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EXTENSIONS OF TOPOLOGICAL FIELDS

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In a paper [3] Knopfmacher and Sinclair showed that a normed field (Λ, v) has a finite number of non-isomorphic normed extensions if and only if $\Lambda = \boldsymbol{C}$ or Λ is a real closed field satisfying $\Lambda(i) \simeq \boldsymbol{C}$, where the isomorphism is topological.

Let $(K, \mathscr{T})$ be a non-discrete topological field. Let us recall that a field topology $\mathscr{T}$ is called *locally bounded* if there exists an open bounded subset A of K, i.e. such that for every neighbourhood V of zero there exists another neighbourhood U with $AU \subset V$, where by AU we mean the set of all elements of K of the form au, $a \in A$, $u \in U$.

The aim of this note is to give a proof of the following

THEOREM. *Let (Λ, v) be a field with valuation v. Then Λ has only a finite number of non-isomorphic valuated extensions if and only if $\Lambda \simeq \boldsymbol{C}$ topologically or Λ is a real closed field such that $\Lambda(i) \simeq \boldsymbol{C}$, where the isomorphism is topological.*

Proof. ($\Leftarrow$) By theorem 5 of [4] $\boldsymbol{R}$ and $\boldsymbol{C}$ are the only locally bounded extensions of the reals $\boldsymbol{R}$. Since every topology induced by a valuation is locally bounded [1], so the result follows. Moreover, every real closed subfield Λ of $\boldsymbol{C}$ with $\Lambda(i) \simeq \boldsymbol{C}$ is order-isomorphic to a subfield of $\boldsymbol{R}$.

($\Rightarrow$) Suppose that the topology is induced in Λ by a Krull valuation, say v, i.e. there exists a multiplicative linearly ordered group Γ such that $v\colon \Lambda \to \Gamma \cup \{0\}$, $v(x) \geqslant 0$, $v(x) = 0$ if and only if $x = 0$; $v(xy) = v(x)v(y)$; $v(x+y) \leqslant \lambda \max\{v(x), v(y)\}$ for some $\lambda \in \Gamma$ and all $x, y \in \Lambda$. We claim that the order of Γ must be Archimedean. Suppose to the contrary that $\Gamma = \Gamma_0$ is a non-Archimedean ordered group. Let $\Gamma_1 = \Gamma_0 \times \boldsymbol{Z}$ be the direct product of Γ_0 and of the integers $\boldsymbol{Z}$ with the lexicographic ordering:

$(a, b) < (c, d)$ if and only if $a < c$ or, for $a = c$, $b < d$.

Similarly, let $\Gamma_{k+1} = \Gamma_k \times \boldsymbol{Z}$ for $k = 0, 1, 2, \ldots$ Obviously, all Γ_k's are the non-Archimedean ordered groups.

Now let us consider an arbitrary sequence of the non-isomorphic fields $\Delta = \Delta_0 \subsetneqq \Delta_1 \subsetneqq \Delta_2 \subsetneqq \ldots$, where $\Delta = \Delta_0$ is the residue class field of A with respect to v.

LEMMA (see [2]). *Let K_0 be a field, v_0 a valuation on K_0 with the value group Γ_0 and the residue class field Δ_0. Let Γ_1 be a linearly ordered abelian group containing Γ_0, and Δ_1 a field containing Δ_0. Then v_0 can be extended to a valuation v_1 on a field K_1 containing K_0 with the value group Γ_1 and the residue class field Δ_1.*

From this lemma it follows that there exist valuated fields (K_i, v_i), $K_i \supset K$, $K_i \neq K_j$ for $i \neq j$, $K_0 = A$, $v_0 = v$, where $\Gamma_j = v_j(K_j^x)$, and Δ_j — the residue class field of K_j, for all $j = 0, 1, 2, \ldots$ (Here L^x denotes the multiplicative group of the field L.) Our conditions on Δ_j's and Γ_j's imply that the fields $K_0 = A, K_1, K_2, \ldots$ are pairwise non-isomorphic — a contradiction with our assumptions.

Hence it follows that Γ is an Archimedean-ordered group, so Γ is order-isomorphic to a subgroup of the multiplicative group of the positive reals $\boldsymbol{R}_+$, i.e. v is the real valuation [5].

Now the result follows from [3].

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