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A generalized hypergeometric polynomial

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1. Introduction. In this paper we study some properties of the hypergeometric polynomial:

(1.1)
$$f_n^{(c,k)}(a_p;\, b_q;\, x) \equiv f_n^{(c,k)}(a_1, a_2, \ldots, a_p;\, b_1, b_2, \ldots, b_q;\, x)$$
$$= \frac{(c)_n}{n!}\, {}_{p+k}F_{q+k}\left[\begin{matrix} -n,\, \Delta(k-1,\, c+n),\, a_1, \ldots, a_p; \\ \Delta(k,\, c),\, b_1, \ldots, b_q; \end{matrix}\ (k-1)^{k-1}x\right],$$

where n, k are non-negative integers and $\Delta(k, c)$ denotes the set of k parameters c/k, $(c+1)/k, \ldots, (c+k-1)/k$. When there are no a's and b's we shall simply write $f_n^{(c,k)}(x)$ instead of $f_n^{(c,k)}(-;\, -;\, x)$.

This polynomial has arisen in the course of an attempt to unify and extend the study of most of the well-known sets of polynomials. When $c = 1$, $k = 2$, the polynomial (1.1) reduces to the polynomial $f_n(a_p;\, b_q;\, x)$ of M. C. Fasenmyer [3]. All those polynomials which are special cases of Fasenmyer's polynomial can also be obtained from (1.1). In addition we have the following particular cases:

(1.2)
$$f_n^{(1+a,1)}(x) = L_n^{(a)}(x) \quad \text{(Laguerre polynomial)}\,.$$

(1.3)
$$f_n^{(1+a+b,2)}(\tfrac{1}{2}+\tfrac{1}{2}a+\tfrac{1}{2}b,\, 1+\tfrac{1}{2}a+\tfrac{1}{2}b;\ 1+a;\, x) = \frac{(1+a+b)_n}{(1+a)_n} P_n^{(a,b)}(1-2x) \quad \text{(Jacobi polynomial)}\,.$$

(1.4)
$$f_n^{(1+a+b,2)}(\tfrac{1}{2}+\tfrac{1}{2}a+\tfrac{1}{2}b,\, 1+\tfrac{1}{2}a+\tfrac{1}{2}b,\, \xi;\, 1+a,\, p;\, v) = \frac{(1+a+b)_n}{(1+a)_n} H_n^{(a,b)}(\xi;\, p;\, v)\,,$$

where $H_n^{(a,b)}(\xi;\, p;\, v)$ is the generalized Rice's polynomial introduced by Khandekar [4].

(1.5)
$$f_n^{(c,2)}(\tfrac{1}{2}c,\, \tfrac{1}{2}c+\tfrac{1}{2};\, -;\, x) = \frac{(c)_n}{n!}\, {}_2F_0(-n,\, c+n;\, -;\, x) \equiv \varphi_n(c,\, x)\,,$$

where $\varphi_n(c, x)$ is taken as the standard form of the generalized Bessel polynomial [6]. In the notation of Krall and Frink [5],

$$\varphi_n(c, x) = \frac{(c)_n}{n!} y_n(x, c+1, -1).$$

$$f_n^{(c,2)}(a; c; x) = \frac{(c)_n}{n!} {}_3F_3\left[\begin{matrix} -n, c+n, a; \\ \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}, c; \end{matrix} x\right] \equiv R_n(a; c; x) . \tag{1.6}$$

This $R_n(a; c; x)$ reduces to Bateman's $Z_n(x)$ when $2a = c = 1$.

2. A generating function. Let $G(y)$, analytic at $y = 0$, have the expansion

$$G(y) = \sum_{n=0}^{\infty} h_n y^n,$$

and let $g_n(x)$ be defined by the relation

$$(1-t)^{-c} G\left[-\frac{k^k xt}{(1-t)^k}\right] = \sum_{n=0}^{\infty} g_n(x) t^n . \tag{2.1}$$

For sufficiently small values of t, L.H.S. of (2.1) can be expanded in an absolutely convergent double series and the terms can be rearranged so as to have a convergent power series in t. Thus, we can easily obtain:

$$g_n(x) = \frac{(c)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r \left(\frac{c+n}{k-1}\right)_r \left(\frac{c+n+1}{k-1}\right)_r \cdots \left(\frac{c+n+k-2}{k-1}\right)_r}{\left(\frac{c}{k}\right)_r \left(\frac{c+1}{k}\right)_r \cdots \left(\frac{c+k-1}{k}\right)_r} \times \tag{2.2}$$

$$\times\, h_r (k-1)^{r(k-1)} x^r, \qquad k > 1;$$

$$= \frac{(c)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r}{(c)_r} x^r, \qquad k = 1 .$$

For the choice

$$G(y) = {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} y\right]$$

whose convergence conditions are well-known, the $g_n(x)$ defined by (2.2) become $f_n^{(c,k)}(a_p; b_q; x)$ of (1.1), i.e.

$$(1-t)^{-c} {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} -\frac{k^k xt}{(1-t)^k}\right] = \sum_{n=0}^{\infty} f_n^{(c,k)}(a_p; b_q; x) t^n . \tag{2.3}$$

This generating relation includes the following:

$$(1-t)^{-1-a}\exp\left(\frac{-xt}{1-t}\right)=\sum_{n=0}^{\infty}L_n^{(a)}(x)t^n.$$

This is formula § 113 (3) in [6].

$$(2.5)\qquad (1-t)^{-c}{}_1F_1\left[\begin{matrix}c; & \dfrac{-xt}{1-t}\\ 1+a; & \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(c)_nL_n^{(a)}(x)}{(1+a)_n}t^n.$$

This is formula § 113 (3) in [6].

$$(2.6)\qquad (1-t)^{-1-a-b}{}_2F_1\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b,\ 1+\frac{1}{2}a+\frac{1}{2}b; & \dfrac{-4xt}{(1-t)^2}\\ 1+a; & \end{matrix}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(1+a+b)_n}{(1+a)_n}P_n^{(a,b)}(1-2x)t^n.$$

This is formula § 136 (1) in [6].

$$(2.7)\qquad (1-t)^{-1-a-b}{}_3F_2\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b,\ 1+\frac{1}{2}a+\frac{1}{2}b,\ \xi; & \dfrac{-4vt}{(1-t)^2}\\ 1+a,\ p; & \end{matrix}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(1+a+b)_n}{(1+a)_n}H_n^{(a,b)}(\xi;p;v)t^n.$$

This is formula (4.2) in [4].

$$(2.8)\qquad (1-t)^{-c}{}_2F_0\left[\begin{matrix}\frac{1}{2}c;\ \frac{1}{2}c+\frac{1}{2}; & \dfrac{-4xt}{(1-t)^2}\\ -; & \end{matrix}\right]=\sum_{n=0}^{\infty}\varphi_n(c,x)t^n.$$

This is formula § 150 (6) in [6].

$$(2.9)\qquad (1-t)^{-c}{}_1F_1\left[\begin{matrix}a; & \dfrac{-4xt}{(1-t)^2}\\ c; & \end{matrix}\right]=\sum_{n=0}^{\infty}R_n(a;c;x)t^n.$$

When $2a=c=1$, (2.9) reduces to

$$(2.10)\qquad (1-t)^{-1}{}_1F_1\left[\begin{matrix}\frac{1}{2}; & \dfrac{-4xt}{(1-t)^2}\\ 1; & \end{matrix}\right]=\sum_{n=0}^{\infty}Z_n(x)t^n.$$

This is formula § 146 (3) in [6].

3. A differential recurrence formula. When there is no restriction on $G(y)$ to be of hypergeometric type, we differentiate separately the two sides of (2.1) w.r.t. x and t and eliminate $G'(y)$ $\left[\equiv \frac{d}{dy} G(y)\right]$ to obtain the recurrence formula:

$$x[g'_n(x) + (k-1)g'_{n-1}(x)] = ng_n(x) - (n-1+c)g_{n-1}(x)\,. \tag{3.1}$$

Applying (3.1) directly to (1.4) and (1.6), we get the following results:

$$v\frac{d}{dv}[H_n^{(a,b)}(\xi;p;v) + H_{n-1}^{(a,b)}(\xi;p;v)] = nH_n^{(a,b)}(\xi;p;v) - (n+a+b)H_{n-1}^{(a,b)}(\xi;p;v)\,. \tag{3.2}$$

$$x[R'_n(a;c;x) + R'_{n-1}(a;c;x)] = nR_n(a;c;x) - (n-1+c)R_{n-1}(a;c;x)\,. \tag{3.3}$$

The recurrence formula (3.1) includes similar results for the Laguerre, Jacobi and Bessel polynomials in addition to the one for Fasenmyer's polynomial.

The formula (3.1) can be used to develop an expression for $xg'_n(x)$ as a series in the $g_n(x)$. First, we put $g_n(x) = (-1)^n p_n(x)$ in (3.1) to get:

$$np_n(x) + (n-1+c)p_{n-1}(x) = x[p'_n(x) - (k-1)p'_{n-1}(x)]\,. \tag{3.4}$$

Now shifting the index several times, we have

$$\begin{aligned}(n-1)p_{n-1}(x) + (n-2+c)p_{n-2}(x) &= x[p'_{n-1}(x) - (k-1)p'_{n-2}(x)]\,,\\ (n-2)p_{n-2}(x) + (n-3+c)p_{n-3}(x) &= x[p'_{n-2}(x) - (k-1)p'_{n-3}(x)]\,,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ 2p_2(x) + (1+c)p_1(x) &= x[p'_2(x) - (k-1)p'_1(x)]\,,\\ p_1(x) + cp_0(x) &= x[p'_1(x) - (k-1)p'_0(x)]\,.\end{aligned} \tag{3.5}$$

Multiplying the equations in (3.5) by $(k-1), (k-1)^2, \dots, (k-1)^{n-2}, (k-1)^{n-1}$ respectively, and adding all these to (3.4), we obtain

$$np_n(x) + [k(n-1)+c]p_{n-1}(x) + (k-1)[k(n-2)+c]p_{n-2}(x) + \\ + \dots + (k-1)^{n-2}(k+c)p_1(x) + c(k-1)^{n-1}p_0(x) = xp'_n(x)\,,$$

or, in terms of $g_n(x)$,

$$xg'_n(x) = ng_n(x) + \sum_{r=0}^{n-1}(-1)^{n-r}(k-1)^{n-r-1}(rk+c)g_r(x)\,, \qquad n \geqslant 1\,. \tag{3.6}$$

Applying this to (1.4) and (1.5), we obtain the formulae:

$$v\frac{d}{dv}\left[\frac{(1+a+b)_n}{(1+a)_n}H_n^{(a,b)}(\xi;p;v)\right] = \frac{(1+a+b)_n}{(1+a)_n}nH_n^{(a,b)}(\xi;p;v) + \\ + \sum_{r=0}^{n-1}(-1)^{n-r}(2r+a+b+1)\frac{(1+a+b)_r}{(1+a)_r}H_r^{(a,b)}(\xi;p;v)\,, \tag{3.7}$$

$$x\varphi'_n(c,x) = n\varphi_n(c,x) + \sum_{r=0}^{n-1}(-1)^{n-r}(2r+c)\varphi_r(c,x)\,. \tag{3.8}$$

4. Integral relations. Inside the region of convergence of the resulting series, we have ([6], p. 104)

$$(4.1)\quad \int_0^u t^{\alpha-1}(u-t)^{\beta-1}{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix}\,xt\right]dt = B(\alpha,\beta)\,u^{\alpha+\beta-1}{}_{p+1}F_{q+1}\left[\begin{matrix}a_1,\ldots,a_p,\alpha;\\ b_1,\ldots,b_q,\alpha+\beta;\end{matrix}\,ux\right],$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$.

Using (4.1), we obtain:

$$(4.2)\quad f_n^{(c,k)}(a_p;\,b_q;\,ux) = \frac{u^{1-b_q}}{B(a_p,\,b_q-a_p)}\int_0^u t^{a_p-1}(u-t)^{b_q-a_p-1}f_n^{(c,k)}(a_{p-1};\,b_{q-1};\,xt)\,dt$$

where $\mathrm{Re}(b_q)>\mathrm{Re}(a_p)>0$, and

$$(4.3)\quad f_n^{(c,k)}(a_{p-1};\,b_{q-1};\,ux) = \frac{u^{1-a_p}}{B(b_q,\,a_p-b_q)}\int_0^u t^{b_q-1}(u-t)^{a_p-b_q-1}f_n^{(c,k)}(a_p;\,b_q;\,xt)\,dt\,,$$

where $\mathrm{Re}(a_p)>\mathrm{Re}(b_q)>0$.

Putting $u=1$ in (4.2) and (4.3), and changing t to $t/(t+1)$, we get:

$$(4.4)\quad f_n^{(c,k)}(a_p;\,b_q;\,x) = \frac{1}{B(a_p,\,b_q-a_p)}\int_0^\infty t^{a_p-1}(1+t)^{-b_q}f_n^{(c,k)}\left(a_{p-1};\,b_{q-1};\,\frac{xt}{1+t}\right)dt$$

and

$$(4.5)\quad f_n^{(c,k)}(a_{p-1};\,b_{q-1};\,x) = \frac{1}{B(b_q,\,a_p-b_q)}\int_0^\infty t^{b_q-1}(1+t)^{-a_p}f_n^{(c,k)}\left(a_p;\,b_q;\,\frac{xt}{1+t}\right)dt\,.$$

Again, from the formula (19) ([2], p. 220)

$$(4.6)\quad {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix}\,x\right] = \frac{1}{\Gamma(a_p)}\int_0^\infty e^{-t}t^{a_p-1}{}_{p-1}F_q\left[\begin{matrix}a_1,\ldots,a_{p-1};\\ b_1,\ldots,b_q;\end{matrix}\,xt\right]dt\,,$$

where $p\leqslant q+1$, $\mathrm{Re}(a_p)>0$, we obtain:

$$(4.7)\quad f_n^{(c,k)}(a_p;\,b_q;\,x) = \frac{1}{\Gamma(a_p)}\int_0^\infty e^{-t}t^{a_p-1}f_n^{(c,k)}(a_{p-1};\,b_q;\,xt)\,dt\,.$$

The relations (4.2), (4.3), (4.4), (4.5) and (4.7) include the following results:

$$(4.8)\qquad \frac{(1+a)_n}{(1+a+b)_n}\, u^{a+b} L_n^{(a+b)}(u) = \frac{1}{B(1+a,b)} \int_0^u t^a (u-t)^{b-1} L_n^{(a)}(t)\,dt\,.$$

This is formula (9), in [6] p. 216.

$$(4.9)\qquad H_n^{(a,b)}(\xi;p;v) = \frac{\Gamma(p)}{\Gamma(\xi)\Gamma(p-\xi)} \int_0^1 t^{\xi-1}(1-t)^{p-\xi-1} P_n^{(a,b)}(1-2vt)\,dt\,,$$

where $\mathrm{Re}(p) > \mathrm{Re}(\xi) > 0$. This is formula (2.1) in [4].

$$(4.10)\qquad P_n^{(a,b)}(1-2v) = \frac{\Gamma(\xi)}{\Gamma(p)\Gamma(\xi-p)} \int_0^1 t^{p-1}(1-t)^{\xi-p-1} H_n^{(a,b)}(\xi;p;vt)\,dt\,,$$

where $\mathrm{Re}(\xi) > \mathrm{Re}(p) > 0$. This is formula (2.3) in [4] with a minor correction.

$$(4.11)\qquad H_n^{(a,b)}(\xi;p;v) = \frac{\Gamma(p)}{\Gamma(\xi)\Gamma(p-\xi)} \int_0^\infty t^{\xi-1}(1+t)^{-p} P_n^{(a,b)}\left(1-\frac{2vt}{1+t}\right)dt\,.$$

This is formula (3.1) in [4].

$$(4.12)\qquad P_n^{(a,b)}(1-2v) = \frac{\Gamma(\xi)}{\Gamma(p)\Gamma(\xi-p)} \int_0^\infty t^{p-1}(1+t)^{-\xi} H_n^{(a,b)}\left(\xi;p;\frac{vt}{1+t}\right)dt\,.$$

$$(4.13)\qquad H_n^{(a,b)}(\xi;p;v) = \frac{(1+a)_n}{n!\,\Gamma(\xi)} \int_0^\infty e^{-t} t^{\xi-1}\, {}_2F_2\left[\begin{matrix}-n,\ n+a+b+1; \\ 1+a,\ p;\end{matrix}\ vt\right] dt\,.$$

This is formula (3.3) in [4] with a correction.

$$(4.14)\qquad P_n^{(a,b)}(1-2v) = \frac{(1+a)_n}{n!\,\Gamma(p)} \int_0^\infty e^{-t} t^{p-1}\, {}_2F_2\left[\begin{matrix}-n,\ n+a+b+1; \\ 1+a,\ p;\end{matrix}\ vt\right] dt\,.$$

5. The polynomial $R_n(a;c;x)$ and Laguerre polynomials. From (4.7) we have

$$(5.1)\qquad f_n^{(1+a,2)}(a;1+a;x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} f_n^{(1+a;2)}(-;1+a;xt)\,dt\,.$$

Now, by Ramanujan's theorem ([6], p. 106, Ex. 5), we have

$$(5.2)\qquad L_n^{(a)}(x)L_n^{(a)}(-x)=\left[\frac{(1+a)_n}{n!}\right]^2 {}_2F_3\left[\begin{matrix}-n,\,1+a+n;\\ \frac12+\frac12a,\,1+\frac12a,\,1+a;\end{matrix}\ \tfrac14x^2\right]$$
$$=\frac{(1+a)_n}{n!}f_n^{(1+a,2)}(-;\,1+a;\,\tfrac14x^2)\,.$$

From (5.1), we have

$$R_n(a;\,1+a;\,x^2)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-t}t^{a-1}f_n^{(1+a,2)}(-;\,1+a;\,x^2t)\,dt$$
$$=\frac{1}{\Gamma(a)}\int_0^\infty e^{-\frac14t^2}(\tfrac12t)^{2a-1}f_n^{(1+a,2)}(-;\,1+a,\,\tfrac14x^2t^2)\,dt\,.$$

Hence, using (5.2), we obtain:

$$(5.3)\qquad \frac{(1+a)_n}{n!}R_n(a;\,1+a;\,x)=\frac{1}{2^{2a-1}\Gamma(a)}\int_0^\infty e^{-\frac14t^2}t^{2a-1}L_n^{(a)}(tx)L_n^{(a)}(-tx)\,dt\,.$$

When $a=\frac12$, $a=0$, (5.3) reduces to

$$Z_n(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\frac14t^2}L_n(tx)L_n(-tx)\,dt\,,$$

which is formula § 149 (10) in [6].

6. Transformation of a special ${}_3F_2$. From (2.3) we have

(6.1)

$$(1-t)^{-c}{}_3F_2\left[\begin{matrix}\frac12c,\,\Delta(2,\,a+1);\\ a+1,\,\frac12c+\frac12;\end{matrix}\ \frac{-4t}{(1-t)^2}\right]=\sum_{n=0}^\infty\frac{(c)_n}{n!}\,{}_4F_3\left[\begin{matrix}-n,\,c+n,\,\Delta(2,\,a+1);\\ a+1,\,\Delta(2,\,c+1);\end{matrix}\ 1\right]t^n.$$

Carlitz [1] has given the following formula:

$$(6.2)\qquad {}_4F_3\left[\begin{matrix}-n,\,c+n,\,\Delta(2,\,a+1);\\ a+1,\,\Delta(2,\,c+1);\end{matrix}\ 1\right]=\frac{c(c-a)_n}{(c+2n)(c)_n}\,.$$

From (6.1) and (6.2), we obtain the formula:

$$(6.3)\qquad (1-t)^{-c}{}_3F_2\left[\begin{matrix}\frac12c,\,\Delta(2,\,a+1);\\ a+1,\,\frac12c+\frac12;\end{matrix}\ \frac{-4t}{(1-t)^2}\right]$$
$$=\sum_{n=0}^\infty\frac{(c)_n}{n!}\cdot\frac{c(c-a)_n}{(c+2n)(c)_n}t^n=\sum_{n=0}^\infty\frac{(\frac12c)_n(c-a)_n}{n!(\frac12c+1)_n}t^n={}_2F_1\left[\begin{matrix}\frac12c,\,c-a;\\ \frac12c+1;\end{matrix}\ t\right].$$

When $t\to -1$, this reduces to

$$ {}_3F_2\left[\begin{matrix}\tfrac{1}{2}c,\ \tfrac{1}{2}a+\tfrac{1}{2},\ \tfrac{1}{2}a+1; \\ a+1,\ \tfrac{1}{2}c+\tfrac{1}{2};\end{matrix}\ 1\right] = 2^c\,{}_2F_1\left[\begin{matrix}\tfrac{1}{2}c,\ c-a; \\ \tfrac{1}{2}c+1;\end{matrix}\ -1\right]. $$

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