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# Some applications of the theorem on rational approximation in the mean

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**Abstract.** For a compact set $X \subset C$ denote by $L_a^p(X)$ the subspace in $L^2(X)$ which consists of functions holomorphic in the interior of $X$. A subspace $R(X) \subset L_a^p(X)$ consists of functions which are holomorphic in a neighbourhood of $X$. The functions in $L_a^p(X)$ $(R(X))$ which possess a single valued-integral in $\mathring{X}$ (in a neighbourhood of $X$) form a subset denoted by $\tilde{L}_a^p(X)$ $(\tilde{R}(X))$. The paper gives a proof of the following statement. If $\partial X$ has countably many components, $2 \leqslant p < \infty$, and $\tilde{R}(X)$ is dense in $\tilde{L}_a^p(X)$, then $R(X)$ is dense in $L_a^p(X)$. Conversely, if $\partial X$ has finitely many components and $R(X)$ is dense in $L_a^p(X)$, then $\tilde{R}(X)$ is dense in $\tilde{L}_a^p(X)$. For $p = 2$ this result yields a theorem on the convergence of reduced Bergman functions for a decreasing sequence of domains.

**1. Introduction.** The rational approximation in the mean was studied by various authors, e.g. by Bers [3] and Sinanian [10] in 1965, but the problem attracted great attention when it turned out that the solution can be stated in terms of potential theory. In 1968 Havin [5] pointed out that for $p = 2$ the family of subsets $E \subset C$ on which such approximation is possible is characterized in terms of logarithmic capacity. A complete solution in terms of $q$-capacity was given in 1972 by Bagby [1] and Hedberg [6]. In particular the following result was proved by Hedberg [7].

THEOREM 1. *Let $X$ be an arbitrary compact subset of the complex plane. Denote by $R(X)$ the set of functions which are holomorphic in a neighbourhood of $X$, and by $L_a^p(X)$ the Banach space of functions which are holomorphic in the interior $\mathring{X}$ and belong to $L^p(X)$. The following conditions are equivalent* $(1 < p < \infty,\ q = p/(p-1))$:

(i) *$R(X)$ is dense in $L_a^p(X)$.*

(ii) *Every function $g$ in the Sobolev space $\mathring{W}_1^q(C)$ $q$-quasi-continuous and such that $g = 0$ $q$-quasi-everywhere on $C \setminus X$ vanishes $q-q\cdot e$, on the boundary of $X$.*

(iii) Every function $g$ in the Sobolev space $\mathring{W}_1^q(C)$ $q$-quasi-continuous and such that $g = 0$ $q$-quasi-everywhere on $C \setminus X$ belongs to $\mathring{W}_1^q(\mathring{X})$.

In the present paper we are concerned with the approximation problem within the class of functions with a single-valued integral. We consider therefore the subspace $\tilde{L}_a^p(X) \subset L_a^p(X)$ consisting of functions with a single-valued integral in $\mathring{X}$, and the set $\tilde{R}(X) \subset R(X)$ consisting of functions with a single-valued integral in some neighbourhood of $X$. For $2 \leqslant p < \infty$ it is shown that if $\tilde{R}(X)$ is dense in $\tilde{L}_a^p(X)$ and $\partial X$ has countably many components, then $R(X)$ is dense in $L_a^p(X)$. On the other hand, if $R(X)$ is dense in $L_a^p(X)$ and $\partial X$ has a finite number of components, then $R(X)$ is dense in $\tilde{L}_a^p(X)$. We also show that for $p = 2$ the above approximation problem is related to the theorem on a sequence of reduced Bergman functions in a way analogous to that in [8].

**2. Definition of $q$-capacity.** We recall briefly this fundamental notion. Let $\Omega$ be an open subset of $C$. The Sobolev space $W_1^q(\Omega)$ consists of functions which together with partial derivatives of the first order are in $L^q(\Omega)$.

$L^q(\Omega)$ is a Banach space with the norm

$$\|f\|_q = \Big\{\int_\Omega (|f|^2 + |\mathrm{grad} f|^2)^{q/2} dm\Big\}^{1/q}.$$

The set of all real-valued smooth functions with compact support in $\Omega$ will be denoted by $\mathscr{D}(\Omega)$. The closure of $\mathscr{D}(\Omega)$ in $W_1^q(\Omega)$ will be denoted by $\mathring{W}_1^q(\Omega)$. If $E$ is a compact subset of $\Omega$, the number

$$\gamma_\Omega^q(E) = \inf\{\|f\|_q,\ f \geqslant 1 \text{ on } E,\ f \in \mathscr{D}(\Omega)\}$$

s called the *$q$-capacity of $E$*. The (outer) $q$-capacity of an arbitrary set $E \subset \Omega$ is defined in a standard way (Bagby [1]). Condition $\gamma_\Omega^q(E) = 0$ is independent of $\Omega$. A property which a point may or may not have is said to hold *$q$-quasi-everywhere* if it holds for all points with the exception of a set of (outer) capacity zero. A function is $q$-quasi-continuous if it is continuous outside an open set with an arbitrarily small capacity. The following lemma is due to Deny [4]:

LEMMA 1. *If a sequence $\varphi_n \in \mathscr{D}(C)$ converges in $W_1^q(C)$, then for each $\varepsilon > 0$ there exists an open set $U$ such that $\gamma_C^q(U) < \varepsilon$ and the convergence is uniform in $C \setminus U$.*

We see in particular that every function in $\mathring{W}_1^q(\Omega)$ is equal almost everywhere to a quasi-continuous function.

**3. Annihilating functionals.** Let $\Omega$ be an open subset of $C$. Consider the family $V(\Omega)$ of smooth functions defined by the following property. For each $\varphi \in V(\Omega)$ there exists a compact set $K \subset \Omega$ such that $\varphi = 0$ in the unbounded component of $C \setminus K$, and is constant in every other component. The closure of $V(\Omega)$ in the Sobolev space $W_1^q(\Omega)$ will be denoted by $\mathring{V}(\Omega)$.

LEMMA 2. *Let* $g \in L^q(X)$. *The condition*

$$\int_X f\bar{g}\,dm = 0$$

*is satisfied for all* $f \in \tilde{L}_a^p(X)$ *if and only if*

(1) $$g = \partial\varphi/\partial z,$$

*where* $\varphi \in \mathring{V}(\mathring{X})$.

Proof. It is easy to see that a function $f \in L^p(X)$ is orthogonal to every function $\partial\varphi/\partial z$, $\varphi \in \mathring{V}(\mathring{X})$ if and only if $f \in \tilde{L}_a^p(X)$. Therefore $g$ is orthogonal to $\tilde{L}_a^p(X)$ if and only if it belongs to the closure of the set $\{\partial\varphi/\partial z\}$ in $L^q$ norm. Since $\mathring{X}$ is bounded, the latter is equivalent to (1).

LEMMA 3. *Let* $g \in L^q(X)$. *The condition*

$$\int_X f\bar{g}\,dm = 0$$

*is satisfied for all* $f \in \tilde{R}(X)$ *if and only if* $g = \partial\varphi/\partial z$, *where* $\varphi \in \mathring{W}_1^q(C)$ *vanishes quasi-everywhere in the unbounded component of* $C\setminus X$, *and in every other component is equal to a constant quasi-everywhere.*

Proof. *Necessity.* Extend $g$ trivially to all of $C$. If $Y$ is any compact neighbourhood of $X$, Lemma 2 says that $g = \partial\varphi/\partial z$, $\varphi \in \mathring{V}(\mathring{Y})$, and we conclude from Lemma 1 that the condition on $\varphi$ holds when $X$ is replaced by $Y$. Since $Y$ is arbitrary, this completes the first part of the proof. For sufficiency assume $f$ holomorphic in an open set $G \supset X$. Take $\varrho \in \mathcal{D}(G)$ such that $\varrho = 1$ in some neighbourhood of $X$, and $D = \{0 < \varrho < 1\}$ is a Stokes domain bounded by finitely many closed curves $\gamma_i$. Then

$$\int_X f\bar{g}\,d\bar{z}\,dz = \int_G \varrho f\frac{\partial\varphi}{\partial\bar{z}}\,d\bar{z}\,dz = -\int_G \varphi f\frac{\partial\varrho}{\partial z}\,d\bar{z}\,dz$$

$$= -\int_D \varphi f\frac{\partial\varrho}{\partial\bar{z}} = -\sum_i c_i \int_{D\cap\{\varphi=c_i\}} f\frac{\partial\varrho}{\partial\bar{z}}\,d\bar{z}\,dz = \sum_i C_i \int_{\gamma_i} f\,dz = 0.$$

**4. The approximation problem.** We can now state

THEOREM 2. *The space* $\tilde{R}(X)$ *is dense in* $\tilde{L}_a^p(X)$ *if and only if each function* $\varphi \in \mathring{W}_1^q(C)$ *such that* $\varphi = \mathrm{const}$ *quasi-everywhere in each component of* $C\setminus X$ *belongs to* $\mathring{V}(\mathring{X})$.

Proof. The proof follows immediately from Lemmas 2 and 3. We need only to notice that the equality $\partial\varphi/\partial z = \partial\psi/\partial z$, where $\varphi$ and $\psi$ both belong to $\mathring{W}_1^q(\Omega)$ with $\Omega$ bounded, implies that $\varphi = \psi$.

COROLLARY 1. *If* $\tilde{R}(X)$ *is dense in* $\tilde{L}_a^p(X)$, *then every function* $\varphi \in \mathring{W}_1^q(C)$ *which on every component of* $C\setminus X$ *is equal to a constant quasi-everywhere* *must be equal to a constant quasi-everywhere on each component of* $C\setminus\mathring{X}$.

Proof. In view of Lemma 1 each function in $\mathring{V}(\mathring{X})$ is equal to a constant quasi-everywhere on each component of $C\setminus\mathring{X}$.

THEOREM 3. *If* $\tilde{R}(X)$ *is dense in* $\tilde{L}_a^p(X)$, $2 \leqslant p < \infty$, *and* $\partial X$ *has at most countably many components, then* $R(X)$ *is dense in* $L_a^p(X)$.

Proof. By contradiction. Suppose that $R(X)$ is not dense in $L_a^p(X)$ By (ii) in Theorem 1 there exists a function $\varphi \in \mathring{W}_1^q(C)$ which vanishes quasi-everywhere in $C\setminus X$ but is positive on a subset $E \subset \partial X$, and $\gamma_C^q(E) > 0$. Since $\partial X$ has at most countably many components, we may assume that $E$ is contained in some component $F$.

For $2 \leqslant p < \infty$ a single point has $q$-capacity zero [7] and we can find two compact disjoint subsets $K_0, K_1 \subset E$, both of positive capacity. Take $\varrho \in \mathscr{D}(C)$ such that $\varrho = 0$ on $K_0$ and $\varrho = 1$ on $K_1$. The function $\varrho\varphi \in \mathring{W}_1^q(C)$ vanishes quasi everywhere in $C\setminus X$ but is not equal to a constant quasi-everywhere on $F \subset C\setminus\mathring{X}$. This contradicts Theorem 2.

The next theorem shows that the inverse implication holds if the number of components of $\partial X$ is finite. The general case, however, seems to be less obvious.

THEOREM 4. *If* $R(X)$ *is dense in* $L_a^p(X)$ *and* $\partial X$ *has a finite number of components, then* $R(X)$ *is dense in* $\tilde{L}_a^p(X)$.

Proof. Let $F_i$, $i = 1, 2, \ldots, m$, be the components of $C\setminus\mathring{X}$. We can find $m$ disjoint open sets $G_i \supset F_i$ and functions $\varphi_i \in \mathscr{D}(G_i)$ such that $\varphi_i = 1$ in a neighbourhood of $F_i$. Consider an arbitrary $\varphi \in \mathring{W}_1^q(C)$ equal to $d_i$ quasi-everywhere on that component of $C\setminus X$ which is contained in $F_i$. The function $\psi = \varphi - \sum d_i\varphi_i$ vanishes quasi-everywhere on $C\setminus X$ and by (iii) in Theorem 1 $\psi \in \mathring{W}_1^q(\mathring{X})$. Since $\mathring{W}_1^q(\mathring{X}) \subset \mathring{V}(\mathring{X})$ and $d_i\varphi_i \in \mathring{V}(\mathring{X})$ for $i = 1, 2, \ldots, m$, it follows that $\varphi \in \mathring{V}(\mathring{X})$.

We may now apply Theorem 2.

**5. Relation to the theory of the Bergman function.** In the following we restrict ourselves to the case $p = 2$. Consider a bounded domain $D \subset C$ such that the boundary of $D$ is equal to the boundary of its exterior. Thus $D = \mathring{X}$, where $X = \bar{D}$. By $K_D$ and $\tilde{K}_D$ we shall denote the Bergman function and the reduced Bergman function of $D$.

The following theorem holds:

THEOREM 5. *Suppose that the boundary of a bounded domain* $D$ *has measure zero and equals the boundary of the exterior of* $D$. *The following conditions are equivalent*:

(i) $R(X)$ *is dense in* $L_a^2(X)$.

(ii) *For every sequence of domains $D_n$, $n = 1, 2, \ldots$, such that $D_{n+1} \Subset D_n$ and $X = \bigcap_{n=1}^{\infty} D_n$*

$$\lim_{n\to\infty} K_{D_n}(t, \bar{t}) = K_D(t, \bar{t}) \quad \textit{for each } t \in D.$$

*Moreover, if* (ii) *holds for one sequence $D_n$, then it holds for all.*

For the proof see [8]. In exactly the same way one can prove

THEOREM 6. *Suppose that the boundary of a bounded domain $D$ has measure zero and equals the boundary of the exterior of $D$. The following conditions are equivalent:*

(i) *$\tilde{R}(X)$ is dense in $\tilde{L}_a^2(X)$.*

(ii) *For every sequence of domains $D_n$, $n = 1, 2, \ldots$, such that $D_{n+1} \Subset D_n$ and $X = \bigcap_{n=1}^{\infty} D_n$*

$$\lim_{n\to\infty} \tilde{K}_{D_n}(t, \bar{t}) = \tilde{K}_D(t, \bar{t}) \quad \textit{for each } t \in D.$$

*Moreover, if* (ii) *holds for one sequence, then it holds for all.*

Theorem 3 now gives the following

COROLLARY 2. *Suppose that the boundary of a bounded domain $D$ has measure zero, consists of countably many components and equals the boundary of the exterior of $D$. Let $D_n$ be any sequence of domains such that $D_{n+1} \Subset D_n$, $\bigcap_{n=1}^{\infty} D_n = \bar{D}$. Then $\lim_{n\to\infty} \tilde{K}_{D_n} = \tilde{K}_D$ implies $\lim_{n\to\infty} K_{D_n} = K_D$. If $\partial X$ has a finite number of components, the inverse implication is also true and therefore $\lim_{n\to\infty} \tilde{K}_{D_n} = \tilde{K}_D$.*

The proof is obvious. The last part of the statement follows from the result of Sinanian [9], who proved that condition (i) of Theorem 1 is always satisfied provided that $\partial X$ has a finite number of components.

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