(a_\xi, \xi < n(g)) \in A^{n(g)}$ if $g_{\boldsymbol{A}}$ is defined for $(a_\xi, \xi < n(g))$, then $g_{\boldsymbol{B}}$ is defined for the sequence $(h(a_\xi), \xi < n(g))$, and, moreover, that we have

$$h\big(g_{\boldsymbol{A}}(a_\xi, \xi < n(g))\big) = g_{\boldsymbol{B}}\big(h(a_\xi), \xi < n(g)\big).$$

A homomorphism h of $\boldsymbol{A}$ into $\boldsymbol{B}$ is called *strong* provided that, for all $g \in G$ and all sequences $(a_\xi, \xi < n(g)) \in A^{n(g)}$, if $g_{\boldsymbol{B}}$ is defined for $(h(a_\xi), \xi < n(g))$, then there are elements $a'_\xi \in A$, $\xi < n(g)$ such that $h(a'_\xi) = h(a_\xi)$ for $\xi < n(g)$, and that $g_{\boldsymbol{A}}$ is defined for $(a'_\xi, \xi < n(g))$. The one-to-one homomorphisms are *isomorphisms*. Homomorphisms between algebras are always strong.

Let T be any set and let $\boldsymbol{A}_t = \langle A_t, (g_{\boldsymbol{A}_t}, g \in G)\rangle$, for $t \in T$, be any quasi-algebra of the type G. Let us denote by $A = \mathop{\mathrm{P}}_{t \in T} A_t$ and $A' = \mathop{\mathrm{S}}_{t \in T} A_t$ *the cartesian product* and *direct sum of sets* A_t, i. e. A is the set of all mappings $\varphi\colon T \to \bigcup_{t \in T} A_t$ with $\varphi(t) \in A_t$ for $t \in T$, and $A' = \{(t, a)\colon t \in T,$

$a \in A_t\}$ is the set of all pairs (t, a), where $t \in T$ and $a \in A_t$. The quasi-algebra $\boldsymbol{A} = \langle A, (g_A, g \in G)\rangle$, where g_A, for $g \in G$, is an $n(g)$-ary partial operation in A such that g_A is defined for $(\varphi_\xi, \xi < n(g))$ if and only if, for all $t \in T$, g_{A_t} is defined for $(\varphi_\xi(t), \xi < n(g))$, and in which we have

$$g_A(\varphi_\xi, \xi < n(g)) = \varphi \quad \text{with} \quad \varphi(t) = g_{A_t}(\varphi_\xi(t), \xi < n(g))$$

for all $t \in T$, is called the *direct product of quasi-algebras* $\boldsymbol{A}_t$, $t \in T$. The quasi-algebra $\boldsymbol{A}' = \langle A', (g_{A'}, g \in G)\rangle$, where $g_{A'}$, for $g \in G$, is defined for a sequence $((t_\xi, a_\xi), \xi < n(g))$ if and only if there exists an element $t_0 \in T$ such that $(t_\xi, a_\xi) = (t_0, a_\xi)$, $\xi < n(g)$, and where $g_{A_{t_0}}$ is defined for $(a_\xi, \xi < n(g))$, and in which, moreover, we have

$$g_{A'}((t_\xi, a_\xi), \xi < n(g)) = (t_0, g_{A_{t_0}}(a_\xi, \xi < n(g))),$$

is called the *direct sum of quasi-algebras* $\boldsymbol{A}_t$, $t \in T$. The direct product and direct sum of quasi-algebras $\boldsymbol{A}_t$, $t \in T$, will be denoted by $\underset{t \in T}{\mathrm{P}} \boldsymbol{A}_t$ and $\underset{t \in T}{\mathrm{S}} \boldsymbol{A}_t$ respectively. Let p_t be the *natural projection* of A onto A_t (i. e. $p_t(\varphi) = \varphi(t)$ for $\varphi \in A$) and let i_t be the *natural injection of* A_t *into* A (i. e. $i_t(a) = (t, a)$). Then p_t is a homomorphism of $\boldsymbol{A} = \underset{t \in T}{\mathrm{P}} \boldsymbol{A}_t$ onto $\boldsymbol{A}_t$ and i_t is an isomorphism of $\boldsymbol{A}_t$ into $\boldsymbol{A}' = \underset{t \in T}{\mathrm{S}} \boldsymbol{A}_t$. Moreover, i_t is a strong isomorphism of $\boldsymbol{A}_t$ onto $i_t(\boldsymbol{A}_t)$.

If h is a homomorphism of a quasi-algebra $\boldsymbol{A}$ into a quasi-algebra $\boldsymbol{B}$, then h considered as a subset of $A \times B$ is a subquasi-algebra of the direct product $\boldsymbol{A} \times \boldsymbol{B}$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ such that

(*) for all $a \in A$ there exists one and only one element $b \in B$ with $\langle a, b\rangle \in h$.

The converse is not always true. Any subquasi-algebra h of $\boldsymbol{A} \times \boldsymbol{B}$ which has the property (*) is said to be a *full-homomorphism of* $\boldsymbol{A}$ *into* $\boldsymbol{B}$. Every homomorphism h of $\boldsymbol{A}$ into $\boldsymbol{B}$ is a full-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$, but a full-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ is not always a homomorphism of $\boldsymbol{A}$ and $\boldsymbol{B}$. It is easy to verify that for algebras the notions of homomorphism, strong homomorphism and full-homomorphism are identical. The subquasi-algebras h of the direct product $\boldsymbol{A} \times \boldsymbol{B}$ of quasi-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ such that

(**) for all $a \in A$ there exists at most one element $b \in B$ with $\langle a, b\rangle \in h$

are called *partial-homomorphisms of* $\boldsymbol{A}$ *into* $\boldsymbol{B}$. Let h be a partial-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. Then the sets $p_1(h)$ and $p_2(h)$, where p_1 and p_2 are the natural projections of $A \times B$ onto A and B, are said to be the *domain and the image of* h. If $\langle a, b\rangle \in h$, then $a \in p_1(h)$, $b \in p_2(h)$, and the element b will be denoted by $h(a)$. If the domain of h is the whole set A, then h is a full-homomorphism of $\boldsymbol{A}$ and $\boldsymbol{B}$. Now we observe that

(1.1) *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are algebras and h is a partial-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ such that a set of generators for A is contained in the domain of h, then h is a full-homomorphism and also a homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$.*

(1.2) *If $\boldsymbol{A}$ is an algebra and h is a partial-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ such that the image of h contains a set of generators for $\boldsymbol{B}$, then h is a partial-homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$.*

Now we shall give a definition of *Peano-algebra*. An algebra

$$\boldsymbol{G}^* = \langle G^*, (g_{G^*}, g \in G)\rangle$$

of the type G is said to be a *Peano-algebra* of the type G generated by a set Y if it has the following properties:

(1.a) the elements in Y are not values of the operations $g_{G^*}, g \in G$, for elements in G^*,

(1.b) for all $g, g' \in G$, and all sequences $(w_\xi, \xi < n(g))$ and $(w'_\xi, \xi < n(g'))$ of elements in G^* the relation

$$g_{G^*}(w_\xi, \xi < n(g)) = g'_{G^*}(w'_\xi, \xi < n(g'))$$

implies $g = g'$ and $w_\xi = w'_\xi$ for $\xi < n(g) = n(g')$,

(1.c) the set Y generates the algebra $\boldsymbol{G}^*$.

There are Peano-algebras of the type G generated by arbitrary sets. For a construction of Peano-algebras see my paper [4]. The Peano-algebras have an important property which is given in the following theorem:

THEOREM 1. *Let $\boldsymbol{G}^* = \langle G^*, (g_{G^*}, g \in G)\rangle$ be a Peano-algebra of the type G generated by a set Y and let $\boldsymbol{A} = \langle A, (g_A, g \in G)\rangle$ be an arbitrary quasi-algebra of the type G. Then for every mapping φ: $Y \to A$ the subquasi-algebra $\bar{\varphi}$ of $\boldsymbol{G}^* \times \boldsymbol{A}$ generated by φ is a partial-homomorphism of $\boldsymbol{G}^*$ into $\boldsymbol{A}$. If $\boldsymbol{A}$ is an algebra, then $\bar{\varphi}$ is a homomoprhism of $\boldsymbol{G}^*$ into $\boldsymbol{A}$.*

For a proof of Theorem 1 see my paper [4] (proof of theorem (2.7)).

From Theorem 1 it follows that the Peano-algebra of the type G generated by a set Y is uniquely determined up to isomomorphisms by the cardinal number of set Y, and since it is the absolutely free algebra of the type G freely generated by Y, it is denoted by Free (G, Y). By virtue of Theorem 1, Peano-algebra of the type G may be considered as an *algebra of G-terms*. Let $\boldsymbol{G}^* = \mathrm{Free}(G, X)$ be a fixed Peano-algebra of the type G generated by the set $X = (x_0, x_1, \ldots, x_\xi, \ldots, \xi < \beta)$ composed of different elements x_ξ. The elements in $\boldsymbol{G}^*$ are called *G-terms*, the elements in X may be considered as *individuum-variables*. If a G-term τ

belongs to the subalgebra of G^* generated by variables $(x_0, x_1, \ldots, x_\xi, \ldots, \xi < \varrho)$, then we shall write $\tau = \tau(x_\xi, \xi < \varrho)$ (1). Let $\tau = \tau(x_\xi, \xi < \varrho)$ be any G-term and let $\boldsymbol{A} = \langle A, (g_A, g \in G)\rangle$ be an arbitrary quasi-algebra of the type G. Then the G-term τ defines in the set A a ϱ-ary partial operation. We define τ_A as follows. Let $(a_\xi, \xi < \varrho)$ be a sequence of the type ϱ in A and let ψ be a mapping of X into A such that $\psi(x_\xi) = a_\xi$ for $\xi < \varrho$. By Theorem 1, the subquasi-algebra $\bar{\psi}$ of $\boldsymbol{G}^* \times \boldsymbol{A}$ generated by ψ is a partial-homomorphism of $\boldsymbol{G}^*$ into $\boldsymbol{A}$. The partial operation τ_A is defined for $(a_\xi, \xi < \varrho)$ if and only if the G-term τ belongs to the domain of $\bar{\psi}$. Moreover, we put $\tau_A(a_\xi, \xi < \varrho) = \bar{\psi}(\tau)$. The partial operation τ_A defined above is said to be *defined by G-term τ in quasi-algebra $\boldsymbol{A}$*. If $\boldsymbol{A}$ is an algebra, then τ_A is an operation. The partial operation τ_A may be also considered as one of the type X, i. e. τ_A is defined for $\psi \in A^X$ if and only if τ belongs to the domain of $\bar{\psi}$ and if, moreover, we have $\tau_A(\psi) = \bar{\psi}(\tau)$. The pairs $\langle \tau, \vartheta \rangle$, where τ and ϑ are G-terms, are called *G-equations*. The G-equation $\langle \tau, \vartheta \rangle$ will be also denoted by $\ulcorner \tau = \vartheta \urcorner$. A G-equation $\ulcorner \tau = \vartheta \urcorner$ is said to be *valid in a quasi-algebra $\boldsymbol{A}$ of the type G* if $\tau_A = \vartheta_A$, i. e. if for all $\psi \in A^X$ we have $\bar{\psi}(\tau) = \bar{\psi}(\vartheta)$ provided that τ_A and ϑ_A are defined for ψ. The set of all G-equations which are valid in a quasi-algebra $\boldsymbol{A}$ of the type G will be denoted by $E(\boldsymbol{A})$. Let E_0 be a set of G-equations. By $G(E_0)$ will be denoted the class of all quasi-algebras $\boldsymbol{A}$ of the type G such that $E_0 \subset E(\boldsymbol{A})$. The classes of the form $G(E_0)$ are called *equationally definable*.

§ 2. A theory of P-mappings between quasi-algebras. Let $\Phi = (\varphi_\sigma, \sigma < \alpha)$ and $X = (x_\xi, \xi < \beta)$ be arbitrary sets. The elements in Φ and in X may be considered as *mapping* and *individuum-variables*. The pairs $\langle \varphi_\sigma, x_\xi \rangle \in \Phi \times X$ will be also denoted by $\varphi_\sigma(x_\xi)$. Let us denote by $\boldsymbol{G}_\Phi^* = \mathrm{Free}(G, \Phi \times X)$ the Peano-algebra of the type G generated by the set $\Phi \times X$. The elements in the algebra $\boldsymbol{G}_\Phi^*$ are called *G-mapping-terms*. If a G-mapping-term τ belongs to the subalgebra of $\boldsymbol{G}_\Phi^*$ generated by elements $\varphi_\sigma(x_\xi)$, $\sigma < \alpha_1$, $\xi < \beta_1$, then we shall write

(***) $\quad \tau = \tau(\varphi_\sigma(x_\xi), \sigma < \alpha_1, \xi < \beta_1)$.

Let $\boldsymbol{B} = \langle B, (g_B, g \in G)\rangle$ be an arbitrary quasi-algebra of the type G and let τ be any G-mapping-term which fulfils the relation (***). Then the G-mapping-term τ defines in B a partial operation τ_B, the domain of which is a set of some $\alpha_1 \times \beta_1$-matrices over B. Let $(b_{\sigma\xi}, \sigma < \alpha_1, \xi < \beta_1)$ be any $\alpha_1 \times \beta_1$-matrix over set B and let $\psi\colon \Phi \times X \to B$ be a mapping

(1) Let us observe that the meaning of the notation $\tau = \tau(x_\xi, \xi < \varrho)$ given in this paper is different from that in my paper [4]. In [4] the relation $\tau = \tau(x_\xi, \xi < \varrho)$ means that the set $(x_\xi, \xi < \varrho)$ is the support of the term τ. Obviously, if the relation $\tau = \tau(x_\xi, \xi < \varrho)$ holds in the sense of [4], then the relation $\tau = \tau(x_\xi, \xi < \varrho)$ holds also in the sense of this paper, but the converse is not true.

such that $\varphi_\sigma(x_\xi) = b_{\sigma\xi}$ for $\sigma < \alpha_1$, $\xi < \beta_1$. By Theorem 1, the subquasi-algebra $\bar{\psi}$ of $\boldsymbol{G}_\Phi^* \times \boldsymbol{B}$ generetad by ψ is a partial-homomorphism of $\boldsymbol{G}_\Phi^*$ into $\boldsymbol{B}$. The partial operation $\tau_{\boldsymbol{B}}$ is defined for $(b_{\sigma\xi}, \sigma < \alpha_1, \xi < \beta_1)$ if and only if τ belongs to the domain of $\bar{\psi}$. Moreover, we put

$$\tau_{\boldsymbol{B}}(b_{\sigma\xi}, \sigma < \alpha_1, \xi < \beta_1) = \bar{\psi}(\tau).$$

The partial operation $\tau_{\boldsymbol{B}}$ defined above is called *defined in quasi-algebra* $\boldsymbol{B}$ *by G-mapping-term* τ. The partial operation $\tau_{\boldsymbol{B}}$ may be also considered as one of the type $\Phi \times X$, i. e. for all mapping ψ of $\Phi \times X$ into B, $\tau_{\boldsymbol{B}}$ is defined for ψ if and only if τ belongs to the domain of $\bar{\psi}$, and if, moreover, we have $\tau_{\boldsymbol{B}}(\psi) = \bar{\psi}(\tau)$. The pairs $\langle \tau, \vartheta \rangle$, where τ and ϑ are G-mapping-terms, are called *G-mapping-equations*. A G-mapping-equation $\langle \tau, \vartheta \rangle$ will be also denoted by $\ulcorner \tau = \vartheta \urcorner$. A G-mapping-equation $\ulcorner \tau = \vartheta \urcorner$ is said to be *valid in a quasi-algebra* $\boldsymbol{B}$ *of the type* G, if $\tau_{\boldsymbol{B}} = \vartheta_{\boldsymbol{B}}$, i. e. if for all mappings ψ of $\Phi \times X$ into $\boldsymbol{B}$ we have $\bar{\psi}(\tau) = \bar{\psi}(\vartheta)$ provided that $\tau_{\boldsymbol{B}}$ and $\vartheta_{\boldsymbol{B}}$ are defined for ψ.

Now let us consider two sets $F = \{f, \ldots\}$ and $G = \{g, \ldots\}$ of operator symbols. The elements of the Peano-algebra $\boldsymbol{F}^* = \mathrm{Free}(F, X)$ of the type F generated by X we shall call *F-terms* and the pairs of F-terms we shall call *F-equations*.

Now we shall give a definition of *basic mapping-formulas*. An identity of the form

$$\text{(i)} \qquad \varphi_\mu\big(f(x_\xi, \xi < n(f))\big) = \tau\big(\varphi_\sigma(x_\xi), \sigma < \alpha, \xi < n(f)\big),$$

where $\mu < \alpha$, $f \in F$, and τ is a G-mapping-term which fulfils the relation (***) for $\alpha_1 = \alpha$, and $\beta_1 = n(f)$, is called a *basic mapping-formula* (*of* φ_μ *concerning* f).

Let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F) \rangle$ and $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g \in G) \rangle$ be two quasi-algebras of the type F and G, respectively, and let $H = \{h_\sigma, \sigma < \alpha\}$ be a system of mappings h_σ of A into B. We say that the system H of mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$ *fulfils the basic mapping-formula* (i) provided that for every sequence $\big(a_\xi, \xi < n(f)\big) \in A^{n(f)}$, if $f_{\boldsymbol{A}}$ is defined for $\big(a_\xi, \xi < n(f)\big)$, then $\tau_{\boldsymbol{B}}$ is defined for $\big(h_\sigma(a_\xi), \sigma < \alpha, \xi < n(f)\big)$ and that, moreover, we have

$$h_\mu\big(f_{\boldsymbol{A}}(a_\xi, \xi < n(f))\big) = \tau_{\boldsymbol{B}}\big(h_\sigma(a_\xi), \sigma < \alpha, \xi < n(f)\big).$$

Let P be any family of basic mapping-formulas (see (i)). Then P is said to be a $P_{F,G}$ $(\varphi_\sigma, \sigma < \alpha)$*-family of basic mapping-formulas*. Let P be any $P_{F,G}(\varphi_\sigma, \sigma < \alpha)$-family of basic mapping-formulas and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any quasi-algebras of the type F and G. A system $H = \{h_\sigma, \sigma < \alpha\}$ of mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$ is called a *system of P-mappings of quasi-algebra* $\boldsymbol{A}$ *into quasi-algebra* $\boldsymbol{B}$ if system H fulfils every basic mapping-formula (i) belonging to P. Now we observe that

(2.1) *If $H = \{h_\sigma, \sigma < a\}$ is a system of P-mappings of a quasi-algebra $\boldsymbol{A}$ of the type F into a quasi-algebra $\boldsymbol{B}$ of the type G, and q is a homomorphism of $\boldsymbol{B}$ into a quasi-algebra $\boldsymbol{C}$ of the type G, then the system $q\cdot H = \{q\cdot h_\sigma, \sigma < a\}$ is a system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{C}$.*

Proof. Let (i) be any basic mapping-formula belonging to P. We have

$$qh_\mu\big(f_{\boldsymbol{A}}(a_\xi, \xi < n(f))\big) = q\big(\tau_{\boldsymbol{B}}(h_\sigma(a_\xi), \sigma < a, \xi < n(f))\big)$$
$$= \tau_{\boldsymbol{C}}\big(qh_\sigma(a_\xi), \sigma < a, \xi < n(f)\big),$$

provided that $f_{\boldsymbol{A}}$ is defined for $(a_\xi, \xi < n(f))$, i. e. H fulfils the basic mapping-formula (i). Thus $q\cdot H$ is a system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{C}$.

(2.2) *If q is a homomorphism of a quasi-algebra $\boldsymbol{A}$ of the type F into a quasi-algebra $\boldsymbol{A}'$ of the type F, and $H = \{h_\sigma, \sigma < a\}$ is a system of P-mappings of $\boldsymbol{A}'$ into a quasi-algebra $\boldsymbol{B}$ of the type G, then the system $H\cdot q = \{h_\sigma q, \sigma < a\}$ is a system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$.*

Proof. Let (i) be any basic mapping-formula belonging to P. We have

$$h_\sigma\big(q(f_{\boldsymbol{A}}(a_\xi, \xi < n(f)))\big) = h_\sigma\big(f_{\boldsymbol{A}'}(q(a_\xi), \xi < n(f))\big)$$
$$= \tau_{\boldsymbol{B}}\big(h_\sigma(q(a_\xi)), \sigma < a, \xi < n(f)\big) = \tau_{\boldsymbol{B}}\big(h_\sigma q(a_\xi), \sigma < a, \xi < n(f)\big)$$

provided that $f_{\boldsymbol{A}}$ is defined for $(a_\xi, \xi < n(f))$, i. e. the system $H\cdot q$ fulfils the basic mapping-formula (i). Thus $H\cdot q$ is a system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$, and theorem (2.2) is proved.

Let P be any $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas and let, for each pair $(\varphi_\mu, f)\in\Phi\times F$, $P_{\varphi_\mu,f}$ be the set of basic mapping-formulas of φ_μ concerning f, which is an element of P. The family P is said to be *proper* if for each pair $(\varphi_\mu, f)\in\Phi\times F$, $P_{\varphi_\mu,f}$ is a one-element set. If P is a proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas, then for any $\mu < a$ and any $f\in F$ there exists one and only one basic mapping-formula (i) in P of φ_μ concerning f. In this case the G-mapping-term τ which appears in (i) and which is uniquely determined will be denoted by $P(\varphi_\mu, f)$. Hence the basic mapping-formulas in the proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family P have the following form:

(i′) $$\varphi_\mu\big(f(x_\xi, \xi < n(f))\big) = P(\varphi_\mu, f)\big(\varphi_\sigma(x_\xi), \sigma < a, \xi < n(f)\big),$$

where $\mu < a$, $f\in F$ and $P(\varphi_\mu, f)$ is a G-mapping-term belonging to the subalgebra of $\boldsymbol{G}^*_\Phi$ generated by $\big(\varphi_\sigma(x_\xi), \sigma < a, \xi < n(f)\big)$.

Let P be any proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas and let $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g\in G)\rangle$ be an arbitrary quasi-algebra of the type G. We define a quasi-algebra $P(\boldsymbol{B})$ of the type F, which will be called *the P-product system over* $\boldsymbol{B}$, as follows. At first let us consider the set $P(B) = B^{a}$ of all sequences $(b_{\xi}, \xi < a)$ of the type a in B. In the set $P(B) = B^{a}$ we introduce the $n(f)$-ary partial operations $f_{P(\boldsymbol{B})}$ in the following way. The partial operation $f_{P(\boldsymbol{B})}$ is defined for a sequence $\big((b_{\sigma\xi}, \sigma < a), \xi < n(f)\big)$ of the type $n(f)$ in B^{a} if and only if $P(\varphi_{\mu}, f)_{\boldsymbol{B}}$ is defined for the $a \times n(f)$-matrix $\big(b_{\sigma\xi}, \sigma < a, \xi < n(f)\big)$ for all $\mu < a$. Moreover, we put

(4) $$f_{P(\boldsymbol{B})}\big((b_{\sigma\xi}, \sigma < a), \xi < n(f)\big) = \big(P(\varphi_{\mu}, f)_{\boldsymbol{B}}(b_{\sigma\xi}, \sigma < a, \xi < n(f)), \mu < a\big).$$

The quasi-algebra $\boldsymbol{P}(\boldsymbol{B}) = \langle P(B), (f_{P(\boldsymbol{B})}, f \in F)\rangle$ of the type F is said to be the *P-product system over* $\boldsymbol{B}$. Let us denote by p_{σ}, $\sigma < a$, the natural projections of $P(B) = B^{a}$ onto $B_{\sigma} = B$. Now we prove

(2.3) *For every proper $P_{F,G}(\varphi_{\sigma}, \sigma < a)$-family P of basic mapping-formulas and for every quasi-algebra $\boldsymbol{B}$ of the type G the system $Pr = \{p_{\sigma}, \sigma < a\}$ of natural projections p_{σ} of $P(B)$ onto $B_{\sigma} = B$ is a system of P-mappings of the P-product system $P(\boldsymbol{B})$ over B into $\boldsymbol{B}$.*

Proof. By (4), we have

$$\begin{aligned} & p_{\mu}\Big(f_{P(\boldsymbol{B})}\big((b_{\sigma\xi}, \sigma < a), \xi < n(f)\big)\Big) \\ &= p_{\mu}\Big(P(\varphi_{\mu}, f)_{\boldsymbol{B}}\big(b_{\sigma\xi}, \sigma < a, \xi < n(f)\big), \mu < a\Big) \\ &= P(\varphi_{\mu}, f)_{\boldsymbol{B}}\big(b_{\sigma\xi}, \sigma < a, \xi < n(f)\big) \\ &= P(\varphi_{\mu}, f)_{\boldsymbol{B}}\Big(p_{\sigma}\big((b_{\sigma\xi}, \sigma < a)\big), \sigma < a, \xi < n(f)\Big), \end{aligned}$$

i. e. the system Pr fulfils the basic mapping-formula (i). Thus (2.3) is proved.

Now we shall show a fundamental theorem:

THEOREM 2. *Let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F)\rangle$ and $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g \in G)\rangle$ be any quasi-algebras of the type F and G, and let P be an arbitrary proper $P_{F,G}(\varphi_{\sigma}, \sigma < a)$-family of basic mapping-formulas. Then a system $H = \{h_{\sigma}, \sigma < a\}$ is a system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$ if and only if the direct product h of h_{σ}, $\sigma < a$, i. e. the mapping $h: A \to P(B) = B^{a}$, such that*

$$h(a) = \big(h_{\sigma}(a), \sigma < a\big) \quad \textit{for} \quad a \in A,$$

is a homomorphism of $\boldsymbol{A}$ into $P(\boldsymbol{B})$, where $P(\boldsymbol{B})$ is the P-product system over $\boldsymbol{B}$.

Proof. Let us suppose that h is a homomorphism of $\boldsymbol{A}$ into $P(\boldsymbol{B})$. Then we have $h_{\sigma} = p_{\sigma}h$, where p_{σ} is the natural projection of $P(B) = B^{a}$ onto $B_{\sigma} = B$, for $\sigma < a$. Hence it follows that $H = Pr \cdot h$, where $Pr = \{p_{\sigma}, \sigma < a\}$. By theorem (2.3), the system Pr is a system of P-mappings of $P(\boldsymbol{B})$ into $\boldsymbol{B}$, and thus, by theorem (2.2), the system $H = Pr \cdot h$ is a system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$. Conversely, assume that H is a system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$. Then we have

$$h\big(f_{\boldsymbol{A}}(a_\xi,\ \xi < n(f)\big) = \big(h_\mu\big(f_{\boldsymbol{A}}(a_\xi,\ \xi < n(f)\big),\ \mu < a\big)$$

$$= \big(P(\varphi_\mu, f)_{\boldsymbol{B}}\big(h_\sigma(a_\xi,\ \sigma < a,\ \xi < n(f),\ \mu < a\big)$$

$$= f_{P(\boldsymbol{B})}\big(\big(h_\sigma(a_\xi),\ \sigma < a\big),\ \xi < n(f)\big) = f_{P(\boldsymbol{B})}\big(h(a_\xi),\ \xi < n(f)\big)$$

provided that $f_{\boldsymbol{A}}$ is defined for $\big(a_\xi,\ \xi < n(f)\big)$, i. e. h is a homomorphism of $\boldsymbol{A}$ into $P(\boldsymbol{B})$. Thus Theorem 2 is proved.

From Theorem 2 if follows

(2.4) *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two quasi-algebras of the type F and G and let P be any proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas. If $H = \{h_\sigma, \sigma < a\}$ and $H' = \{h'_\sigma, \sigma < a\}$ are two systems of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$, and if H and H' are the same on a set of generators for $\boldsymbol{A}$, then H and H' are identical, i. e. $H = H'$.*

Proof. Let h and h' be the direct products of $h_\sigma, \sigma < a$, and h'_σ, $\sigma < a$, respectively. By Theorem 2, h and h' are two homomorphisms of $\boldsymbol{A}$ into $P(\boldsymbol{B})$. But by the assumption of (2.4), h and h' are the same on a set of generators for $\boldsymbol{A}$, and thus $h = h'$. Hence it follows that $H = H'$ and theorem (2.4) is proved.

If in Theorem 2 we assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ are algebras with finitary operations, then from Theorem 2 we obtain Theorem 1.1 in paper [1] of Fujiwara.

4. Direct products of P-mappings. Let P be an arbitrary $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas and let $H_t = \{h_{t\sigma}, \sigma < a\}$, for $t \in T$, be a system of P-mappings of a quasi-algebra $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F)\rangle$ of the type F into a quasi-algebra $\boldsymbol{B}_t = \langle B_t, (g_{\boldsymbol{B}_t}, g \in G)\rangle$ of the type G. Let $H = \{h_\sigma, \sigma < a\}$ be the *direct product of systems* H_t, $t \in T$, i. e. H is a system of mappings of A into $B = \mathrm{P}_{t\in T} B_t$ such that, for all $\sigma < a$ and for all $a \in A$, we have

$$h_\sigma(a) = \varphi \quad \text{with} \quad \varphi(t) = h_{t\sigma}(a) \text{ for all } t \in T.$$

The direct product H of the systems H_t, $t \in T$, of P-mappings is also a system of P-mappings. This follows from the theorem:

THEOREM 3. *The direct product $H = \{h_\sigma, \sigma < a\}$ of the systems $H_t = \{h_{t\sigma}, \sigma < a\}$, $t \in T$, of P-mappings of a quasi-algebra $\boldsymbol{A}$ of the type F into quasi-algebra $\boldsymbol{B}_t$, $t \in T$, of the type G is a unique system of P-mappings of $\boldsymbol{A}$ into the direct product $\boldsymbol{B} = \mathrm{P}_{t\in T} \boldsymbol{B}_t$ such that $H_t = p_t \cdot H$ for $t \in T$ (i. e. $h_{t\sigma} = p_t h_\sigma$ for all $\sigma < a$ and all $t \in T$), where p_t is the natural projection of $\boldsymbol{B}$ onto $\boldsymbol{B}_t$.*

Proof. Let (i) be any basic mapping-formula belonging to P. The system H fulfils the basic mapping-formula (i). Indeed, for all sequences $(a_\xi, \xi < n(f))$ belonging to the domain of $f_{\boldsymbol{A}}$, we have $h_\mu\big(f_{\boldsymbol{A}}(a_\xi, \xi < n(f))\big) = \varphi$ with $\varphi(t) = h_{t\mu}\big(f_{\boldsymbol{A}}(a_\xi, \xi < n(f))\big) = \tau_{\boldsymbol{B}_t}\big(h_{t\sigma}(a_\xi), \sigma < a, \xi < n(f)\big)$ for all $t \in T$, and thus, by the definition of direct product of quasi-algebras, $\varphi = \tau_{\boldsymbol{B}}\big(h_\sigma(a_\xi), \sigma < a, \xi < n(f)\big)$; therefore

$$h_\mu\big(f_{\boldsymbol{A}}(a_\xi, \xi < n(f))\big) = \tau_{\boldsymbol{B}}\big(h_\sigma(a_\xi), \sigma < a, \xi < n(f)\big),$$

i. e. we have proved that the system H fulfils the basic mapping-formula (i). Hence H is a system of of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$. Obviously $H_t = p_t H$ for $t \in T$, and thus Theorem 3 is proved.

B. Direct sums of P-mappings. Let P be an arbitrary $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas and let $H_t = \{h_{t\sigma}, \sigma < a\}$, for $t \in T$, be a system of P-mappings of a quasi-algebra $\boldsymbol{A}_t = \langle A_t, (f_{\boldsymbol{A}_t}, f \in F)\rangle$ of the type F into a quasi-algebra $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g \in G)\rangle$ of the type G. Let $H = \{h_\sigma, \sigma < a\}$ be the *direct sum of systems* H_t, $t \in T$, i. e. H is a system of mappings of the direct sum $A = \underset{t \in T}{\mathrm{S}} A_t$ of sets A_t into the set B such that, for all $\sigma < a$, all elements $t \in T$ and all $(t, a) \in A$, we have $h_\sigma\big((t, a)\big) = h_{t\sigma}(a)$. The direct sum H of systems H_t, $t \in T$, of P-mappings is also a system of P-mappings. This follows from the next theorem.

THEOREM 4. *The direct sum $H = \{h_\sigma, \sigma < a\}$ of the systems $H_t = \{h_{t\sigma}, \sigma < a\}$, $t \in T$, of P-mappings of quasi-algebras $\boldsymbol{A}_t$ of the type F into quasi-algebra $\boldsymbol{B}$ of the type G is a unique system of P-mappings of the direct sum $\boldsymbol{A} = \underset{t \in T}{\mathrm{S}} \boldsymbol{A}_t$ of quasi-algebras $\boldsymbol{A}_t$ into the quasi-algebra $\boldsymbol{B}$ such that $H_t = H \cdot i_t$ for $t \in T$ (i. e. $h_{t\sigma} = h_\sigma i_t$ for all $t \in T$ and all $\sigma < a$), where i_t is the natural injection of A_t into A.*

Proof. Let (i) be an arbitrary basic mapping-formula in P. The system H fulfils this basic mapping-formula. Indeed, we have

$$\begin{aligned} h_\mu\Big(f_{\boldsymbol{A}}\big((t_\xi, a_\xi), \xi < n(f)\big)\Big) &= h_\mu\Big(f_{\boldsymbol{A}}\big((t_0, a_\xi), \xi < n(f)\big)\Big) \\ = h_\mu\Big(\big(t_0, f_{\boldsymbol{A}_{t_0}}(a_\xi, \xi < (f))\big)\Big) &= h_{t_0\mu}\Big(f_{\boldsymbol{A}_{t_0}}\big(a_\xi, \xi < n(f)\big)\Big) \\ = \tau_{\boldsymbol{B}}\big(h_{t_0,\sigma}(a_\xi), \sigma < a, \xi < n(f)\big) &= \tau_{\boldsymbol{B}}\Big(h_\sigma\big((t_0, a_\xi)\big), \sigma < a, \xi < n(f)\Big) \\ = \tau_{\boldsymbol{B}}\Big(h_\sigma\big((t_\xi, a_\xi)\big), \sigma < a, \xi < n(f)\Big) & \end{aligned}$$

for all sequences $(t_\xi, a_\xi) = (t_0, a_\xi)$, $\xi < n(f)$ belonging to the domain of $f_{\boldsymbol{A}}$, i. e. we have proved that H fulfils the basic mapping-formula (i). Thus H is a system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$. Obviously, we have $H_t = H \cdot i_t$ for all $t \in T$, and therefore Theorem 4 is proved.

C. $\mathfrak{V}$-P-mappings. Let P be any $P_{F,G}(\varphi_\sigma, (\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas and let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F)\rangle$ be any quasi-algebra of the type F. Moreover, let $\mathfrak{B}$ be an arbitrary class of quasi-algebras of the type G. The pairs $(H, \boldsymbol{B})$, where $\boldsymbol{B} \in \mathfrak{B}$ and $H = \{h_\sigma, \sigma < \alpha\}$ is a system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$, are called *systems of $\mathfrak{B}$-P-mappings of quasi-algebra $\boldsymbol{A}$*. Now we introduce some relations between systems of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$. Let $(H, \boldsymbol{B})$ and $(H', \boldsymbol{B}')$, where $H = \{h_\sigma, \sigma < \alpha\}$ and $H' = \{h'_\sigma, \sigma < \alpha\}$, be two systems of $\mathfrak{B}$-P-mappings of quasi-algebra $\boldsymbol{A}$. We say that:

1. $(H, \boldsymbol{B}) \leqslant (H', \boldsymbol{B}')$ *if there exists exactly one homomorphism q of $\boldsymbol{B}$ into $\boldsymbol{B}'$ with $H' = q \cdot H$ (i. e. with $h'_\sigma = q \cdot h_\sigma$ for all $\sigma < \alpha$)*,

2. $(H, \boldsymbol{B}) \equiv (H', \boldsymbol{B}')$ *if there exists exactly one strong isomorphism q of $\boldsymbol{B}$ onto $\boldsymbol{B}'$ with $H' = q \cdot H$.*

A system $(H, \boldsymbol{B})$ of $\mathfrak{B}$-P-mappings of quasi-algebra $\boldsymbol{A}$ is said to be *$\mathfrak{B}$-free* if for every system $(H', \boldsymbol{B}')$ of $\mathfrak{B}$-P-mappings of quasi-algebra $\boldsymbol{A}$ we have $(H, \boldsymbol{B}) \leqslant (H', \boldsymbol{B}')$. Now we prove that

(2.5) *If there exists an $\mathfrak{B}$-free system of $\mathfrak{B}$-P-mappings of quasi-algebra $\boldsymbol{A}$, then it is uniquely determined up to the relation $\equiv$*

Proof. Let $(H, \boldsymbol{B})$ and $(H', \boldsymbol{B}')$ be two $\mathfrak{B}$-free systems of $\mathfrak{B}$-P-mappings of quasi-algebra $\boldsymbol{A}$. Then $H' = q \cdot H$ and $H = q' \cdot H'$, where q and q' are homomorphisms of $\boldsymbol{B}$ into $\boldsymbol{B}'$ and of $\boldsymbol{B}'$ into $\boldsymbol{B}$, respectively. Hence $H' = q \cdot q' \cdot H'$ and $H = q' \cdot q \cdot H$. But we also have $H' = i' \cdot H'$ and $H = i \cdot H$, where i' and i are the identity isomorphism of $\boldsymbol{B}'$ onto $\boldsymbol{B}'$ and of $\boldsymbol{B}$ onto $\boldsymbol{B}$, respectively, and thus, by 1, $q \cdot q' = i'$ and $q' \cdot q = i$. Hence it follows that q' and q are one-to-one and onto, and, moreover, that $q' = q^{-1}$. Therefore q is a strong isomorphism of $\boldsymbol{B}$ onto $\boldsymbol{B}'$ and we obtain the relation $(H, \boldsymbol{B}) \equiv (H', \boldsymbol{B}')$. Hence (2.5) is proved.

A class $\mathfrak{B}$ of quasi-algebras of the type G is called *quasi-primitive* if it is closed with respect to direct products, subquasi-algebras and strong isomorphic images. Now we prove a general existence theorem.

THEOREM 5. *Let P be any $P_{F,G}(\varphi_\sigma, \sigma < \alpha)$-family of basic mapping-formulas and let $\mathfrak{B}$ be any quasi-primitive class of quasi-algebras of the type G. Moreover, let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F) \rangle$ be an arbitrary quasi-algebra of the type F. Then there exists the $\mathfrak{B}$-free system of $\mathfrak{B}$-P-mappings of quasi-algebra $\boldsymbol{A}$.*

Proof. By virtue of theorem (2.3) of my paper [4] there exists a number $\overline{m}$ such that $|\boldsymbol{B}| \leqslant \overline{\mathfrak{m}}$ for all quasi-algebras $\boldsymbol{B}$ of the type generated by sets M with $|M| \leqslant |A| \cdot \bar{\alpha}$ (where $|Y|$ and α denote the cardinal numbers of the set Y and of ordinal number α, resp.). Let E be an arbitrary set with $|E| \geqslant \mathfrak{m}$. Let us denote by $P(\boldsymbol{A}, \boldsymbol{B})$, where $\boldsymbol{B}$ is a quasi-algebra of the type G such that $B \subset E$, the set of all systems λ of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$, and let $H_{\boldsymbol{B}}$ be the direct product of all systems $\lambda \in P(\boldsymbol{A}, \boldsymbol{B})$. By Theorem 3, $H_{\boldsymbol{B}}$ is a unique system of P-mappings of $\boldsymbol{A}$ into the direct power $\boldsymbol{B}^{P(\boldsymbol{A},\boldsymbol{B})}$ such that $p_\lambda H_{\boldsymbol{B}} = \lambda$, where $\lambda \in P(\boldsymbol{A}, \boldsymbol{B})$

and p_λ is the natural projection of $\boldsymbol{B}^{P(\boldsymbol{A},\boldsymbol{B})}$ onto $\boldsymbol{B}_\lambda = \boldsymbol{B}$. Let H be the direct product of all systems $H_{\boldsymbol{B}}$ of P-mappings, where $\boldsymbol{B} \epsilon \mathfrak{B}$ and $\boldsymbol{B} \subset E$. By virtue of Theorem 3, H is a unique system of P-mappings of $\boldsymbol{A}$ into direct product $\mathrm{P}\, \boldsymbol{B}^{P(\boldsymbol{A},\boldsymbol{B})}$ of all direct powers $\boldsymbol{B}^{P(\boldsymbol{A},\boldsymbol{B})}$, where $\boldsymbol{B} \epsilon \mathfrak{B}$ and $\boldsymbol{B} \subset E$, such that $H_{\boldsymbol{B}} = q_{\boldsymbol{B}} \cdot H$, where $q_{\boldsymbol{B}}$ is the natural projection of $\mathrm{P}\, \boldsymbol{B}^{P(\boldsymbol{A},\boldsymbol{B})}$ onto $\boldsymbol{B}^{P(\boldsymbol{A},\boldsymbol{B})}$. Let $\boldsymbol{C}$ be the subquasi-algebra of $\mathrm{P}\, \boldsymbol{B}^{P(\boldsymbol{A},\boldsymbol{B})}$ generated by $\bigcup_{\sigma<a} h_\sigma(A)$, where h_σ, $\sigma < a$, are the mappings of the system H, i. e. $H = \{h_\sigma, \sigma < a\}$. Obviously, $\boldsymbol{C} \epsilon \mathfrak{B}$. Now we prove that the pair $(H, \boldsymbol{C})$ is the $\mathfrak{B}$-free system of $\mathfrak{B}$-P-mappings of quasi-algebra $\boldsymbol{A}$. Let $(H', \boldsymbol{B}')$, where $H' = \{h'_\sigma, \sigma < a\}$, be an arbitrary system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$. Let us denote by $\mathrm{D} = \overline{\bigcup_{\sigma<a} h'_\sigma(A)}$ the subquasi-algebra of $\boldsymbol{B}'$ generated by $\bigcup_{\sigma<a} h'_\sigma(A)$. Obviously, $\boldsymbol{D} \epsilon \mathfrak{B}$ and $|\boldsymbol{D}| \leqslant \overline{\mathfrak{m}}$. Hence it follows that there exists a quasi-algebra $\boldsymbol{B}$ with $B \subset E$ such that $\boldsymbol{B}$ is strongly isomorphic to $\boldsymbol{D}$. Let i be a strong isomorphism of $\boldsymbol{B}$ onto $\boldsymbol{D}$. By the definition of quasi-primitive class, $\boldsymbol{B} \epsilon \mathfrak{B}$. Then we have $H' = q \cdot H$, where $q = ip_\lambda q_{\boldsymbol{B}}|C$ with $\lambda = i^{-1}H'$, and thus we obtain the relation $(H, \boldsymbol{C}) \leqslant (H', \boldsymbol{B}')$, i. e. $(H, \boldsymbol{C})$ is the $\mathfrak{B}$-free system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$. Theorem 5 is proved.

D. $\mathfrak{B}$-P-direct sums of quasi-algebras. Let P be an arbitrary $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas and let $\mathfrak{B}$ be any quasi-primitive class of quasi-algebras of the type G. Let T be any set and let $\boldsymbol{A}_t$, $t \epsilon T$, be any family of quasi-algebras of the type F. Moreover let $\boldsymbol{A} = \mathop{\mathrm{S}}_{t \epsilon T} \boldsymbol{A}$ be the direct sum of quasi-algebras $\boldsymbol{A}_t$. By virtue of Theorem 5 there exists the $\mathfrak{B}$-free system $(H, \boldsymbol{C})$, where $H = \{h_\sigma, \sigma < a\}$, of $\mathfrak{B}$-P-mappings of quasi-algebra $\boldsymbol{A}$. The quasi-algebra $\boldsymbol{C}$ is called the *$\mathfrak{B}$-P-direct sum of quasi-algebras $\boldsymbol{A}_t$, $t \epsilon T$*, and we denote $\boldsymbol{C} = \mathfrak{B}\text{-}P\text{-}\mathop{\mathrm{S}}_{t \epsilon T} \boldsymbol{A}_t$. The $\mathfrak{B}$-P-direct sum of quasi-algebras $\boldsymbol{A}_t$, $t \epsilon T$, is, by (2.5), uniquely determined up to isomorphisms. Now we prove

THEOREM 6. *Putting for all $t \epsilon T$, $\boldsymbol{H}_t = \boldsymbol{H} \cdot i_t = \{h_\sigma i_t, \sigma < a\}$, where i_t is the natural injection of A_t into $A = \mathop{\mathrm{S}}_{t \epsilon T} A_t$, we obtain a family of systems of P-mappings of quasi-algebras $\boldsymbol{A}_t$ into $\boldsymbol{C} = \mathfrak{B}\text{-}P\text{-}\mathop{\mathrm{S}}_{t \epsilon T} \boldsymbol{A}_t$ which has the following property*:

(5) *for each quasi-algebra $\boldsymbol{B} \epsilon \mathfrak{B}$ and each family H'_t, $t \epsilon T$, of systems of P-mappings of quasi-algebras $\boldsymbol{A}_t$ into quasi-algebra $\boldsymbol{B}$, there exists one and only one homomorphism q of $\boldsymbol{C}$ into $\boldsymbol{B}$ such that $H'_t = q \cdot \boldsymbol{H}_t$ for all $t \epsilon T$.*

Proof. By theorem (2.2), $\boldsymbol{H}_t = \boldsymbol{H} \cdot i_t$ are systems of P-mappings for $t \epsilon T$. Let H' be the direct sum of systems H'_t, $t \epsilon T$, of P-mappings.

By Theorem 4, H' is a unique system of P-mappings of $\boldsymbol{A} = \mathop{\mathrm{S}}_{t \in T} \boldsymbol{A}_t$ into $\boldsymbol{B}$ such that $H'_t = H' \cdot i_t$ for $t \in T$. The pair $(H', \boldsymbol{B})$ is a system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$. Since $(H, \boldsymbol{C})$ is the $\mathfrak{B}$-free system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$, then we have the relation $(H, \boldsymbol{C}) \leqslant (H', \boldsymbol{B})$. Thus, by the definition of relation $\leqslant$, there exists exactly one homomorphism q of $\boldsymbol{C}$ into $\boldsymbol{B}$ such that $H' = q \cdot H$. Hence we have $H'_t = H' \cdot i_t = q \cdot H \cdot i_t = q \cdot H_t$ for all $t \in T$, and thus Theorem 6 is proved.

If H is *one-to-one*, i. e. if each mapping h_σ, $\sigma < \alpha$, in H is one-to-one, then the $\mathfrak{B}$-P-direct sum $\boldsymbol{C}$ of quasi-algebras $\boldsymbol{A}_t$ is said to be *proper*. In this case the systems H_t, $t \in T$, given in Theorem 6, are also one-to-one. The one-to-one systems of $\mathfrak{B}$-P-mappings of quasi-algebras are called the *systems of $\mathfrak{B}$-P-extensions of* those quasi-algebras. Hence, by Theorem 6, we obtain immediately

(2.6) *The proper $\mathfrak{B}$-P-direct sum of quasi-algebras $\boldsymbol{A}_t$ $t \in T$ exists, if and only if the direct sum $\boldsymbol{A}$ of quasi-algebras $\boldsymbol{A}_t$, $t \in T$, has a system of $\mathfrak{B}$-P-extensions.*

A pair $\langle \{H_t\}_{t \in T}, \boldsymbol{B} \rangle$, where $\boldsymbol{B} \in \mathfrak{B}$ and $\{H_t\}_{t \in T}$ is a family of systems of P-mappings of quasi-algebras $\boldsymbol{A}_t$, $t \in T$, of the type F into quasi-algebra $\boldsymbol{B}$, is called a *system of common $\mathfrak{B}$-P-mappings of quasi-algebras $\boldsymbol{A}_t$, $t \in T$.* If moreover, all systems H_t, $t \in T$, are one-to-one, then this pair is said to be a *system of common $\mathfrak{B}$-P-extensions of quasi-algebras $\boldsymbol{A}_t$, $t \in T$.* Let $H = \langle \{H_t\}_{t \in T}, B \rangle$ and $\boldsymbol{H}' = \langle \{H'_t\}_{t \in T}, B' \rangle$ be two systems of common $\mathfrak{B}$-P-mappings of quasi-algebras $\boldsymbol{A}_t$, $t \in T$.

We say that:

1. $\boldsymbol{H} \leqslant \boldsymbol{H}'$ if and only if there exists exactly one homomorphism q of $\boldsymbol{B}$ into $\boldsymbol{B}'$ such that $H'_t = q \cdot H_t$ for all $t \in T$,

2. $\boldsymbol{H} \equiv \boldsymbol{H}'$ if and only if there exists exactly one strong isomorphism q of $\boldsymbol{B}$ onto $\boldsymbol{B}'$ with $H'_t = q \cdot H_t$ for all $t \in T$.

A system $\boldsymbol{H}$ of common $\mathfrak{B}$-P-mappings of quasi-algebras $\boldsymbol{A}_t$, $t \in T$, is said to be $\mathfrak{B}$-*free* if, for every system $\boldsymbol{H}'$ of common $\mathfrak{B}$-P-mappings of quasi-algebras $\boldsymbol{A}_t$, $t \in T$, we have the relation $\boldsymbol{H} \leqslant \boldsymbol{H}'$. The $\mathfrak{B}$-free system of common $\mathfrak{B}$-P-mappings of quasi-algebras $\boldsymbol{A}_t$, $t \in T$, is uniquely determined up to the relation $\equiv$. Now we prove

THEOREM 7. *Let P be any $P_{F,G}(\varphi_\sigma, \sigma < \alpha)$-family of basic mapping-formulas and let $\mathfrak{B}$ be any quasi-primitive class of quasi-algebras of the type G. Moreover, let $\boldsymbol{A}_t$, $t \in T$, be an arbitrary family of quasi-algebras of the type F. Then there exists the $\mathfrak{B}$-free system of common $\mathfrak{B}$-P-mappings of quasi-algebras $\boldsymbol{A}_t$, $t \in T$.*

Proof. Let $(H, \boldsymbol{C})$ be the free system, which exists by Theorem 5, of $\mathfrak{B}$-P-mappings of the direct sum $\boldsymbol{A}$ of quasi-algebras $\boldsymbol{A}_t$, $t \in T$. The quasi-algebra $\boldsymbol{C}$ is the $\mathfrak{B}$-P-direct sum of quasi-algebras $\boldsymbol{A}_t$, $t \in T$. By

Theorem 6 the pair $\boldsymbol{H} = \langle\{H_t\}_{t\epsilon T}, \boldsymbol{C}\rangle$, where $H_t = H\cdot i_t$, is the $\mathfrak{V}$-free system of common $\mathfrak{V}$-P-mappings of quasi-algebras $\boldsymbol{A}_t, t\epsilon T$. Thus Theorem 7 is proved.

From (2.6) immediately results

(2.7) *There exists the proper $\mathfrak{V}$-P-direct sum of quasi-algebras $\boldsymbol{A}_t, t\epsilon T$, if and only if there exists a system of common $\mathfrak{V}$-P-extensions of $\boldsymbol{A}_t, t\epsilon T$.*

E. P-independence. Let P be an arbitrary $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas. Let us consider *the notion of independence with respect to P-mappings, i. e. the notion of P-independence.* Let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f\epsilon F)\rangle$ and $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g\epsilon G)\rangle$ be any quasi-algebras of the type F and G. A subset M of A is called *$\boldsymbol{B}$-P-independent* if every system $\Psi = \{\psi_\sigma, \sigma < a\}$ of mappings of M into B can be extended to a system $H = \{h_\sigma, \sigma < a\}$ of P-mappings of $\overline{\boldsymbol{M}}$ into $\boldsymbol{B}$ (i. e. h_σ is an extension of ψ_σ for all $\sigma < a$), where $\overline{\boldsymbol{M}}$ is the subquasi-algebra of $\boldsymbol{A}$ generated by M. Let us denote by P-ind* M and P-ind M the class of all quasi-algebras $\boldsymbol{B}$ of the type G such that M is $\boldsymbol{B}$-P-independent and, respectively the class of all algebras $\boldsymbol{B}$ of the type G such that M is $\boldsymbol{B}$-P-independent. Then we have

THEOREM 8. *The classes P-ind* M and P-ind M are primitive, i. e. closed with respect to subquasi-algebras, direct products, and homomorphic images.*

Proof. Obviously, these classes are closed with respect to subquasi-algebras. Let us suppose that $\boldsymbol{B}_t\epsilon P$-ind* M ($\boldsymbol{B}_t\epsilon P$-ind M) for $t\epsilon T$. Let $\boldsymbol{B} = \mathrm{P}_{t\epsilon T} \boldsymbol{B}_t$ be the direct product of $\boldsymbol{B}_t$ and let $\Psi = \{\psi_\sigma, \sigma < a\}$ be any system of mappings of M into $\boldsymbol{B}$. Let us consider the systems $\Psi_t = \{p_t\psi_\sigma, \sigma < a\} = p_t\cdot\Psi$, where $t\epsilon T$ and p_t is the natural projection of $\boldsymbol{B}$ onto $\boldsymbol{B}_t$, of mappings of M into $\boldsymbol{B}_t$. These systems can be, by the supposition, extended to systems $H_t, t\epsilon T$, of P-mappings of $\overline{\boldsymbol{M}}$ into $\boldsymbol{B}_t$. Let H be the direct product of systems $H_t, t\epsilon T$. By Theorem 3, H is a unique system of P-mappings of $\overline{\boldsymbol{M}}$ into $\boldsymbol{B}$ such that $H_t = p_t\cdot H$ for $t\epsilon T$. Hence it follows that H is an extension of Ψ, and thus $\boldsymbol{B}\epsilon P$-ind* M ($\boldsymbol{B}\epsilon P$-ind M). Therefore the classes P-ind* M and P-ind M are closed with respect to direct products. Now we prove that these classes are closed with respect to homomorphic images. Let us assume that $\boldsymbol{B}\epsilon P$-ind* M ($\boldsymbol{B}\epsilon P$-ind M) and that q is a homomorphism of $\boldsymbol{B}$ onto $\boldsymbol{C}$. Let $\Psi = \{\Psi_\sigma, \sigma < a\}$ be a system of mappings of M into $\boldsymbol{C}$. Let us denote by $\chi = \{\chi_\sigma, \sigma < a\}$ a system of mappings of M into $\boldsymbol{B}$ such that $\Psi = q\cdot\chi$, i. e. $\psi_\sigma = q\chi_\sigma$, for $\sigma < a$. Let H be a system of P-mappings of $\overline{\boldsymbol{M}}$ into $\boldsymbol{B}$ being an extension of χ. Then, by theorem (2.1), the system $q\cdot H$ is a system of P-mappings of $\overline{\boldsymbol{M}}$ into $\boldsymbol{C}$ which is obviously an extension of Ψ, and thus $C \epsilon P$-ind* M ($C \epsilon P$-ind M). This completes the proof of Theorem 8.

From Theorem 2 results

(2.8) *If P is a proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas, then a subset M of a quasi-algebra $\boldsymbol{A}$ of the type F is $\boldsymbol{B}$-P-independent, where $\boldsymbol{B}$ is any quasi-algebra of the type G, if and only if M is $P(\boldsymbol{B})$-independent (i. e. independent with respect to ordinary homomorphisms), where $P(\boldsymbol{B})$ is the P-product system over $\boldsymbol{B}$.*

Proof. Assume that M is $P(\boldsymbol{B})$-independent. Let $\Psi = \{\psi_\sigma, \sigma < a\}$ be any system of mappings of M into B. The direct product h of all mappings ψ_σ, $\sigma < a$, is a mapping of M into $P(B) = B^a$. Let $\bar{h}$ be the homomorphism of $\bar{M}$ into $P(\boldsymbol{B})$ being an extension of h. By Theorem 2 (see also (2.3) and (2.2)) the system $H = Pr \cdot \bar{h} = \{p_\sigma \bar{h}, \sigma < a\}$, where p_σ is the natural projection of $P(B) = B^a$ onto $B_\sigma = B$, is a system of P-mappings of M into $\boldsymbol{B}$ which is obviously an extension of Ψ. Thus M is $\boldsymbol{B}$-P-independent. Conversely, assume that M is $\boldsymbol{B}$-P-independent. Let ψ be any mapping of M into $P(B) = B^a$. The system $Pr \cdot \psi = \{p_\sigma \psi, \sigma < a\}$, where p_σ is the natural projection of $P(B) = B^a$ onto $B_\sigma = B$, is a system of mappings of M into B. Let $H = \{h_\sigma, \sigma < a\}$ be the system of P-mappings of $\bar{M}$ into $\boldsymbol{B}$ being an extension of the system $Pr \cdot \psi$ and let h be the direct product of all mappings h_σ, $\sigma < a$. By Theorem 2, h is a homomorphism of $\bar{M}$ into $P(\boldsymbol{B})$. Obviously, h is an extension of ψ. Therefore M is $P(\boldsymbol{B})$-independent. This completes the proof of theorem (2.8).

From (2.8) results

(2.9) *If M is an absolutely free of the type F subset of an algebra $\boldsymbol{A}$ of the type F (i. e. M is $\boldsymbol{D}$-independent for all algebras $\boldsymbol{D}$ of the type F, or, in other words, every mapping of M into D can be extended to a homomorphism of $\bar{M}$ into $\boldsymbol{D}$), then M is $\boldsymbol{B}$-P-independent for each algebra $\boldsymbol{B}$ of the type G and for each proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family P of basic mapping-formulas.*

Proof. The P-product system $P(\boldsymbol{B})$ over any algebra $\boldsymbol{B}$ of the type G is an algebra of the type F, and therefore M is $P(\boldsymbol{B})$-independent. Hence, by (2.8), M is $\boldsymbol{B}$-P-independent. Thus theorem (2.9) is proved.

In the sequel we shall consider the notion of P-independence with respect to an arbitrary proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family P of basic mapping-formulas only. The algebra $\boldsymbol{F}^* = \text{Free}(F, X)$ of F-terms as a Peano-algebra is, by Theorem 1, absolutely free of the type F freely generated by X and thus theorem (2.9) may be applied to this algebra. By theorem (2.9), X is a set of generators for $\boldsymbol{F}^*$ such that X is $\boldsymbol{B}$-P-independent for any algebra $\boldsymbol{B}$ of the type G, in particular X is $\boldsymbol{G}^*_\Phi$-P-independent, where

$G^*_\Phi = \mathrm{Free}(G, \Phi \times X)$ is the algebra of G-mapping-terms. Let us remind that $\Phi = \{\varphi_\sigma, \sigma < a\}$ and $X = \{x_\xi, \xi < \beta\}$, and let us consider the mappings $i_\sigma, \sigma < a$, of X into G^*_Φ such that $i_\sigma(x_\xi) = \varphi_\sigma(x_\xi) = \langle \varphi_\sigma, x_\xi \rangle$ for all $\sigma < a$ and $\xi < \beta$. The system $I = \{i_\sigma, \sigma < a\}$ can be, by (2.9), extended to a system $J_p = \{j_\sigma, \sigma < a\}$ of P-mappings of the algebra F^* into the algebra G^*_Φ. Hence we have $j_\sigma(x_\xi) = \varphi_\sigma(x_\xi)$ for all $\sigma < a$ and $\xi < \beta$. For any $\mu < a$ and any F-term $\tau \in F^*$, we put $P(\varphi_\mu, \tau) = j_\mu(\tau)$.

Let us observe that

(2.10) *For any system $H = \{h_\sigma, \sigma < a\}$ of P-mappings of a quasi-algebra A of the type F into a quasi-algebra B of the type G, and for every F-term $\tau = \tau(x_\xi, \xi < \varrho)$, we have*

$$h_\mu\big(\tau_A(a_\xi, \xi < \varrho)\big) = P(\varphi_\mu, \tau)_B\big(h_\sigma(a_\xi), \sigma < a, \xi < \varrho\big)$$

provided that τ_A is defined for $(a_\xi, \xi < \varrho)$ and $\mu < a$.

Proof. At first we remark that

(6) $$\tau_{P(B)}\big((b_{\sigma\xi}, \sigma < a), \xi < \varrho\big) = \big(P(\varphi_\mu, \tau)_B(b_{\sigma\xi}, \sigma < a, \xi < \varrho), \mu < a\big)$$

provided that $P(B)$ is the P-product system over B, and that $\tau_{P(B)}$ is defined for $\big((b_{\sigma\xi}, \sigma < a), \xi < \varrho\big)$. Indeed, by the definition of partial operations defined by terms in quasi-algebras, we have

$$\tau_{P(B)}\big((b_{\sigma\xi}, \sigma < a), \xi < \varrho\big) = \bar{\psi}(\tau),$$

where $\bar{\psi}$ is the homomorphism of F^* into $P(B)$ being an extension of a mapping ψ: $X \to P(B)$ such that $\psi(x_\xi) = (b_{\sigma\xi}, \sigma < a)$ for $\xi < \varrho$. Let us denote by $\bar{\psi}'$ the homomorphism of G^*_Φ into B, which is an extension of a mapping ψ': $\Phi \times X \to B$ such that $\psi'\big(\varphi_\sigma(x_\xi)\big) = b_{\sigma\xi}$ for $\sigma < a$ and $\xi < \varrho$. Let $Pr = \{p_\sigma, \sigma < a\}$ be the system of natural projections p_σ of $P(B) = B^a$ onto $B_\sigma = B$. The systems $H = Pr\cdot\bar{\psi} = \{p_\sigma\bar{\psi}, \sigma < a\}$ and $H' = \bar{\psi}'\cdot J_p = \{\bar{\psi}' j_\sigma, \sigma < a\}$, are, by theorems (2.3), (2.2) and (2.1), systems of P-mappings of F^* into B which are the same on the set X of generators for F^*. Hence, by (2.4), $H = H'$, i. e. we have $p_\mu\bar{\psi} = \bar{\psi}'\cdot j_\mu$ for all $\mu < a$, and thus for $\tau \in F^*$ we obtain $p_\mu\bar{\psi}(\tau) = \bar{\psi}' j_\mu(\tau)$. Then $p_\mu\tau_{P(B)}(\psi) = \bar{\psi}'\big(P(\varphi_\mu, \tau)\big) = P(\varphi_\mu, \tau)_B(\psi')$ for all $\mu < a$, whence

$$p_\mu\Big(\tau_{P(B)}\big((b_{\sigma\xi}, \sigma < a), \xi < \varrho\big)\Big) = P(\varphi_\mu, \tau)_B(b_{\sigma\xi}, \sigma < a, \xi < \varrho) \quad \text{for all} \quad \mu < a.$$

Hence we obtain relation (6). By Theorem 2, the direct product h of $h_\sigma, \sigma < a$, is a homomorphism of A into $P(B)$. Hence, using (6), we have

$$h\big(\tau_A(a_\xi, \xi < \varrho)\big) = \tau_{P(B)}\big(h(a_\xi), \xi < \varrho\big) = \tau_{P(B)}\Big(\big(h_\sigma(a_\xi), \sigma < a\big), \xi < \varrho\Big)$$
$$= \Big(P(\varphi_\mu, \tau)_B\big(h_\sigma(a_\xi), \sigma < a, \xi < \varrho\big), \mu < a\Big).$$

But, by the definition of. h,

$$h(\tau_A(a_\xi, \xi < \varrho)) = (h_\mu(\tau_A(a_\xi, \xi < \varrho)), \mu < a)$$

and thus

$$h_\mu(\tau_A(a_\xi, \xi < \varrho)) = P(\varphi_\mu, \tau)_B(h_\sigma(a_\xi), \sigma < a, \xi < \varrho).$$

Theorem (2.10) is proved.

Now we prove

THEOREM 9. *A subset M of a quasi-algebra A of the type F is B-P-independent, where B is an algebra of the type G and P is any proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas, if and only if each equality $\tau_A(m_\xi, \xi < \varrho) = \vartheta_A(m_\xi, \xi < \varrho)$, where m_ξ are different elements in M and τ and ϑ are F-terms, implies the following equalities: $P(\varphi_\mu, \tau)_B = P(\varphi_\mu, \vartheta)_B$, $\mu < a$ (resp. the G-mapping-equations $\ulcorner P(\varphi_\mu, \tau) = P(\varphi_\mu, \vartheta)\urcorner$ for $\mu < a$ are valid in B).*

Proof. Let us assume that M is B-P-independent. Let $(b_{\sigma\xi}, \sigma < a, \xi < \varrho)$ be any $a \times \varrho$-matrix over B and let ψ_σ, $\sigma < a$, be mappings of M into B such that $\psi_\sigma(m_\xi) = b_{\sigma\xi}$ for $\sigma < a$ and $\xi < \varrho$. The system $\Psi = \{\psi_\sigma, \sigma < a\}$ can be extended to a system $\overline{\Psi} = \{\overline{\psi}_\sigma, \sigma < a\}$ of P-mappings of A into B. But, by (2.10), we have

$$\begin{aligned}\overline{\psi}_\mu(\tau_A(m_\xi, \xi < \varrho)) &= P(\varphi_\mu, \tau)_B(\overline{\psi}_\sigma(m_\xi), \sigma < a, \xi < \varrho) \\ &= P(\varphi_\mu, \tau)_B(b_{\sigma\xi}, \sigma < a, \xi < \varrho)\end{aligned}$$

and also

$$\overline{\psi}_\mu(\tau_A(m_\xi, \xi < \varrho)) = \overline{\psi}_\mu(\vartheta_A(m_\xi, \xi < \varrho)) = P(\varphi_\mu, \vartheta)_B(b_{\sigma\xi}, \sigma < a, \xi < \varrho).$$

Hence $P(\varphi_\mu, \tau)_B(b_{\sigma\xi}, \sigma < a, \xi < \varrho) = P(\varphi_\mu, \vartheta)_B(b_{\sigma\xi}, \sigma < a, \xi < \varrho)$, i. e. $P(\varphi_\mu, \tau)_B = P(\varphi_\mu, \vartheta)_B$ for all $\mu < a$. This completes the proof of necessity. Now we give a proof of sufficiency. Let $F_0^* = \mathrm{Free}(F, M)$ be the Peano-algebra of the type F generated by M, i. e. the absolutely free algebra of the type F freely generated by the set M. Let $\Psi = \{\psi_\sigma, \sigma < a\}$ be any system of mappings of M into B. By theorem (2.9), the system Ψ can be extended to a system $\overline{\Psi} = \{\overline{\psi}_\sigma, \sigma < a\}$ of P-mappings of F_0^* into B. Now let us consider for $m \in M$ the identity mapping $m \to m$. By Theorem 1, there exists a partial-homomorphism χ of F_0^* into A (onto the subquasi-algebra $\overline{M}$ of A generated by M, see (1.2)) such that $\chi(m) = m$ for $m \in M$. Let us observe that

(7) if $\chi(w) = \chi(w')$, then $\overline{\psi}_\mu(w) = \overline{\psi}_\mu(w')$ for all $\mu < a$.

Indeed, let χ be defined for w and w' and let $\chi(w) = \chi(w')$. The elements w and w' can be represented in the form

$$w = \tau_{F_0^*}(m_\xi, \xi < \varrho) \quad \text{and} \quad w' = \vartheta_{F_0^*}(m_\xi, \xi < \varrho).$$

Then $\chi(w) = \tau_{\boldsymbol{A}}(m_\xi, \xi < \varrho)$ and $\chi(w') = \vartheta_{\boldsymbol{A}}(m_\xi, \xi < \varrho)$. Since $\chi(w) = \chi(w')$, we have the equality $\tau_{\boldsymbol{A}}(m_\xi, \xi < \varrho) = \vartheta_{\boldsymbol{A}}(m_\xi, \xi < \varrho)$. Hence, by the supposition, $P(\varphi_\mu, \tau)_{\boldsymbol{B}} = P(\varphi_\mu, \vartheta)_{\boldsymbol{B}}$ for all $\mu < a$. But, by (2.10), we have

$$\begin{aligned}\overline{\psi}_\mu(w) &= \overline{\psi}_\mu\big(\tau_{\boldsymbol{F}_0^*}(m_\xi, \xi < \varrho)\big) = P(\varphi_\mu, \tau)_{\boldsymbol{B}}\big(\overline{\psi}_\sigma(m_\xi), \sigma < a, \xi < \varrho\big)\\ &= P(\varphi_\mu, \vartheta)_{\boldsymbol{B}}\big(\overline{\psi}_\sigma(m_\xi), \sigma < a, \xi < \varrho\big) = \overline{\psi}_\mu\big(\vartheta_{\boldsymbol{F}_0^*}(m_\xi, \xi < \varrho)\big)\\ &= \overline{\psi}_\mu(w') \quad \text{for all} \quad \mu < a.\end{aligned}$$

Hence lemma (7) is proved. From (7) it follows that the mappings h_σ: $\overline{M} \to B$ such that $h_\sigma(a) = \overline{\psi}_\sigma(w)$, where $a = \chi(w)$, may be considered as functions on $\overline{M}$, where $\overline{M}$ is the subquasi-algebra of $\boldsymbol{A}$ generated by M. It may be verified that the system $H = \{h_\sigma, \sigma < a\}$ is a system of P-mappings of $\overline{M}$ into $\boldsymbol{B}$ being extension of system Ψ, i. e. the set M is $\boldsymbol{B}$-P-independent. This completes the proof of sufficiency and thus also the proof of Theorem 9.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be any classes of quasi-algebras of the type F and G. And let P be any proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas. A subset M of a quasi-algebra $\boldsymbol{A}$ of the type F is called:

1. $\mathfrak{B}$-*P-free* if M is $\boldsymbol{B}$-P-independent for all $\boldsymbol{B} \in \mathfrak{B}$,
2. $\mathfrak{A}$-*free* if M is $\boldsymbol{B}$-independent for all $\boldsymbol{B} \in \mathfrak{A}$.

The family P is said to be:

3. $(\mathfrak{A}, \mathfrak{B})$-*universal* if every $\mathfrak{A}$-free set is $\mathfrak{B}$-*P-free*,
4. $(\mathfrak{A}, \mathfrak{B})$-*constructor* if $P(\boldsymbol{B}) \in \mathfrak{A}$ for all $\boldsymbol{B} \in \mathfrak{B}$, where $P(\boldsymbol{B})$ is the P-product system over $\boldsymbol{B}$.

Let us observe that

(2.11) *If P is $(\mathfrak{A}, \mathfrak{B})$-constructor, then P is $(\mathfrak{A}, \mathfrak{B})$-universal.*

Proof. Let a set M be $\mathfrak{A}$-free and let $\boldsymbol{B} \in \mathfrak{B}$. Since $P(\boldsymbol{B}) \in \mathfrak{A}$, then the set M is $P(\boldsymbol{B})$-independent and thus, by (2.8), M is $\boldsymbol{B}$-P-independent, i. e. M is $\mathfrak{B}$-P-free. This completes the proof of (2.11).

Now let us assume that $\mathfrak{A}$ is an equationally definable class of algebras of the type F. Then we have the following theorems.

(2.12) *The family P is $(\mathfrak{A}, \mathfrak{B})$-universal if and only if P is $(\mathfrak{A}, \mathfrak{B})$-constructor.*

Proof. Assume that P is $(\mathfrak{A}, \mathfrak{B})$-universal. Let $\ulcorner \tau(x_\xi, \xi < \varrho) = \vartheta(x_\xi, \xi < \varrho) \urcorner$ be an arbitrary F-equation valid in $\mathfrak{A}$ (i. e. valid in each algebra in $\mathfrak{A}$) and let $\boldsymbol{B} \in \mathfrak{B}$. Let $\big((b_{\sigma\xi}, \sigma < a), \xi < \varrho\big)$ be any elements in $P(\boldsymbol{B})$, which is the P-product system over $\boldsymbol{B}$. Let us denote by $\boldsymbol{W} = \mathrm{Free}(\mathfrak{A}, M)$ the $\mathfrak{A}$-free algebra freely generated by $M = (m_\xi, \xi < \varrho)$, where m_ξ are different elements. Since P is $(\mathfrak{A}, \mathfrak{B})$-universal, M is $\mathfrak{B}$-P-free. Hence there exists a system $H = \{h_\sigma, \sigma < a\}$ of P-mappings of W into B such that $h_\sigma(m_\xi) = b_{\sigma\xi}$ for $\sigma < a$ and $\xi < \varrho$. By Theorem 2, the direct product h of all h_σ, $\sigma < a$, is a homomorphism of W into $P(B)$. Hence we have

$$\begin{aligned}\tau_{P(B)}\big((b_{\sigma\xi}, \sigma < a), \xi < \varrho\big) &= \tau_{P(B)}\Big(\big(h_\sigma(m_\xi), \sigma < a\big), \xi < \varrho\Big)\\ &= \tau_{P(B)}\big(h(m_\xi), \xi < \varrho\big) = h\big(\tau_W(m_\xi, \xi < \varrho)\big) = h\big(\vartheta_W(m_\xi, \xi < \varrho)\big)\\ &= \vartheta_{P(B)}\big(h(m_\xi), \xi < \varrho\big) = \vartheta_{P(B)}\Big(\big(h_\sigma(m_\xi), \sigma < a\big), \xi < \varrho\Big)\\ &= \vartheta_{P(B)}\big(b_{\sigma\xi}, \sigma < a), \xi < \varrho\big),\end{aligned}$$

i. e. the F-equation $\ulcorner\tau = \vartheta\urcorner$ is valid in $P(B)$ and thus $P(B) \in \mathfrak{A}$. The converse implication follows from (2.11) and thus theorem (2.12) is proved. Let us assume that $\mathfrak{A}$ and $\mathfrak{B}$ are any equationally definable classes of algebras of the type F and G respectively. Then we have

(2.13) *The family P is $(\mathfrak{A}, \mathfrak{B})$-universal (resp. $(\mathfrak{A}, \mathfrak{B})$-constructor) if and only if, for all F-terms τ and ϑ, the validity of the F-equation $\ulcorner\tau = \vartheta\urcorner$ in the class $\mathfrak{A}$ implies the validity of G-mapping-equations $\ulcorner P(\varphi_\mu, \tau) = P(\varphi_\mu, \vartheta)\urcorner$ in the class $\mathfrak{B}$ for all $\mu < a$.*

Proof. This follows from Theorem 9.

Let us denote by $\mathfrak{B}(P, \mathfrak{A})$ the class of all $B \in \mathfrak{B}$ such that the G-mapping-equations $\ulcorner P(\varphi_\mu, \tau) = P(\varphi_\mu, \vartheta)\urcorner$, $\mu < a$, are valid in B provided the F-equation $\ulcorner\tau = \vartheta\urcorner$ is valid in the class $\mathfrak{A}$. By $\mathfrak{B}(\mathfrak{A})$ will be denoted the intersection of all classes $\mathfrak{B}(P, \mathfrak{A})$, where P is proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family of basic mapping-formulas.

(2.14) *For every pair $(\mathfrak{A}, \mathfrak{B})$ and every proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family P of basic mapping-formulas there exists a maximal subclass $\mathfrak{B}_P \subset \mathfrak{B}$ such that P is $(\mathfrak{A}, \mathfrak{B}_P)$-universal (resp. $(\mathfrak{A}, \mathfrak{B}_P)$-constructor).*

Proof. The class $\mathfrak{B}_P = \mathfrak{B}(P, \mathfrak{A})$ fulfils, by (2.13), the thesis of (2.14).

The pair $(\mathfrak{A}, \mathfrak{B})$ is said to *be universal* (resp. *constructor*) if for every proper $P_{F,G}(\varphi_\sigma, \sigma < a)$-family P of basic mapping-formulas P is $(\mathfrak{A}, \mathfrak{B})$-universal (resp. $(\mathfrak{A}, \mathfrak{B})$-constructor).

(2.15) *For every pair $(\mathfrak{A}, \mathfrak{B})$ there exists a maximal subclass $\mathfrak{B}_0 \subset \mathfrak{B}$ such that the pair $(\mathfrak{A}, \mathfrak{B}_0)$ is universal (resp. constructor).*

Proof. The class $\mathfrak{B}_0 = \mathfrak{B}(\mathfrak{A})$ fulfils the thesis of (2.15).

Let us assume that $F = G$. Then we have

(2.16) *For every equationally definable class $\mathfrak{A}$ of algebras of the type F there exists a maximal subclass $\mathfrak{A}_0 \subset \mathfrak{A}$ such that the pair $(\mathfrak{A}_0, \mathfrak{A}_0)$ is universal (resp. constructor).*

Proof. The class $\mathfrak{A}_0 = \mathfrak{A}(\mathfrak{A})$ fulfils the thesis of (2.16), since we have $\mathfrak{A}_0 = \mathfrak{A}_0(\mathfrak{A}_0)$.

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